Schramm - Loewner Evolution.

(1) <u>Def and Properties:</u>

① <u>Def</u>: i) r.v.'s $(K_t)_{t\geq 0} \subset \mathcal{L}$ is a $SLE(\kappa)$ if its Loewner transform $(\xi_t)_{t\geq 0} = (\kappa^{\frac{1}{2}} B_t)_{t\geq 0}$ where B_t is SBM.

<u>Rmk</u>: i) The Loewner - Kufarev Thm guarantees the existence of $SLE(\kappa)$.

ii) The motivation of introduction of SLE is for describing scaling limits of some lattice-based planar random system.

ii) r.v.'s (K_t) in $\mathcal{L}$ is scale-invariant if $(K_t^\lambda)_{t\geq 0} = (\lambda K_{\lambda^{-2}t})_{t\geq 0} \overset{\mathcal{L}}{\sim} (K_t)_{t\geq 0}$. for $\forall \lambda > 0$

<u>Rmk</u>: It's natural : $hcap(K_t) = hcap(K_t^\lambda)$. with $\xi_t^\lambda = \lambda \xi_{\lambda^{-2}t}$. $\Rightarrow (K_t^\lambda) \subset \mathcal{L}$.

iii) r.v.'s (K_t) in $\mathcal{L}$ has domain Markov property if $(K_t^{(s)}) = (K_{s,s+t} - \xi_s) \overset{\mathcal{L}}{\sim} (K_t)$ and indept of $\mathcal{F}_s = \sigma(\xi_r : r \leq s)$.

Rmk: $K_t^{(s)}$ has Loewner flow $g_t(z) =$

$$g_{K_{s,s+t}-\xi_s} = g_{K_{s,s+t}}(z+\xi_s) - \xi_s$$

and $hcap(K_t^{(s)}) = hcap(K_{s,s+t}) = 2t$

satisfies local growth prop. with

$$\xi_t^{(s)} = \xi_{s+t} - \xi_s. \Rightarrow (K_t^{(s)}) \subset \mathcal{L}.$$

iv) r.v. (K_t) is symmetric under reflection w.r.t y-axis if $(m(K_t))_{t\geq 0} \overset{d}{\sim} (K_t)_{t\geq 0}$, where $m(z) = -\bar{z}$.

Rmk: Note $hcap(m(K_t)) = \lim_{y\to\infty} \mathbb{E}_{iy}(\operatorname{Im} B_{T(\mathbb{H}/m(K_t))})$

$= hcap(K_t)$. by symmetric. $\Rightarrow m(K_t) \subset \mathcal{L}$

Thm. For (K_t). r.v.'s in $\mathcal{L}$. Then (K_t) is SLE

$\Leftrightarrow$ (K_t) is scale-invariant and has the domain Markov property.

Pf: $\Leftrightarrow$ $(\xi_t^{(s)}) \sim (\xi_t)$, indept of $\mathcal{F}_s$. (*)

and $(\xi_t^{\lambda}) \sim (\xi_t)$

(*) For general stopping time S $\Rightarrow$ Thm. holds as well!

$\Leftrightarrow$ (ξ_t) is conti. Lévy process.

and invariant under Brown-scaling.

Since $\xi_t = at + bB_t$. $\xi_t \sim \lambda \xi_{\lambda^{-2}t}$.

$\Rightarrow \xi_t = bB_t$. for some $b\geq 0$.

Prop. If (K_t) is SLE. Then $(m(K_t)) \overset{d}{\sim} (K_t)_{t\geq 0}$

② Def: Conti path $(\gamma_t)_{t\geq 0}$ in $\overline{\mathbb{H}}$ generates an increasing family of cpt $\mathbb{H}$-hulls (K_t) if $H_t = \mathbb{H}/K_t$ is the unbdd component of $\mathbb{H}/\gamma[0,t]$.

Thm. (Rohde - Schramm)

$(K_t)_{t\geq 0}$ is $SLE(\kappa)$. $\kappa \geq 0$. with g_t. ξ_t.

Consider $g_t^{-1} : \mathbb{H} \to H_t$ extend conti to $\bar{g}_t^{-1}$

Then $\gamma_t = \bar{g}_t^{-1}(\xi_t)$ is conti and generates $(K_t)_{t\geq 0}$. a.s.

Rmk: We call γ_t by $SLE(\kappa)$ path.

③ Two - point Domain:

Def: i) $D = (D, z_0, z_\infty)$ is a two-point domain if D is proper simply connected. $z_0 \neq z_\infty \in \partial D$. Denote $\mathcal{D}$ is set of such domains.

Rmk: For D'. D. two-point domains.

$\exists \phi : D' \xrightarrow{\sim} D$. conformal. and.

: $\phi(z_0) = z_0'$. $\phi(z_\infty) = z_\infty'$.

ii) We call $\sigma : D \xrightarrow{\sim} (\mathbb{H}, 0, \infty)$ a scale for D.

Rmk: $\lambda \sigma(z)$ is also a scale. $\forall \lambda > 0$.

iii) Fix $\mathcal{D} = (D, z_1, z_\infty) \in \mathcal{D}$. and scale σ.

$K \subseteq D$ is $\mathcal{D}$-hull if D/K is simply connected nbd of z_∞ in D.

Denote $\mathcal{K}(\mathcal{D})$ is set of $\mathcal{D}$-hull.

Rmk: $K \mapsto \sigma(K)$. is bijection of $\mathcal{K}(D) \to \mathcal{K}$ $\triangleq \mathcal{K}(\mathbb{H}, 0, \infty)$. We can define the caratheodory metric on $\mathcal{K}(D)$. indept of choice of σ.

iv) $\mathcal{L}(\mathcal{D}, \sigma) = \{ (K_t)_{t \geq 0} \mid (K_t)$ is increasing family of $\mathcal{D}$-hulls have local growth prop. and $\mathrm{hcap}(\sigma(K_t)) = 2t \}$.

Rmk: Similar as $\mathcal{L}$: give it metric of u.c.c.

v) r.v. (K_t) in $\mathcal{L}(\mathcal{D}, \sigma)$ is $SLE(\kappa)$ in $\mathcal{D}$ of scale σ if (ξ_t) of $(\sigma(K_t))$ is $\kappa^{\frac{1}{2}} B_t$.

Prop. (Conformal Invariance of SLE)

$\phi : D \xrightarrow{\sim} D'$ conformal between $\mathcal{D}, \mathcal{D}' \in \mathcal{D}$.

For σ, σ' scale of $\mathcal{D}, \mathcal{D}'$. set $\lambda = \sigma' \circ \phi \circ \sigma^{-1}$ (map $\mathbb{R}' \to \mathbb{R}'$) If (K_t) is $SLE(\kappa)$ in D of σ. Then $\phi(K_{\lambda^{-2} t}) = K'_t$ is $SLE(\kappa)$ in D' of σ'.

Pf: $\sigma'(K_t') = \sigma'\circ\phi\circ\sigma^{-1}(\sigma(K_{\lambda^{-2}t}))$

$= \lambda(\sigma(K_{\lambda^{-2}t})) \sim \sigma(K_t)$

Rmk: Set $\sigma' = z$, $\sigma = \phi$, $D = D' = \mathbb{H}$. $\Rightarrow$ We have: conformal invariance of SLE in $\mathbb{H}$.

prop. (K_t) is $SLE(k)$ in D of σ. T is bdd stopping time. Set $\tilde{K}_t = K_{T+t}/K_T$. $D_t = D/K_t$

$g_t = g_{\sigma(K_t)}\circ\sigma$. Def: $\sigma_T : D_T \to \mathbb{H}$ by:

$\sigma_T = g_T - \xi_T$, $z_T = g_T^{-1}(\xi_T)$. Then:

$(D_T, z_T, z_\infty) \in \mathcal{D}$. σ_T is scale of it.

Besides. $(\tilde{K}_t)_{t\geq 0} \mid \mathcal{F}_T = \sigma(\xi_s, s\leq T)$ is $SLE(k)$ in D_T of σ_T

(2) Bessel Flow and Hitting Prob.

① Bessel Equation:

Consider $X = (B^1 \cdots B^d)$ d-dim SBM.

Set $Z_t = \|X_t\|_2^2 = \sum_1^d (B^i)^2$.

By Itô's Formula:

$Z_t = Z_0 + 2\sum_1^d \int_0^t B_s^i \, dB_s^i + d\cdot t$.

Set $Y_t = \sum_1^d \int_0^t B_s^i \, dB_s^i / Z_s^{\frac{1}{2}} \Rightarrow [Y]_t = t$

$\Rightarrow Y_t = \tilde{B}_t$ By Lévy's Charac.

So: $Z_t = Z_0 + 2\int_0^t Z_s^{\frac{1}{2}} \, d\tilde{B}_s + d\cdot t$

Then we have square Bessel SDE of dim d:

$$dZ_t = 2Z_t^{\frac{1}{2}} d\widetilde{B}_t + d \cdot dt.$$

<u>Def</u>: For $d \in \mathbb{R}^1$. we say Z_t is square Bessel process of dimension d.

Denote $Z_t \sim BESQ^d$.

Set $U_t = Z_t^{\frac{1}{2}}$. By Itô Formal. we have:

$$U_t = U_0 + \frac{d-1}{2} \int_0^t ds/U_s + \widetilde{B}_t.$$

i.e. $dU_t = \frac{d-1}{2} dt/U_t + d\widetilde{B}_t$.

<u>Def</u>: We say U_t is Bessel process of dimension $d \in \mathbb{R}^1$. Denote $U_t \sim BES^d$.

<u>prop</u>. For $d \in \mathbb{R}^1$. $U_t \sim BES^d$.

i) $d < 2 \Rightarrow U_t$ hits 0 a.s.

ii) $d \geq 2 \Rightarrow U_t$ doesn't hit 0 a.s.

<u>Pf</u>: Set $M_t = \begin{cases} \log U_t & \text{if } d = 2 \\ U_t^{2-d} & \text{if } d \neq 2 \end{cases}$

By Itô $\Rightarrow M_t$ is c.l.m.

Set $\tau_a = \inf\{t \geq 0 \mid U_t = a\}$.

$\Rightarrow M_0 = \mathbb{E}(M_{t \wedge \tau_a \wedge \tau_b}) = \square$

Set $a \to 0$. $b \to \infty$.

② <u>Bessel Flow:</u>

Consider Loewner flow $(g_t(x))_{t<\tau(x)}$ $x \in \mathbb{R}^1/\{0\}$.

associated to $SLE(k)$. $\xi_s = k^{\frac{1}{2}} B_s$.

$\Rightarrow g_t(x) = x + \int_0^t 2/(g_s(x) - \xi_s)\, ds$.

Set $a = \frac{2}{k}$. $\tilde{B}_t = -\frac{\xi_t}{\sqrt{k}}$. $z(x) = \tau(x\sqrt{k})$.

$X_t = (g_t(x\sqrt{k}) - \xi_t)/\sqrt{k}$. $X_t(x) \xrightarrow{t \to z(x)} 0$ if $z(x) < \infty$

$\Rightarrow X_t(x) = x + \tilde{B}_t + \int_0^t a/X_s(x)\, ds$. Bessel Flow

So $X_t(x) \overset{d}{\sim} BES^{1+\frac{4}{k}}$

<u>Rmk:</u> By prop in ①. $\Rightarrow X_t(x)$ won't hit 0 iff

$k \leq 4$ $\Rightarrow$ $z(x) < \infty$ iff $k > 4$.

<u>prop.</u> i) Monotonicity: For $x, y \in (0, \infty)$. $x \leq y$.

$\Rightarrow z(x) \leq z(y)$. and $X_t(x) < X_t(y)$. for

all $t < z(x)$.

ii) Scaling: For $\lambda > 0$. Set $\hat{B}_t = \lambda \tilde{B}_{\lambda^{-2}t}$

$\tilde{z}(x) = \lambda^2 z(\lambda^{-1}x)$. $\tilde{X}_t(x) = \lambda X_{\lambda^{-2}t}(\lambda^{-1}x)$

$\Rightarrow \tilde{X}_t(x) \sim X_t(x)$ is also Bessel flow

of parameter a driven by $\hat{B}_t$.

<u>Pf:</u> i) $z(x) = \inf\{t \geq 0 \mid X_t(x) = 0\}$

$= \inf\{t \geq 0 \mid g_t(x\sqrt{k}) - \xi_t = 0\}$

Note $\eta_t(x) \uparrow$ on $\mathbb{R}^1$ w.r.t. x.

$\overset{\xi_0=0}{\underset{\eta_0=z}{\Rightarrow}} z(x) \le z(\eta)$. $X_t(x) \le X_t(\eta)$.

ii) By uniqueness of solutions.

<u>prop.</u> For $x, \eta > 0$. $x < \eta$. Then:

i) For $a \in (0, \frac{1}{4}]$ $\Rightarrow$ $\mathbb{P}(Z(x) < Z(\eta) < \infty) = 1$.

ii) For $a \in (\frac{1}{4}, \frac{1}{2})$ $\Rightarrow$ $\mathbb{P}(Z(x) < \infty) = 1$ and

$\mathbb{P}(Z(x) < Z(\eta)) = \phi((\eta - x)/\eta)$ where

$\phi(\theta) \propto \int_0^\theta du / (1-u)^{2a} \cdot u^{2-4a}$. $\phi(1) = 1$.

iii) For $a \in [\frac{1}{2}, \infty)$ $\Rightarrow$ $\mathbb{P}(Z(x) = \infty) = 1$. and

for $a \in (\frac{1}{2}, \infty)$. we have $X_t(x) \xrightarrow{t \to \infty} \infty$. a.s.

<u>Pf:</u> i) Set $M_t = X_t^{1-2a}$. $a > \frac{1}{2}$.

By Itô $\Rightarrow$ M_t is c.l.m.

Note $M_t \ge 0$ $\Rightarrow$ M_t is supermart.

$\therefore$ M_t converges a.s. $\Rightarrow$ $[M]_\infty = c \int_0^\infty X_t^{-4a} < \infty$

$\Rightarrow$ $X_t \to \infty$. as $t \to \infty$. a.s.

2) Set $\chi(\theta) = \int_\theta^1 du / u^{2-4a} (1-u)^{2a}$.

$\Rightarrow$ $\chi(0) < \infty$ for $a \in (\frac{1}{4}, \frac{1}{2})$.

$\chi(0) = \infty$ for $a \in (0, \frac{1}{4})$

Fix $\eta > x$. Set $X_t(\eta) = Y_t$. $Z \triangleq Z(x)$

Def $R_t = Y_t - X_t$. $\theta_t = R_t / Y_t$. $N_t = \chi(\theta_t)$

By Itô: $(N_t)_{t<z}$ is c.l.m.

Note $N_t \geq 0 \Rightarrow N_t \xrightarrow{t\to z} N_z \Rightarrow \theta_t \xrightarrow{t\to z} \theta_z$.

follows from $\chi \downarrow$. conti.

3) Claim: $z = z(\eta) \Rightarrow \theta_z = 0$ a.s.

If $\theta_z > 0$. Note $[N]_z = \int_0^z \frac{\chi'(\theta_s)^2 \theta_s^2}{Y_s^2} ds$

Then: $\int_0^z ds/Y_s^2 = \int_0^{z_\eta} ds/Y_s^2 < \infty$. ($Y_t > X_t \Rightarrow \theta_t > 0$)

But $A(\eta) = RHS = \sum_1^\infty A_n(\eta)$.

where $A_n(\eta) = \int_{T(2^{-n+1}\eta)}^{T(2^{-n}\eta)} ds/Y_s^2 > 0$. i.i.d. (SMP)

$\Rightarrow A(\eta) = \infty$ a.s. contradict!

4) For $a \in (0, \frac{1}{4}]$. If $z = z_\eta$. Then $N_t \xrightarrow{t\to z} \infty$

Since $\theta_z = 0$. $\Rightarrow$ contradict!

For $a \in (\frac{1}{4}, \frac{1}{2})$. N^z is bdd mart.

$\Rightarrow \chi(\frac{\eta - x}{\eta}) = N_0 = \mathbb{E}(N_z) = \chi(0)\, \mathbb{P}(z = z_\eta)$.

<u>Prop.</u> For $a \in (0, \frac{1}{2})$, $x, \eta \in (0, \infty)$. Then we have:

$\mathbb{P}(z(x) < z(-\eta)) = \psi(\eta/(x+\eta))$. where $\psi(1) = 1$.

$\psi(\theta) \propto \int_0^\theta du / u^{2a}(1-u)^{2a}$.

<u>Pf</u>: Set $X_t = X_t(x)$. $Y_t = -X_t(-\eta)$. $R_t = X_t + Y_t$. $\theta_t = Y_t/R_t$

Ref: $Q_t = \psi(\theta_{t \wedge z(-\eta) \wedge z(x)})$. conti. bdd.

By Itô $\Rightarrow Q_t$ is c.l.m. So bdd mart.

$\Rightarrow \mathbb{P}(z_x < z_{-\eta}) = \mathbb{E}(Q_{z(-\eta)\wedge z(x)}) = Q_0 = \psi(\frac{\eta}{x+\eta})$

③ Hitting Prob.

Prop. For γ_t is $SLE(\kappa)$ path.

Then: i) $\kappa \in (0,4] \Rightarrow \gamma[0,\infty) \cap \mathbb{R} = \{0\}$. a.s.

ii) $\kappa \in (4,8) \Rightarrow \forall x, \eta \in (0,\infty)$. γ hits both $[x,\infty)$, $(-\infty,-\eta]$. a.s. and

$\mathbb{P}(\gamma \text{ hits } [x, x+\eta)) = \phi(\eta/(x+\eta))$

$\mathbb{P}(\gamma \text{ hits } [x,\infty) \text{ before } (-\infty,-\eta]) = \psi(\frac{\eta}{x+\eta})$

iii) $\kappa \in [8,\infty) \Rightarrow \mathbb{R} \subseteq \gamma[0,\infty)$. a.s.

Rmk: Extend i):

γ will not intersect $\partial\mathbb{H}$ when $t>0$

since if $\kappa \in (0,4]$, for $\forall x \in \mathbb{R}$,

then $\tau(x) = \infty$. i.e. $x \neq g_t^{-1}(\xi_t)$

$= \gamma_t$ for $\forall x \in \mathbb{R}$, $t > 0$.

With $g_0^{-1}(\xi_0) = 0 = \gamma_0$. the start point.

Pf: **Lemma.** $\{\gamma[0,t] \text{ hits } [x,\infty)\} = \{\tau(x) \le t\}$

Pf: If $\gamma[0,t] \cap [x,\infty) = \emptyset$.

By cpt $\Rightarrow \exists U$. nbd of $[x,\infty)$

s.t. $\gamma[0,t] \cap U = \emptyset$. $\Rightarrow x \notin \overline{K_t}$

So $\tau(x) > t$

If $\exists s < t$. $\gamma_s \in [x,\infty)$. then

$\gamma_s \in \overline{K_t} \Rightarrow \tau(x) \le \tau(\gamma_s) \le t$

Cor. i) $\{ \gamma[0,t] \text{ hits } (-\infty, -x] \} = \{ \tau(-x) \le t \}$. (*)

ii) $\{ \gamma \text{ hits } [x, x+\eta) \} = \{ \tau(x) < \tau(x+\eta) \}$.

(*) Rmk: For $x > 0$:

$\{ \gamma[0,t) \text{ hit } (-\infty, x) \} = \{ \tau(x) > t \}$. Since $\gamma(0) = 0 \Rightarrow (-\infty, x) \cap \gamma[0,t] = \emptyset \Leftrightarrow x \in \bar{K}_t \Leftrightarrow \tau(x) \le t$

Pf: ii) $LHS = \bigcup_{t \ge 0} \{ \gamma[0,t] \text{ hits } [x, x+\eta) \}$

$= \bigcup_{t \ge 0} (\{ \gamma[0,t] \text{ hits } [x, \infty) \} \cap \{ \gamma[0,t) \text{ hits } (-\infty, x+\eta) \})$

$= \bigcup_{t \ge 0} \{ \tau(x) \le t < \tau(x+\eta) \} = \{ \tau(x) < \tau(x+\eta) \}$

Cor. iii) $\{ \gamma \text{ hits } [x, +\infty) \text{ before } (-\infty, -\eta] \}$

$= \{ \tau(x) < \tau(-\eta) \}$.

Pf: $LHS = \bigcup_{t \in \mathbb{R}^+} [\{ \gamma[0,t] \text{ hits } [x, \infty) \} / \{ \gamma[0,t] \text{ hits } (-\infty, -\eta] \}]$

$= \bigcup_{t \in \mathbb{R}^+} \{ \tau(x) \le t < \tau(-\eta) \}$.

$= \{ \tau(x) < \tau(-\eta) \}$.

$\Rightarrow$ Combine with prop. in ②.

(3) Phase of SLE:

Thm. For γ is SLE(κ) path, $\kappa > 0$.

Then: $|\gamma_t| \xrightarrow{t \to \infty} \infty$ a.s.

Rmk: It's intuitive since $(K_t)_{t \ge 0}$ is strictly increasing.

<u>Thm.</u> For γ is $SLE(\kappa)$ path.

i) (Simple Phase)

For $\kappa \in [0,4]$. $\Rightarrow$ (γ_t) is simple path. a.s.

ii) (Swallowing Phase)

For $\kappa \in (4,8)$. $\Rightarrow$ $\bigcup_{t\geq 0} K_t = \mathbb{H}$. a.s. and $\forall$ given $z \in \overline{\mathbb{H}}/\{0\}$. (γ_t) doesn't hit z. a.s. Besides. γ_t isn't simple path nor space-filling curve a.s.

iii) For $\kappa \in [8,\infty)$. $\Rightarrow$ $\gamma[0,\infty) = \overline{\mathbb{H}}$. a.s.

<u>Pf:</u> Only prove: γ is simple if $\kappa \leq 4$.

γ is self-intersecting if $\kappa > 4$.

Note γ intersects $\partial\mathbb{H}$ $\Leftrightarrow$ $\kappa > 4$.

Fix $t>0$. $s \mapsto g_t(\gamma(s+t)) - \xi_t$ is a $SLE(\kappa)$. by conformal invariance

With domain markov property:

the past of SLE becomes boundary of the domain where SLE evolves in furture

$\Rightarrow$ $\gamma[0,t] \cap \gamma[t,\infty)$ iff $s \mapsto \square$ hit the boundary iff $\kappa > 4$.

(Or see: $g_t(\gamma(s+t)) - \xi_t = x \Rightarrow \gamma(s+t) - x$

$= g_t^{-1}(\xi_t) - x = \gamma(t) - x \Rightarrow \gamma(s+t) = \gamma(t)$)

$\Rightarrow \forall t_n \in \mathbb{Q}^+$. $\gamma[0,t_n] \cap \gamma(t_n,\infty) = \emptyset$. a.s.

when $\kappa \leq 4$.

(4) Conformal Transformations:

① Def: An initial domain is $N \cup I$ s.t. $N \subseteq \mathbb{H}$ is simply connected and $I \subseteq \mathbb{R}^1$ is a open interval s.t. $I \subset N^\circ$.

For $\phi: N \cup I \xrightarrow{\sim} \tilde{N} \cup \tilde{I}$ conformal between initial domains. Then $\exists \phi^*: N_I^* \xrightarrow{\sim} \tilde{N}_{\tilde{I}}^*$ is reflection-invariant extension.

For cpt $\mathbb{H}$-hull K with $\bar{K} \subset N \cup I$. $I = (x^-, x^+)$

Def: $\tilde{K} = \phi(K)$. $\tilde{H} = \mathbb{H}/\tilde{K}$. $N_K = g_K(N/K)$.

$I_K = (g_K^*(x^-), g_K^*(x^+))$

Rmk: $\tilde{H} \neq \phi(H)$. $I_K \neq g_K^*(I)$.

Prop. $\tilde{K}$ is cpt $\mathbb{H}$-hull with $\bar{\tilde{K}} \subset \tilde{N} \cup \tilde{I}$

$N_K \cup I_K$ is also an initial domain.

prop. Define $(\tilde{N}_{\tilde{K}}, \tilde{I}_{\tilde{K}})$ as above and

$\phi_K: N_K \to \tilde{N}_{\tilde{K}}$ by $\phi_K = g_{\tilde{K}} \circ \phi \circ g_K^{-1}$.

$\Rightarrow \phi_K$ can be extended to isomorphism:

$N_K \cup I_K \xrightarrow{\sim} \tilde{N}_{\tilde{K}} \cup \tilde{I}_{\tilde{K}}$ of initial domain.

prop. (Approxi. of hcap($\phi(K)$))

There exists const. $C \in (0, \infty)$. s.t.

For $\phi: N \cup I \xrightarrow{\sim} \tilde{N} \cup \tilde{I}$. $0 \in I$. $\phi(0) = 0$

and $\phi'(0) = 1$. Let $K \subseteq N$ cpt $\mathbb{H}$-hull

If $\exists\, 0 < r < \varepsilon < R < \infty$. s.t. $K \cup \phi(K) \subseteq r\mathbb{D}$.

and $(\varepsilon \mathbb{D}) \cap \mathbb{H} \subseteq N \cup \tilde{N} \subseteq R\mathbb{D}$. Then:

$1 - CrR/\varepsilon^2 \leq \mathrm{hcap}(\phi(K)) / \mathrm{hcap}(K) \leq 1 + CrR/\varepsilon^2$.

Rmk: For small hull K near $\zeta \in I$. Then

$\phi'(\zeta)^2\, \mathrm{hcap}(K)$ is good approxi. of

$\mathrm{hcap}(\phi(K))$

Cor. For general case:

$\phi: N \cup I \xrightarrow{\sim} \tilde{N} \cup \tilde{I}$. $\zeta \in I$. $K \subset N$

cpt $\mathbb{H}$-hull. For $\bar{C} = C \max\{\phi'(\zeta)^2,$

$\phi'(\zeta)^{-2}\}$. If $K \subseteq \zeta + r\mathbb{D}$. $\phi(K) \subseteq$

$\phi(\zeta) + r\mathbb{D}$, $\zeta + (\varepsilon\mathbb{D}) \cap \bar{\mathbb{H}} \subseteq N \cup I \subseteq \zeta + R\mathbb{D}$

and $\phi(\zeta) + (\varepsilon \mathbb{D}) \cap \bar{\mathbb{H}} \subseteq \tilde{N} \cup \tilde{I} \subseteq \phi(\zeta) + R\mathbb{D}$.

Then: $\mathrm{hcap}(\phi(K)) / \phi'(\zeta)^2\, \mathrm{hcap}(K) \in$

$[1 - \bar{C}rR/\varepsilon^2.\ 1 + \bar{C}rR/\varepsilon^2]$.

③ Next. we consider $(K_t)_{t \geq 0}$ increasing family of cpt $\mathbb{H}$-hulls with local growth prop. with $(\xi_t)_{t \geq 0}$

For $N \cup I$, $\tilde{N} \cup \tilde{I}$ initial domains. $\xi_0 \in I$.

and $\phi : N \cup I \xrightarrow{\sim} \tilde{N} \cup \tilde{I}$. $\tilde{K}_t = \phi(K_t)$.

Set $T = \inf\{t \geq 0 \mid \overline{K_t} \nsubseteq N \cup I\}$ consider $t < T$:

Denote: $g_t = g_{K_t}$. $\tilde{g}_t = g_{\tilde{K}_t}$. $\phi_t = \tilde{g}_t \circ \phi \circ g_t^{-1}$

$\tilde{\xi}_t = \phi(\xi_t)$. $N_t = N_{K_t}$. $I_t = I_{K_t}$.

$\tilde{N}_t = \tilde{N}_{\tilde{K}_t}$. $\tilde{I}_t = \tilde{I}_{\tilde{K}_t}$.

prop. $(\tilde{K}_t)_{t<T}$ is increasing family of cpt $\mathbb{H}$-hulls having local growth prop. and having the Loewner transf. $(\tilde{\xi}_t)$

prop. $\forall t \in [0, T)$. $\Rightarrow$ $\mathrm{hcap}(\tilde{K}_t) = \int_0^t \phi_s'(\xi_s)^2 \, d(\mathrm{hcap}(K_s))$

prop. $S = \{(t, z) \mid t \in [0, T), z \in N_t \cup I_t\} \underset{\text{open}}{\subseteq} [0, \infty) \times \overline{\mathbb{H}}$

$\Rightarrow (t, z) \mapsto \phi_t(z)$ on S is t-differentiable for all z. and satisfies:

$$\begin{cases} \dfrac{\partial \phi_t(z)}{\partial t} = \dfrac{2\phi_t'(\xi_t)^2}{\phi_t(z) - \phi_t(\xi_t)} - \dfrac{2\phi_t'(z)}{z - \xi_t}, & z \in N_t \cup I_t / \xi_t \\ \dfrac{\partial \phi_t(z)}{\partial t} = -3\phi_t''(\xi_t). \end{cases}$$

Besides, $\dfrac{\partial \phi_t(z)}{\partial t}$ is holomorphic on $N_t \cup I_t$, ξ_t.

$$\begin{cases} \left(\dfrac{\partial \phi_t(z)}{\partial t}\right)' = 2\left(-\dfrac{\phi_t'(\xi_t)^2 \phi_t'(z)}{(\phi_t(z) - \phi_t(\xi_t))^2} + \dfrac{\phi_t'(z)}{(z - \xi_t)^2} - \dfrac{\phi_t''(z)}{z - \xi_t}\right) \\ \left(\dfrac{\partial \phi_t(z)}{\partial t}\right)' = \dfrac{1}{2}\dfrac{\phi_t''(\xi_t)^2}{\phi_t'(\xi_t)} - \dfrac{4}{3}\phi_t'''(\xi_t). \end{cases}$$