Chordal Loewner Theory

(1) Loewner Transform:

Def: $(K_t)_{t\geq 0}$ is family of cpt $\mathbb{H}$-hulls.

i) It's increasing if $K_s \subsetneqq K_t$. for $s<t$.

ii) It has local growth property if for

$K_{s,t} = g_{K_s}(K_t / K_s)$. $\mathrm{rad}(K_{s,s+t}) \xrightarrow{t\downarrow 0} 0$

uniformly on cpt set of s.

prop. $(K_t)_{t\geq 0}$ is increasing family of cpt $\mathbb{H}$-hulls having local growth property. Then:

i) $K_{t+} = \bigcap_{s>t} K_s = K_t$. $\forall t$.

ii) $t \longmapsto \mathrm{hcap}(K_t)$ is conti. and strictly increasing on t.

iii) $\forall t\geq 0$. There exists unique $\xi_t \in \mathbb{R}^1$.

s.t. $\xi_t \in \overline{K_{t,t+h}}$. $\forall h>0$. and (ξ_t) is conti.

Pf: i) By prop. before:

$\mathrm{hcap}(K_{t+h}) = \mathrm{hcap}(K_t) + \mathrm{hcap}(K_{t,t+h})$

Note $\mathrm{hcap}(K_{t,t+}) \leq \mathrm{hcap}(K_{t,t+h})$

$\leq \mathrm{rad}(K_{t,t+h})^2 \xrightarrow{h\to 0} 0$

$\Rightarrow K_{t,t+} = \emptyset$. i.e. $K_t = K_{t+}$

ii) By i). Set $h \to 0$. it's conti!

With $K_{t,t+h} \neq \emptyset$. it's strictly increasing.

iii) For $t \geq 0$. $(\overline{K_{t,t+h}})_{h>0}$ is nested seq of cpt set. $\Rightarrow \exists ! \xi_t = \bigcap_{h>0} \overline{K_{t,t+h}}$.

Note $\mathbb{H}/\xi_t = \bigcup_{h>0} \mathbb{H}/\overline{K_{t,t+h}}$ nested union of simply connected sets (open)

$\Rightarrow \mathbb{H}/\xi_t$ is connected $\Rightarrow \xi_t \in \mathbb{R}'$.

For $t \geq 0$. $h > 0$. $z \in K_{t+2h}/K_{t+h}$

Set $w = g_{K_t}(z) \in K_{t,t+2h}$. $w' = g_{K_{t+h}}(z) \in K_{t+h,t+2h}$

Besides. $w' = g_{K_{t+h}}(g_{K_t}^{-1}(w)) = g_{K_{t,t+h}}(w)$.

$$|\xi_{t+h} - \xi_t| \leq \underset{\substack{\wedge\!\!\wedge \\ 2\,\mathrm{rad}(K_{t+h,t+2h})}}{|\xi_{t+h} - w'|} + \underset{\substack{(*)\wedge\!| \\ 3\,\mathrm{rad}(K_{t,t+h})}}{|w' - w|} + |w - \xi_t| \xrightarrow{h \to 0} 0$$

$(*)$ is from conti estimate of g.

Def: We call $(\xi_t)_{t\geq 0}$ is Loewner transform of (K_t).

prop. (Reparametrization)

$T, T' \in (0, \infty]$. $\tau = [0, T') \xrightarrow{\sim} [0, T)$. homeomorphism.

If $(K_t)_{t<T}$ is increasing family of cpt $\mathbb{H}$-hulls having local growth property. and has lowener transf. $(\xi_t)_{t<T}$. Then: $(K_{\tau(t)})_{t<T'}$ is increasing family of cpt $\mathbb{H}$-hulls. having local growth prop. and has Loewner transf. $(\xi_{\tau(t)})_{t<T'}$

Rmk: $t \mapsto \text{hcap}(K_t)/2$ is one homeo- on $[0,T)$. Set $\tau(t)$ is inverse of it.

$\Rightarrow (K_t')_{t<T'} = (K_{\tau(t)})_{t<T}$ satisfies: $\text{hcap}(K_t') = 2t$. called parametrized by half-plane capacity.

Pf: Directly check as prop. above!

(2) Loewner's Differential Equation:

prop. $(K_t)_{t\geq 0}$ increasing family of cpt $\mathbb{H}$-hulls. Satisfy local growth prop. and parametrized by half-plane capacity with Loewner transf $(\xi_t)_{t\geq 0}$. Set $g_t \triangleq g_{K_t}$. $\tau(z) = \inf\{t\geq 0 \mid z \in K_t\}$. Then:

i) $\forall z \in \mathbb{H}$. $(g_t(z))_{t<\zeta(z)}$ is t-differentiable. and satisfies: $\frac{\partial g_t(z)}{\partial t} = 2/(g_t(z)-\xi_t)$

ii) If $\tau(z)<\infty$. Then $g_t(z) \to \xi_t \quad (t \to \tau(z))$

Pf: i) Fix $0 \leq s < t < \tau(z)$. $\text{hcap}(K_{s,t}) = 2(t-s)$

Set $z_t = g_t(z)$. $g_{K_{s,t}}(z_s) = z_t$.

Note $K_{s,t} \subseteq \xi_s + 2\,\text{rad}(K_{s,t})\,\overline{D}$.

By conti. estimate:

$$|z_t - z_s| \le 6\,\mathrm{rad}(K_{s,t}) \xrightarrow{s\to t} 0 \quad \text{So conti.}$$

By differentiable estimate:

$$\left| z_t - z_s - \frac{2(t-s)}{z_s - \xi_s} \right| \le \frac{4C\,\mathrm{rad}(K_{s,t})(t-s)}{|z_s - \xi_s|^2}.$$

So $\dot{z}_t = 2/(z_t - \xi_t)$.

ii) For $s < \tau(z) < t$. $\Rightarrow z \in K_t / K_s$. $z_s \in K_{s,t}$

$$\Rightarrow |z_s - \xi_s| \le 2\,\mathrm{rad}(K_{s,t}) \to 2\,\mathrm{rad}(K_{s,\tau(z)}) \xrightarrow{s\to\tau(z)} 0$$

(3) Inversion:

Def: i) We call conti. real-valued func. $(\xi_t)_{t\ge 0}$ driving function.

ii) $b(t,z) = 2/(z-\xi_t)$ on $\mathbb{C}/\xi_t$

Rmk: $b(t,z)$ is holomorphic on $\mathbb{C}/\xi_t$ and

$$|b(t,z) - b(t,z')| \lesssim |z-z'|. \text{ if } |z-\xi_t|.\ |z'-\xi_t| \ge c$$

prop. For $\forall z \in \mathbb{C}/\{\xi_0\}$. $\exists\, \tau(z) \in (0,\infty]$. unique. and unique conti map: $t \mapsto g_t(z): [0,\tau(z)) \to \mathbb{C}$. s.t. $\forall t \in [0,\tau(z))$, $g_t(z) \ne \xi_t$. and:

$$g_t(z) = z + \int_0^t \frac{2}{g_s(z) - \xi_s}\,ds. \quad \cdots (*)$$

If $\tau(z) < \infty$. Then $g_t(z) \to \xi_t$. as $t \to \tau(z)$.

If set $\tau(\xi_0) = 0$. Then: $\forall t \ge 0$. $C_t = \{\tau(z) > t\}$. is open. $g_t: C_t \to \mathbb{C}$. is holomorphic.

Cor. By unique determined:

$$\overline{g_t(z)} = g_t(\bar{z}). \quad r(z) = r(\bar{z}).$$

Rmk: i) We can recover g_t from ξ_t reversely and uniquely.

ii) Set $K_t = \{ z \in \mathbb{H} \mid r(z) \le t \}$.

$H_t = \mathbb{H} / K_t$.

Fix $z \in \mathbb{H}$. $s \le t < r(z)$. $\delta = \inf_{s \le t} |z_s - \xi_s|$

Take im part of $(*)$:

$\Rightarrow \operatorname{Im}(\dot{g}_s(z)) \ge -2 \operatorname{Im}(g_s(z)) / \delta^2$. $\forall s$

$\Rightarrow \operatorname{Im}(g_t(z)) \ge e^{-2t/\delta} \operatorname{Im}(g_0(z)) > 0$

So: $g_t(H_t) \subseteq \mathbb{H}$.

WLOG. restrict g_t. ξ_t in $\mathbb{H}$.

prop. $(K_t)_{t \ge 0}$ (def in Rmk ii)) is increasing family of cpt $\mathbb{H}$-hulls having local growth property.

satisfies: i) $\operatorname{hcap}(K_t) = 2t$. $g_{K_t} = g_t$. $\forall t$

ii) $(\xi_t)_{t \ge 0}$ is Loewner transf. of $(K_t)_{t \ge 0}$

Pf: Note $\operatorname{Im}(b(t, w)) < 0$ if $t \ge 0$. $w \in \mathbb{H}$.

So the Loewner differential equation has unique (well-def) solution $(w_s)_{s \le t}$

$\Rightarrow \tau(w_0) > t$. $g_t(w_0) = w$.

By uniqueness determine. $w_0 \in \mathbb{H}$ is the unique point satisfies these.

$\Rightarrow g_t : H_t \xrightarrow{\sim} \mathbb{H}$. holomorphic bijection.

<u>Lemma.</u> For (K_t). cpt $\mathbb{H}$-hulls. with (g_t). (ξ_t).

$\Rightarrow (\tilde{K}_t)_t = (K_{s,s+t})_t$ are also cpt $\mathbb{H}$-hulls with Loewner flow $\tilde{g}_t = g_{s+t} \circ g_s^{-1}$. $\tilde{\xi}_t = \xi_{s+t}$

<u>Pf:</u> $(\tilde{K}_t)$ are cpt $\mathbb{H}$-hulls by conformal isomorphism.

$\tilde{g}_t = g_{s+t} \circ g_s^{-1} = g_{K_{s,s+t}}$. Loewner flow of $\tilde{K}_t$.

$\tilde{g}_t(\tilde{K}_{t+h} / \tilde{K}_t) = g_{t+s}(K_{s+t+h} / K_{s+t}) \downarrow \xi_{s+t}$.

(4) <u>Characterization of $\bar{K}_t \cap \mathbb{R}$:</u>

Note that we can extend g_t on H_t to g_t^* on H_t^*

For $z \in \mathbb{C}$. Define $\tau^*(z) = \inf\{t \geq 0 \mid z \notin H_t^*\}$.

<u>prop.</u> $\tau^*(z) = \tau(z)$ on $\mathbb{C}$. $H_t^* = C_t$. $g_t^* = g_t$ on C_t. $\forall t$.

<u>Pf:</u> It follows from cor above: $\overline{g_t(z)} = g_t(\bar{z})$. and $\tau(z) = \tau(\bar{z})$.

<u>prop.</u> For $x \in \mathbb{R}$. $\forall t > 0$. Then: we have.

$x \in \bar{K}_t \iff \tau(x) \leq t$.

<u>Def:</u> i) $\mathcal{K}$ is set of cpt $\mathbb{H}$-hulls.

ii) Fix metric d of u.c.c. in $C(\mathbb{H}, \mathbb{H})$

Set Carathéodory metric d_K on K :

$$d_K(K_1, K_2) = d(g_{K_1}^{-1}, g_{K_2}^{-1}).$$

iii) $\mathcal{L}$ is set of increasing family of cpt $\mathbb{H}$-hulls $(K_t)_{t\geq 0}$ having local growth prop. and $\mathrm{hcap}(K_t) = 2t$. $\forall t$.

Rmk: $\mathcal{L} \subset C([0, \infty), K)$. Fix metric d of u.c.c. on it.

Thm. (Loewner - Kufarev)

There exists a bi-adapted homeomorphism L : $C([0,\infty), \mathbb{R}) \xrightarrow{\sim} \mathcal{L}$. st. $L((\xi_t)_{t\geq 0}) = (K_t)_{t\geq 0}$.

where $K(t) = \{z \mid \tau(z) \leq t\}$. $\tau(z)$ is lifetime of $Z_t = z + \int_0^t 2/(Z_t - \xi_t)\, dt$

Moreover. $\overline{K_t} \cap \mathbb{R} = \{x \in \mathbb{R} \mid \tau(x) \leq t\}$. and

$\xi_t = \bigcap_{s>t} \overline{K_{t,s}}$. $K_{t,s} = g_t(K_s / K_t)$

Rmk: We call L by Loewner map.