SLE(6).

(1) Locality:

Thm $\phi: N\cup I \xrightarrow{\sim} \tilde{N}\cup\tilde{I}$ conformal between two initial domains. s.t. $0\in I$. $\phi(0)=0\in\tilde{I}$.

Set $T=\inf\{t\geq 0 \mid \gamma_t \notin N\cup I\}$. $\tilde{T}=\inf\{t\geq 0\mid \gamma_t\notin \tilde{N}\cup\tilde{I}\}$. $S=\inf\{s\geq 0\mid \tilde{\gamma}_s\notin\tilde{N}\cup\tilde{I}\}$. $z(t)=\inf\{r\geq 0\mid \int_0^r \phi_s'(\xi_s)^2\,ds = t\}$. Then:

$$(\phi(\gamma_{z(t)}))_{t<S} \overset{d}{\sim} (\gamma_t)_{t<\tilde{T}} \overset{d}{\sim} SLE(6).$$

Pf: By prop. above: $hcap(\tilde{K}_t)=2\int_0^t \phi_s'(\xi_s)^2\,ds$.

$\Rightarrow hcap(\tilde{K}_{z(s)})=2s$. $hcap(\tilde{K}_T)=2S$.

By Itô on $\tilde{\xi}_t=\phi_t(\xi_t)$:

$\Rightarrow d\tilde{\xi}_t = \phi_t'(\xi_t)\,d\xi_t$ (from: $\frac{\partial\phi_t(\xi_t)}{\partial t} = -3\phi_t''(\xi_t)$)

So: $[\tilde{\xi}]_t = 3\,hcap(\tilde{K}_t)$.

Set $(\eta_s)_{s<S}=(\tilde{\xi}_{z(s)})_{s<S} \Rightarrow [\eta]_s=6s$.

$\Rightarrow \eta_s\sim\sqrt{6}B_s$. (extend on $\mathbb{R}^+$)

So: $(\tilde{\gamma}_s)_{s<S}=(\phi(\gamma_{z(s)}))_{s<S}\sim(\gamma_s)_{s<\tilde{T}}$.

Def: We say initial domain $N\cup I$ in (D, z_0, z_1) if $N\subset D$. simply connected and $z_0\in I\subset \partial D/\{z_1\}$

Thm. (locality)

γ is SLE(6) path in (D, z_0, z_1) and $\tilde{\gamma}$ is SLE(6) path in $(\tilde{D}, z_0, \tilde{z}_1)$. If D, $\tilde{D}$ share the same initial domain $N_0 \cup I_0$.

Then: $T = \inf\{t \geq 0 \mid \gamma_t \notin N_0 \cup I_0\}$ and $\tilde{T} = \inf\{t \geq 0 \mid \tilde{\gamma}_t \notin N_0 \cup I_0\}$ are para-invariant

Besides, $(\gamma_t)_{t<T} \sim (\tilde{\gamma}_{\tau(t)})_{t<\tilde{T}}$

Pf: Consider $D \xrightarrow[\phi]{\simeq} (\mathbb{H}, 0, \infty) \xleftarrow[\tilde{\phi}]{\simeq} \tilde{D}$.

$\overset{CI.}{\Rightarrow} \phi(\gamma_t)$, $\tilde{\phi}(\tilde{\gamma}_t)$ are SLE(6) in $\mathbb{H}$.

For $\psi = \phi \circ \tilde{\phi}^{-1}$: $\phi(\gamma_t) \sim \tilde{\phi}(\tilde{\gamma}_t) \sim \phi(\tilde{\gamma}_{\tau(t)})$

RMK: i) It means before hitting $\partial(N_0 \cup I_0)$

The two curves in different domain have same dist.

ii) The only SLE(k) can have locality is SLE(6).

(2) SLE(6) in triangle:

Denote Δ is an equilateral triangle with vertices $a=0$, $b=1$, $c=e^{\pi i/3}$.

Thm. (Schwartz - Christoffel Mapping)

U is interior of polygon γ with vertices $(w_k)_1^n$ and interior angles $(\alpha_k \pi)_1^n$. Then:

$\forall f: \mathbb{H} \xrightarrow{\sim} U$ conformal iso. st. $f(\infty) = w_n$ is of form: $f(z) = c_1 + c_2 \int_0^z \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1} d\zeta$

where $w_k = f(z_k)$. c_1, c_2 are const.

Thm. γ is $SLE(6)$ in $(\Delta, 0, 1)$ $\Rightarrow$ The point X at which γ hits $[1, e^{\pi i/3}]$ is uniform dist.

Pf: By S-C Mapping: $\exists f(z) = \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^z \frac{dw}{w^{\frac{2}{3}}(1-w)^{\frac{2}{3}}}$

is conformal: $(\mathbb{H}, 0, 1, \infty) \xrightarrow{\sim} (\Delta, 0, 1, e^{\pi i/3})$.

consider: $\varphi(z) = 1/(1-z)$ permutates $0, 1, \infty$.

$g(z) = 1 + e^{\frac{2iz}{3}} z$ permutates a, b, c.

$\Rightarrow f(\varphi(z)) = g(f(z))$.

So: $f(z) = 1 + e^{\frac{2\pi i}{3}} f(\frac{z-1}{z})$. $\forall z \in \mathbb{H}$.

extend on $\overline{\mathbb{H}}$. For $x \in [0,1]$. Choose η

st. $f(\eta/(1+\eta)) = x \Rightarrow \begin{cases} f(1) = 1 \\ f(1+\eta) = 1 + e^{\frac{2\pi i}{3}} x. \end{cases}$

$\mathbb{P}(X \in [1, 1 + x e^{\frac{2\pi i}{3}}]) = \mathbb{P}(\gamma \text{ hits } [1, 1 + x e^{\frac{2\pi i}{3}}])$

$= \mathbb{P}(f^{-1}(\gamma) \text{ hits } [1, 1+\eta])$

$= \mathbb{P}(SLE(6) \text{ hits } [1, 1+\eta])$

$= \Phi(\eta/(1+\eta)) = x$

follows from conformal invariance of SLE