Mapping-out Functions.

(1) Compact $\mathbb{H}$-hulls:

Def: $K \subseteq \mathbb{H}$ is compact $\mathbb{H}$-hull if K is bdd and $\mathbb{H}/K$ is simply connected.

① Reflection:

Def: For $D \subsetneq \mathbb{H}$. simply connected.

Set $D^{\circ} = \{X \in \mathbb{R}' \mid \exists r > 0.\ s.t.\ B(x,r) \cap \mathbb{H} \subset D\}$.

$D^* = D \cup D^{\circ} \cup \{\bar{z} \mid z \in D\}$.

$D_u^* = D \cup u \cup \{\bar{z} \mid z \in D\}.\ \forall u \underset{open}{\subseteq} D^{\circ}$.

Def: i) $f^*: D_u^* \to \mathbb{C}$ is reflection invariant if

$f^*(\bar{z}) = \overline{f^*(z)}.\ \forall z \in D_u^*$.

ii) For f conti. on D. f^* is conti. extension by reflection of f if it extends f on D_u^*.

Rmk: i) The extension is unique

ii) The extension exists when f can extends on $D \cup u$ continuously and $f(u) \subset \mathbb{R}'$.

prop. $D \subset \mathbb{H}$ simply connected. $I \subseteq D^\circ$ open interval. $x \in I$. Then there exists a unique conformal isomorphism ϕ: $D \to \mathbb{H}$ extends to homeomorphism $\tilde{\phi}$: $D \cup I \to \mathbb{H} \cup (-1,1)$. $\tilde{\phi}(x) = 0$.

Rmk: i) I is natural interval of the martin boundary δD. $\Rightarrow D^\circ \subset \delta D$

ii) ϕ can extend to ϕ^*: D_n^* $\to \mathbb{H}^*(-1,1)$.

prop. $D \subset \mathbb{H}$ simply connected. $\phi : D \xrightarrow{\sim} \mathbb{H}$ is conformal isomorphism. If ϕ is bdd on bdd domain. Then ϕ can be extend by reflec. to conformal isomorphism ϕ^* on D^*.

② Construction:

Thm. K is cpt $\mathbb{H}$-hull. $H = \mathbb{H}/K$. Then: there exists a unique conformal isomorphism $g_K : H \to \mathbb{H}$. st. $|g_K(z) - z| \to 0$. ($|z| \to \infty$) and satisfies:

i) $g_K(z) - z$ is uniform bdd in H.

ii) $\exists\, a_K \in \mathbb{R}^1$. $g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2})$. ($|z| \to \infty$)

Rmk: By i) and prop above, g_K can be extended to g_K^* on H^*. conformal iso. and we call g_K mapping-out func. of K.

Pf: 1°) Existence:

By Riemann mapping: $\exists g: H \xrightarrow{\sim} \mathbb{H}$. s.t. $|g(z)| \to \infty$ as $|z| \to \infty$.

By Schwartz reflec. (Prop above.)

$\Rightarrow$ extend g on $\mathbb{C} / (\bar{A} \cup \{\bar{z} \mid z \in A\})$

Note $h(z) \triangleq 1/g(1/z)$. $h(0)=0$.

$\Rightarrow h(z) = \sum_{n \geq 1} a_n z^n \Rightarrow g(z) = b_{-1} z + b_0 + \sum_{n \geq 1} \frac{b_n}{z^n}$

By $g(\bar{z}) = \overline{g(z)} \Rightarrow b_i \in \mathbb{R}^1$.

Set $g_K(z) = (g(z) - b_0)/b_{-1}$

2°) Uniqueness:

For another $\tilde{g}_K$. Note $\tilde{g}_K \circ g_K^{-1} : \mathbb{H} \xrightarrow{\sim} \mathbb{H}$.

$\Rightarrow \tilde{g}_K \circ g_K^{-1} = \frac{az+b}{cz+d}$ $ad-bc=1$. $a.b.c.d \in \mathbb{R}^1$.

Note $|\tilde{g}_K \circ g_K^{-1}(z) - z| \to 0$. $(z \to \infty)$

$\Rightarrow a=d=1$. $b=c=0$. So $\tilde{g}_K = g_K$.

e.g. $g_{\overline{D} \cap \mathbb{H}} = z + z^{-1}$. $g_{(0,i]} = \sqrt{z^2+1}$

③ Properties:

prop. K is a cpt $\mathbb{H}$-hull. For $x \in \mathbb{R}$, $r > 0$.

Then. rK and $K+x$ are cpt $\mathbb{H}$-hulls

and $g_{rK}(z) = r\, g_K(z/r)$. $g_{K+x}(z) = g_K(z-x) + x$

prop. K_0, K_1 are cpt $\mathbb{H}$-hull. Set $K = K_0 \cup g_{K_0}^{-1}(K_1)$

Then K is cpt $\mathbb{H}$-hull containing K_0. and

$g_K = g_{K_1} \circ g_{K_0}$. $a_K = a_{K_0} + a_{K_1}$

Pf: $g_K : \mathbb{H}/K \xrightarrow{\simeq} \mathbb{H}$. $\Rightarrow$ $\mathbb{H}/K$ is simply connect.

Note $g_{K_0}(z) \sim z$. (as $|z| \to \infty$)

$\Rightarrow z_n (g_K(z_n) - z_n) \longrightarrow a_{K_1} + a_{K_0}$. $(z_n \to \infty)$

Cor. $\forall$ cpt $\mathbb{H}$-hull K containing K_0 can be constructed in this way.

Pf: Set $K_1 = g_{K_0}(K/K_0)$. is bdd

$\mathbb{H}_1 = \mathbb{H}/K_1 = g_{K_0}(\mathbb{H}/K)$. Simply connected.

$\Rightarrow K = K_0 \cup g_{K_0}^{-1}(K_1)$. K_1 is cpt $\mathbb{H}$-hull.

④ **Estimate:**

i) **Boundary estimate:**

prop. $S \subseteq \partial\mathbb{H}$. measurable. $\hat{B}_{T(A)}$ is Brownian limit on $\partial\mathbb{H}$. Then: $\lim_{\substack{y \to \infty \\ x/y \to 0}} \pi y\, \mathbb{P}_{x+iy}(\hat{B}_{T(A)} \in S) = \mathrm{Leb}(g_K(S))$

Pf: $g_K(x+iy) =: u+iv \Rightarrow u \sim x$. $v \sim y$.

Note $\mathbb{P}_{x+i\eta}(\hat{B}_{T(H)} \in S) = \mathbb{P}_{u+iv}(B_{T(H)} \in g_K(S))$

$$= \int_{g_K(S)} v/\pi((t-u)^2+v^2)\, dt$$

with $u/\eta \to 0$. $v/\eta \to 1$ when $\eta \to \infty$. $x/\eta \to 0$.

$\Rightarrow$ multiply $\pi\eta$ and set $\eta \to \infty$.

__Cor.__ i) For $(a,b) \subseteq H^\circ$. $\Rightarrow g_K(b) - g_K(a) =$

$$\lim_{\eta\to\infty} \pi\eta\, \mathbb{P}_{i\eta}(B_{T(H)} \in (a,b))$$

ii) $\lim_{\eta\to\infty} \pi\eta\, \mathbb{P}_{i\eta}(B_{T(H)} \in K) = \lim_{\eta\to\infty} \pi\eta\, \mathbb{P}_{i\eta}(B_{T(H)} \notin H^\circ)$

$$= \mathrm{Leb}(\mathbb{R}^1 / g_K(H^\circ)).$$

__Pf:__ i) Set $S = (a,b)$. $x = 0$

ii) $\partial H / K \cup H^\circ$ is countable.

__Rmk:__ by i) $\Rightarrow$ $g_K(x) \uparrow$ on real line.

__Prop.__ K is cpt $\mathbb{H}$-hull. $x \in \mathbb{R}^1$. st. $[x,\infty) \cap \bar{K} = \emptyset$.

Then $g_K(x) \geq x$. if in addition. $K \subseteq \mathbb{D}$. $x > 1$.

then $g_K(x) \leq x + 1/x$.

__Pf:__ i) For $\eta > \mathrm{rad}(K)$. We have:

$\mathbb{P}_{i\eta}(B_{T(H)} \in (x,b)) \geq \mathbb{P}_{i\eta}(B_{T(H)} \in (x,b))$

Apply Cor. above. multiply $\pi\eta$. Set $\eta \to \infty$. $b \to \infty$.

(Note $g_\emptyset(z) = z$.)

2) By $\mathbb{P}_{iy}(B_{T(H/\mathbb{D})}\in(x,b)) \le \mathbb{P}_{iy}(B_{T(H)}\in(x,b))$

multiply πy. Set $y\to\infty$ and $b\to\infty$.

(Note we know $g_{\mathbb{D}\cap H} = 1/z + z$.

ii) <u>Continuity estimate:</u>

<u>prop.</u> K is cpt H-hull. Then $|g_K(z)-z| \le 3\,\mathrm{rad}(K)$. $z\in H$.

<u>Pf</u>: WLOG. Set $K\subseteq\mathbb{D}$. $\mathrm{rad}(K)=1$.

Consider (B_t) BM starts at z.

Note $g_K(B_t)-B_t$ is bdd conti mart.

$\Rightarrow g_K(z)-z = \mathbb{E}_z(g_K(B_T)-B_T)$ ... $(*)$.

By: $x\le g_K(x)\le x+1/x$.

$\Rightarrow \{|x|>2\}\subseteq g_K\{|x|>1\}$. Then:

$$\begin{cases} |g_K(B_T)-B_T| \le 1/|B_T| & \text{if } |B_T|>1. \\ g_K(B_T)\ \&\ g_K(|x|>1) & \text{if } |B_T|\le 1. \end{cases}$$

<u>Cor.</u> From $(*)$: $\mathrm{Im}(z-g_K(z)) = \mathbb{E}_z(\mathrm{Im}(B_T))$

$\Rightarrow a_K = \lim_{y\to\infty} y\,\mathbb{E}_{iy}(\mathrm{Im}(B_T))$.

iii) <u>Differentiable estimate:</u>

<u>prop.</u> $\exists\, c<\infty$. s.t. $\forall r>0$. $\xi\in\mathbb{R}'$. cpt H-hull

$K\subseteq r\mathbb{D}+\xi$. We have:

$$\left|g_K(z)-z-\frac{a_K}{z-\xi}\right| \le \frac{c\,r\,a_K}{|z-\xi|^2}.\quad |z-\xi|\ge 2r.$$

Pf: WLOG. Let $r=1$. $\xi=0$, $K\subseteq\overline{\mathbb{D}}$. $D=\mathbb{H}/\overline{\mathbb{D}}$

Def $a(\theta)=\mathbb{E}_{e^{i\theta}}(\mathrm{Im}(B_{T(H)}))$

By conformal invariance: $\eta=z+z^{-1}: D\xrightarrow{\sim}\mathbb{H}$

$\Rightarrow \mathrm{Im}(z-g_K(z))=\mathrm{Im}\left(\frac{1}{2\cos\theta-\eta(z)}\right)\cdot a$

where $a=\int_0^{\pi}\frac{2\sin\theta}{\pi}a(\theta)\,d\theta$.

Next prove $a=a_K$.

Cor. $a_K=\frac{2}{\pi}\int_0^{\pi}\frac{2\sin\theta}{\pi}\mathbb{E}_{e^{i\theta}}(\mathrm{Im}(B_{T(H)}))\,d\theta$.

Cor. $a_K\geq 0$. and $a_K=0 \Leftrightarrow K=\emptyset$.

Pf: From the expression of a_K above.

(2) Capacity:

① Capacity from ∞ in $\mathbb{H}$:

Def: The capacity from ∞ in $\mathbb{H}$ of a cpt $\mathbb{H}$-hull K is: $\mathrm{cap}(K)=\lim_{y\to\infty}2y\,\mathbb{P}_{iy}(B_{T(H)}\in K)$

Rmk: i) $\mathrm{cap}(K)\leq\mathrm{cap}(\tilde{K})$ if $K\subseteq\tilde{K}$.

ii) By conformal invariance of BM:

$\mathrm{cap}(rK)=r\,\mathrm{cap}(K)$, $\mathrm{cap}(K+x)=\mathrm{cap}(K)$

iii) By Boundary estimate:

$\mathrm{cap}(\mathbb{H}\cap\overline{\mathbb{D}})=4$. $\mathrm{cap}((0,i])=2$.

prop. K is cpt $\mathbb{H}$-hull s.t. $\bar{K}$ is connected.

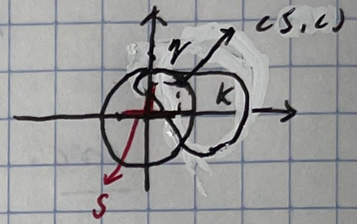
Then: $\mathrm{rad}(K) \le \mathrm{cap}(K) \le 4\,\mathrm{rad}(K)$.

Pf: 1°) RHS: $K \subseteq r\bar{\mathbb{D}} \cap \mathbb{H} + x$. for some $x \in \mathbb{R}$

2°) WLOG. set $r=1$. and assume $\exists\, s \in (0,1]$

and $c \in [0,1]$, s.t. $c^2+s^2=1$. $is \in K$.

$-c$ or $c \in \bar{K}$.



Set $K_0 = (0, is]$.

$\ell(K) = \{-x+iy \mid x+iy \in K\}$

$\sigma(K) = K \cup \ell(K)$. $S = K_0 \cup [-c, c]$.

$\Rightarrow \mathbb{P}^{iy}(B_{T(H_0)} \in S) \le \mathbb{P}^{iy}(B_{T(\sigma(H))} \in \sigma(K))$

$= 2\,\mathbb{P}^{iy}(B_{T(H)} \in K)$

multiply πy. set $y \to \infty$. $(y>1)$

(Note B can't hit S before hit $\sigma(K)$)

prop. $A \cap K = \emptyset$. are disjoint cpt $\mathbb{H}$-hulls

Then: $\mathrm{cap}(g_A(K)) \le \mathrm{cap}(K)$.

Pf: set $g_A(iy) = u+iv$. $\Rightarrow v/y \to 1$. $u \to 0$ as $y \to \infty$.

Note $\mathbb{P}^{u+iv}(B_T \in g_A(K)) =$ (Conformal Inv.)

$\mathbb{P}^{iy}(B$ hits K before $A \cup \mathbb{R})$

$\le \mathbb{P}^{iy}(B_{\tilde{T}} \in K)$

where $T = T(\mathbb{H}/g_A(K))$. $\tilde{T} = T(H)$

① Half-plane capacity:

Def: $a_K = \lim_{|z|\to\infty} z(g_K(z)-z) \geq 0$. is $\mathrm{hcap}(K)$.

Rmk: $\mathrm{hcap}(K) = \lim_{y\to\infty} \mathbb{E}_{iy}(\mathrm{Im}\, B_{\tau(\mathbb{H})})\cdot y$

e.g. $\mathrm{hcap}(\overline{\mathbb{D}}\cap\mathbb{H}) = 1$. $\mathrm{hcap}(0,i] = \frac{1}{2}$.

prop. $\mathrm{hcap}(rK) = r^2\,\mathrm{hcap}(K)$. $\mathrm{hcap}(K+x) = \mathrm{hcap}(K)$.

Pf: Expand $g_{rK}(z) = r\,g_K(z/r)$. $g_{K+x}(z)$.

prop. $K \subseteq K'$. are cpt $\mathbb{H}$-hull. set $\tilde{K} = g_K(K'/K)$

$\Rightarrow \mathrm{hcap}(K) \leq \mathrm{hcap}(K') = \mathrm{hcap}(K) + \mathrm{hcap}(\tilde{K})$.

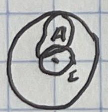
Pf: Expand $g_{K'} = g_{g_K(K'/K)} \circ g_K$.

Cor. By $K \subseteq r\mathbb{D}+x$. $\Rightarrow \mathrm{hcap}(K) \leq \mathrm{rad}(K)^2$.

Lemma (Beurling's estimate)

$A \subset\subset \mathbb{D}$. $\varepsilon \in (0,1)$. If A contains a conti. path from $\{|z|=\varepsilon\}$ to $\partial\mathbb{D}$.

Then: $\mathbb{P}_0(B[0,\tau(\mathbb{D})] \cap A = \emptyset) \leq 2\sqrt{\varepsilon}$.



Rmk: Prob. of SBM hit A connecting $\partial B(0,\varepsilon)$ and $\partial B(0,1)$. Won't be low.

prop. For $K \subseteq K'$ cpt $\mathbb{H}$-hulls. If $d(z, \partial K \cup \mathbb{R}) \leq \varepsilon$. for $\forall z \in \partial K'$. and some $\varepsilon > 0$. Then:

$\mathrm{hcap}(K') \leq \mathrm{hcap}(K) + \frac{16}{\pi}\varepsilon^{\frac{1}{2}}\,\mathrm{rad}(K')^{\frac{3}{2}}$.