$SLE(\frac{8}{3})$

(1) Brownian excursion:

Def: Fix $z = x + iy \in \overline{\mathbb{H}}$. (X_t) is BM in $\mathbb{R}^1$ start from x. (W_t) is BM in $\mathbb{R}^3$. Start from $(y,0,0)$ indept of X_t. Set $R_t = \|W_t\|_2$. $E_t = X_t + iR_t$. Then:
We say: $(E_t)_{t\geq 0}$ is Brownian excursion in $\mathbb{H}$ starts from z.

prop. A is cpt $\mathbb{H}$-hull with $x \in \mathbb{R}^1/\bar{A}$. For (E_t) is Brownian excursion start at x. Then:
$\mathbb{P}_x((E_t)$ doesn't hit $A) = g_A'(x)$

Pf: Wlog. set $x = 0$.
Set: (Z_t) is complex BM starts from $z = x + iy \in \mathbb{H}/A$. Write $Z_t = X_t + iY_t$.
(E_t) is Brownian excursion starts from z as well.

Def: $T_r = \inf\{t \geq 0 \mid Y_t = r\}$. $T_A = \inf\{t \geq 0 \mid Z_t \in A\}$.
$S_r = \inf\{t \geq 0 \mid \operatorname{Im} E_t = r\}$. for $r \geq 0$

1°) Fix $r > y$. Set $M_t = Y_{T_0 \wedge T_r \wedge t}/y \geq 0$. is bdd mart. $M_\infty = r I_{\{T_0 > T_r\}}/y$.

Def: $d\widetilde{\mathbb{P}}/d\mathbb{P} = M_\infty$. $\widetilde{\mathbb{P}}$ is new p.m.

$\Rightarrow \widetilde{\mathbb{P}}(T_0 > T_r) = 1$.

By Girsanov: under $\widetilde{\mathbb{P}}$, for $L_t = \log z + \int_0^t dM_t/M_t$

$$dB_t = dY_t - dL_t\, dY_t$$

$$= dY_t - I_{\{t \le T_r\}}/Y_t \cdot dt \quad B_0 = 0 \text{ is SBM.}$$

Besides, $d\widetilde{Y}_t = dB_t + dt/\widetilde{Y}_t$, $t \le T_r$ has strong solution $(\widetilde{Y}_t)$ under $\widetilde{\mathbb{P}} \overset{d}{\sim} \operatorname{Im}(E_t)$ under $\mathbb{P}$.

(With both BES^3 dist!), $\widetilde{Y}_t = Y_t$ (unique)

$\Rightarrow (X_t + iY_t)_{t \le T_r}$ under $\widetilde{\mathbb{P}} \overset{d}{\sim} (E_t)_{t \le S_r}$ under $\mathbb{P}$.

follows from X_t is still SBM. under $\widetilde{\mathbb{P}}$.

2°) Set $P_r(z) = \mathbb{P}_z((E_t)_{t \le S_r}$ doesn't hit $A)$

$= \widetilde{\mathbb{P}}_z((Z_t)_{t \le T_r}$ doesn't hit $A)$

$= \mathbb{E}_z^{\mathbb{P}}(\eta^{-1} Y_{T_0 \wedge T_r} \cdot I_{\{T_A > T_r\}})$

$= r\,\mathbb{P}_z(T_r < T_0 \wedge T_A)/\eta$.

Note: $\operatorname{Im} g_A(z)/(r+1) = \mathbb{P}_{\operatorname{Im} g_A(z)}(T_{r+1} < T_0)$

$\le \mathbb{P}_z(T_r < T_0 \wedge T_A)$

$\le \mathbb{P}_{\operatorname{Im} g_A(z)}(T_{r-1} < T_0) = \dfrac{\operatorname{Im} g_A(z)}{r-1}$

by conformal invariance of Bm.

$\Rightarrow \mathbb{P}_z((E_t)_t$ doesn't hit $A) = \lim_{r \to \infty} P_r(z)$

$= \operatorname{Im} g_A(z)/\eta \xrightarrow{z \to 0} g_A'(0)$.

(2) Restriction Property:

Prop. A is cpt $\mathbb{H}$-hulls with $0 \notin \bar{A}$

If (γ_t) is SLE(8/3) path. Then:

$\mathbb{P}(\gamma_t$ doesn't hit $A) = g_A'(0)^{\frac{5}{8}}$

Thm. For $\Phi_A(z) = g_A(z) - g_A(0)$. A is cpt $\mathbb{H}$-hull with $0 \notin \bar{A}$. If (γ_t) is SLE(8/3). Then $(\Phi_A(\gamma_{z(t)}))_{t\geq 0}$ condition on $\{\gamma$ doesn't hit $A\}$ $\sim$ SLE(8/3).

Rmk: i) $(\gamma_{z(t)}) \mid \{\gamma$ doesn't hit $A\}$ still dist. as γ in $\mathbb{H}/A$. it's called restrition property

ii) The only SLE(k) has the restriction prop. is SLE(8/3)

Thm. (Intepretation)

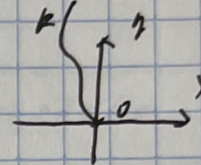
The cpt hull generated by 8 SLE($\frac{8}{3}$) $\overset{d}{\sim}$ The cpt hull generated by 5 indept Brownian excursion.

Rmk: It's intuitive by exponent of hitting prob. 5/8.

(3) Restriction Measure:

Def: i) A filling is connected set K in $\mathbb{H}$ having 0 and ∞ as limit points in $\partial\mathbb{H}$. St. $\mathbb{H}/K = D^- \cup D^+$. two simply connected component of neighbour of $(-\infty, 0)$, $(0, \infty)$.

Denote S is set of such fillings.



ii) $N = \{A \mid A$ is simply connected nbd of $0, \infty\}$

$S_D = \{K \in S \mid K \subseteq D\}$. $\mathcal{A} = \{S_D \mid D \in N\}$. $\mathcal{S} = \sigma(\mathcal{A})$.

iii) Random filling K ($(S, \mathcal{S})$-r.v.) has restriction prop. if $\forall D, D' \in N$, $D' \subseteq D$. We have:

$\mathbb{P}(K \subseteq D' \mid K \subseteq D) = \mathbb{P}(K \subseteq \phi_{\mathbb{H}/D}(D'))$

The law of K on $(S, \mathcal{S})$ is called restriction measure.

Thm. γ is SLE(8/3) path. $K = \gamma(0, \infty)$. Then: K is a random filling and has restrict prop.

Pf: Lemma. If $\mathbb{P}(\gamma$ doesn't hit $A) = \phi_A'(0)^\alpha$ for $\forall \mathbb{H}/A \in N$. Then γ has restrict property.

Pf: Note $\phi_{\mathbb{H}/D'} = \phi_{\mathbb{H}/\phi_{\mathbb{H}/D}(D')} \circ \phi_{\mathbb{H}/D}$

$\Rightarrow \mathbb{P}(K \subseteq D') = \phi_{\mathbb{H}/D'}'(0)^\alpha = \phi_F'(0)^\alpha$.

$\phi_{\mathbb{H}/D}(0)^\alpha = \mathbb{P}(K \subseteq D)\,\mathbb{P}(F)$.

Cor. $K_0 = \{E_t \mid t > 0\}$ is random filling. having restriction property. for E_t: Brownian excursion.