Topological Groups

Def: A topo group $(G, \cdot)$ consists of a topo space G and a binary operation $\cdot$ st.

i) G satisfies T_1 axiom.

ii) G is a group under "$\cdot$".

iii) $\begin{array}{ccc} G \times G & \xrightarrow{f} & G \\ (x, y) & \longmapsto & xy \end{array}$ $\quad$ $\begin{array}{ccc} G & \xrightarrow{g} & G \\ x & \longmapsto & x^{-1} \end{array}$ are conti

Rmk: Def iii) $\Leftrightarrow$ $\varphi : G \times G \to G$, $\varphi(x, y) = xy^{-1}$ is conti map.

pf: $(\Rightarrow)$ $\varphi = f \circ (id \times g)$

$(\Leftarrow)$ $g = \varphi|_{e \times G} \circ (\pi_2|_{e \times G})^{-1}$ with equation of $\varphi = f \circ (id \times g)$

e.g. i) $(\mathbb{Z}, \cdot)$, $(GL(n, \mathbb{R}), \cdot)$

ii) $(G_i)_{i \in J}$ is family of topo groups

$\Rightarrow$ $\prod_{i \in J} G_i$ with componentwise binary operation is a topo group.

Def: For a topo group G, $g \in G$. Let $l_g : x \in G \longmapsto g \cdot x \in G$, $r_g : x \in G \longmapsto x \cdot g \in G$.

Rmk: l_g, r_g are homeomorphisms.

Pf: Bijection is trivial.

$$l_g = f \circ (i|_{g\times G})^{-1}$$

prop. $H < G$. $H \subseteq G$ subspace $\Rightarrow H, \bar{H}$ are topo groups.

Pf: 1) $i: H \to G$, $h \mapsto h$. i is conti.

$\Rightarrow \varphi \circ (i \times i)$ conti: $H \times H \to H$.
$(x, y) \mapsto xy^{-1}$.

2) Besides, $\varphi(\bar{H} \times \bar{H}) = \varphi(\overline{H \times H})$
$\subset \overline{\varphi(H \times H)} = \bar{H} \Rightarrow \bar{H} < G$. subspace.

Def: $H < G$. $xH = \{x \cdot h \mid h \in H\}$ is called left coset. Denote G/H is collection of left-cosets of H in G. with quotient topo.

Rmk: $\tilde{l}_a: xH \mapsto (a \cdot x)H$. on G/H. is also a homeomorphism.

prop. $H < G$. $H \underset{\text{close}}{\subseteq} G$. $\Rightarrow \forall \{xH\}$ is closed in G/H.

Pf: **Lemma.** $G \xrightarrow{p} G/H$ canonical projection p is open (Note p is quotient)

Rmk: It won't be used in the pf.

Pf: For $U \underset{\text{open}}{\subseteq} G$. Prove $p(U)$ is open

$\Leftrightarrow$ $p^{-1}(p(U)) = \bigcup_{x \in H} l_x(U)$. open.

$\Rightarrow$ $p^{-1}(\{xH\}) = xH = l_x(H) \simeq H$. with p quotient

Cor. For $H \lhd G$. $H \leq G$ closed. Then:

G/H is topo group.

Pf: $G/H \times G/H \xrightarrow{\tilde{\varphi}} G/H$

$(g_1 H, g_2 H) \longmapsto g_1 H g_2^{-1} H = g_1 g_2^{-1} H$

$\Rightarrow \tilde{\varphi} = p \circ \varphi \circ (p \times p)^{-1}$. conti.

prop. $A, B \subseteq G$. A is closed. B is cpt. Then:

$A \cdot B$ is closed in G.

Pf: $\forall x \in G - AB$. $\Rightarrow$ $A^{-1}x \cap B = \emptyset$.

Lemma. For M cpt. N closed in t.g. G.

s.t. $M \cap N = \emptyset$. Then $\exists U$ nbd of e.

s.t. i) $MU \cap NU = \emptyset$. ii) $UN \cap UM = \emptyset$.

Pf: By Thm$^{(*)}$. we have G is T_3

separate single point in M and

N. $\Rightarrow$ Cover M by finite cover.

Apply the more general lemma. $\exists U$ nbd of e.

s.t. $A^{-1}xU \cap BU = \emptyset$. i.e. $xU \cap ABU = \emptyset$.

Rmk: Thm$^{(*)}$ is proved below.

Cor. i) $H < G$. H is cpt. $\Rightarrow$ $p: G \to G/H$ is a closed map.

ii) $H \underset{cpt}{<} G$. G/H is cpt. $\Rightarrow$ G is cpt.

Pf: i) $U \underset{closed}{\subseteq} G$. $\Rightarrow$ $p(U) = U \cdot H$ closed

ii) is from perfect map.

Def: A neighbour V of id. e is symmetric if $V = V^{-1}$. (Note: $W \cdot W^{-1}$ is symmetric)

Rmk: $\forall U$ nbd of e. $\exists$ symmetric nbd V of e. st. $V \cdot V \subset U$.

Pf: $\varphi^{-1}(U)$ is open for $e \in U \underset{open}{\subseteq} G$.

$\exists$ Basis $U_1 \times U_2 \subset \varphi^{-1}(U)$.

$\exists W \subset U_1 \cap U_2$. open. Note $W \times W \subset \varphi^{-1}(U)$

$\Rightarrow W \cdot W^{-1} \subset U$. Set $V = W \cdot W^{-1}$.

(*)

Thm: Actually. $(\underline{G}, \cdot)$ satisfies T_2. even T_3.

Pf: i) Prove: $\exists U_1, U_2$ open. st. $x \in U_1 \cap U_2 \ni y = \emptyset$.

$\Leftrightarrow$ $\exists U$ nbd of e. $xU \cap yU = \emptyset$.

$\Leftrightarrow$ $xy^{-1} \cap UU^{-1} = \emptyset$.

Note $\{xy^{-1}\}$ is close in G. $\exists V$ nbd of e. st. $x^{-1}y \notin V$. By Rmk above. $\exists UU^{-1} \subset V$ st. $x^{-1}y \notin UU^{-1}$. $\Rightarrow$ G satisfies T_2.

2) As for T_3: Note $A^{-1}x$ is closed. $e \notin A^{-1}x$.

$\Rightarrow \exists U$ nbd of e. $e \in U \subset G - A^{-1}x$

$\exists V$ open nbd of e. $A^{-1}x \cap VV^{-1} = \emptyset$.

i.e. $xV \cap AV = \emptyset$. $AV = \bigcup_{v \in V} Av$ open

<u>Cor.</u> If $H \underset{\text{close}}{<} G$. Then: G/H satisfies T_3.

<u>Thm:</u> (Lebesgue Number Lemma)

$\forall K \underset{cpt}{\subseteq} G$. with a open cover $(U_j)_{j \in J}$.

Then $\exists V$. nbd of e. st. $\forall x \in K$. $\exists j \in J$.

$xV \subset U_j$.

<u>Pf:</u> $\forall x \in K$. $\exists U_x$. nbd of e. st. $xU_x \subset U_j$

for some $j \in J$.

Choose V_x symmetric. $e \in V_x \subset U_x$. nbd.

Since K is cpt. $\Rightarrow \exists (V_j)_1^n$. $K \subset \bigcup_1^n x_j V_j$.

Set $V = \bigcap_1^n V_i$. (Note $V_x \cdot V_x \subset U_x$)

<u>Def:</u> $f: T \subset G \longrightarrow Y$ (metric space) is left / right

uniform conti if $\forall \varepsilon > 0$. $\exists V$ nbd of e. st. $\forall x, y \in T$.

$x^{-1}y \in V$ / $xy^{-1} \in V \Rightarrow d(f(x), f(y)) < \varepsilon$.

<u>Rmk:</u> f conti. T cpt $\Rightarrow f|_T$ is left and

right uniform conti (Lebesgue Number Lemma)

<u>Cor.</u> $f \in C_c(G) \Rightarrow f$ is right. left uniform

conti.