

Diffusion Theory

To describe motion of a small particle suspended in a moving liquid, subject to random molecular bombardments. If $b(t, x)$ is velocity of fluid at time t and point x . Establish SDE:

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t) W_t.$$

Interpret in Itô:

$$dX_t = b(t, X_t) dt + \sigma(t, X_t) dB_t$$

Remark: b is drift coefficient and σ is diffusion coefficient

Def: (Time-homogeneous) Itô diffusion is a stochastic process $X_t(\omega) = (s, \infty) \times \mathcal{N} \rightarrow \mathbb{R}^n$ satisfies: $dX_t = b(X_t) dt + \sigma(X_t) dB_t$, $t \geq s$ and $X_s = x$.

where $\vec{B}_t$ is m -dim BM, $b: \mathbb{R}^n \rightarrow \mathbb{R}^n$.

$\sigma: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ satisfies Lipschitz condition:

$$\|b(x) - b(y)\| + \|\sigma(x) - \sigma(y)\| \leq D \|x - y\|.$$

Rmk: i) In fact, we can conclude X_t is strong Feller process by Lipschitz.

ii) Note it satisfies E & U Thm. for any hdd interval. Denote the unique solution by $X_t = X_t^{s,x}$, $t \geq s$.

iii) Replace $\sigma(X_t)$, $b(X_t)$ by $\sigma(X_t, t)$, $b(X_t, t)$. it becomes inhomogeneous case.

(1) Properties:

① Time-homogeneous:

prop. $(X_t)_{t \geq 0}$ is time-homogeneous.

Pf:
$$\begin{aligned} X_{s+h}^{s,x} &= x + \int_s^{s+h} b(X_u^{s,x}) du + \int_s^{s+h} \sigma(X_u^{s,x}) dB_u \\ &\stackrel{u=s+v}{=} x + \int_0^h b(X_{s+v}^{s,x}) dv + \int_0^h \sigma(X_{s+v}^{s,x}) d\widetilde{B}_v \end{aligned}$$

where $\widetilde{B}_v = B_{s+v} - B_s$.

$$X_h^{0,x} = x + \int_0^h b(X_v^{0,x}) dv + \int_0^h \sigma(X_v^{0,x}) d\widetilde{B}_v$$

also satisfies identical SDE.

By strong Uniqueness:

$$(X_{s+h}^{s,x})_{h \geq 0} \stackrel{d}{\sim} (X_h^{0,x})_{h \geq 0}$$

Rmk: Moreover, we have $(X_{s+h}^{s,x})_{h \geq 0}$ indpt with $\sigma \circ B_r$, $r \leq s$.

② Markov Property:

Recall: i) $\mathbb{Q}^x$ is law of $(X_s)_{s \geq 0}$.

ii) $\mathcal{F}_t^{(m)} = \sigma(B_s, s \leq t)$, $\mathcal{M}_t = \sigma(X_s, s \leq t)$

Rank: $\mathcal{M}_t \subset \mathcal{F}_t^{(m)}$ by E & μ . Then

Thm: (Simple Markov)

f is bdd Borel $:\mathbb{R}^d \rightarrow \mathbb{R}$. Then $\forall t, h \geq 0$.

$$\mathbb{E}^x (f(X_{t+h}) | \mathcal{F}_t^{(m)}) (\omega) = \mathbb{E}^{X_t(\omega)} (f(X_h))$$

Pf: $F(x, t, r, \omega) \stackrel{\Delta}{=} X_r^{t,x}(\omega)$, $r \geq t$, indep of $\mathcal{F}_t^{(m)}$

$$\Rightarrow X_r(\omega) = F(X_t, t, r, \omega)$$

$$\text{prove: } \mathbb{E} (f(F(X_t, t, t+h, \omega)) | \mathcal{F}_t^{(m)}) =$$

$$\mathbb{E} (f(F(x, 0, h, \omega))) |_{x=X_t(\omega)}$$

$$\text{Set } g(x, \omega) = f \circ F(x, t, t+h, \omega)$$

$$\exists \sum \phi_k(\omega) \varphi_k(x) \uparrow g(x, \omega).$$

$$\mathbb{E} (\sum \phi_k(\omega) \varphi_k(X_t) | \mathcal{F}_t^{(m)}) =$$

$$\sum \varphi_k(X_t) \mathbb{E} (\phi_k(\omega) | \mathcal{F}_t^{(m)}) =$$

$$\sum \varphi_k(\eta) \mathbb{E} (\phi_k(\omega) | \mathcal{F}_t^{(m)}) |_{\eta=X_t}$$

$$\Rightarrow \mathbb{E} (g(X_t, \omega) | \mathcal{F}_t^{(m)}) = \mathbb{E} (g(\eta, \omega) | \mathcal{F}_t^{(m)}) |_{\eta=X_t}$$

$$= \mathbb{E} (g(\eta, \omega)) |_{\eta=X_t} \text{ by indep!}$$

Finally, apply time-homogeneous of (X_t) .

Rank: Since $M_t \in \mathcal{F}_t^{(n)}$, $S_t(X_t)$ is also
 M_t - Markov process.

Thm (Strong Markov)

$f: \mathbb{R}^n \rightarrow \mathbb{R}^k$ bdd, Borel. τ is stopping time
w.r.t $\mathcal{F}_t^{(n)}$. $\tau < \infty$ a.s. Then:

$$\mathbb{E}^x (f(X_{\tau+h}) | \mathcal{F}_\tau^{(n)}) = \mathbb{E}^{X_{\tau}} (f(X_h))$$

Pf: Set $F(x, t, r, w) = X_{r+w}^{x,w}$. $r \geq t$. again,

Argue as in Time-homo. by strong unique:

$F(X_\tau, \tau, \tau+h, w)$ indapt with $\mathcal{F}_\tau^{(n)}$.

$$\text{prove: } \mathbb{E} (f(F(X_\tau, \tau, \tau+h, w)) | \mathcal{F}_\tau^{(n)}) = \mathbb{E} (f(F(X_\tau, \tau, h, w)) | \mathcal{F}_\tau^{(n)})$$

similar as before: $\mathbb{E} f_k(x) \varphi_k(t, r, w) \uparrow \varphi(x, t, r, w)$
 $=: f(F(x, t, r, w))$.

Cor. $(f_k)_k$ are bdd, Borel on $\mathbb{R}^k$. Then:

$$\mathbb{E}^x \left(\prod_{i=1}^n f_i(X_{\tau+h_i}) \mid \mathcal{F}_\tau^{(n)} \right) = \mathbb{E}^{X_\tau} \left(\prod_{k=1}^n f_k(X_{h_k}) \right)$$

Pf: By induction. Directly.

Cor. $\forall \eta \in M_\infty$ bdd. Then:

$$\mathbb{E}^x (\theta_\tau \eta \mid \mathcal{F}_\tau^{(n)}) = \mathbb{E}^{X_\tau} (\eta)$$

Pf: Approx by Cor. above.

Rmk: In inhomogeneous case. If

$$i) |b(x,t) - b(y,t)| \vee |b(x,t) - b(y,t)| \leq K_1 |x - y|$$

$$ii) |b(x,t)| \vee |b(x,t)| \leq K_2 (|x| + 1)$$

Then (X_t) is strong Markov process.

③ Mean Value Property:

Defn: $M \in \mathcal{B}_d^d$. $\tau_M = \inf \{t > 0 \mid X_t \notin M\}$.

where τ is another stopping time.

$$g \in C_b(\mathbb{R}^d). \quad \eta = g(X_{\tau_M}) I_{\{\tau_M < \infty\}}$$

Lemma. $\theta_\alpha \eta I_{\{\tau < \infty\}} = g(X_{\tau_M^\alpha}) I_{\{\tau_M^\alpha < \infty\}}$.

Pf: consider $\sum g(X_{t_j}) X_{(t_j, t_{j+1}]}(z_M) \rightarrow \eta$.

Cor. For $G \subset M$. $G \in \mathcal{B}_d^d$. If $\tau_M < \infty$ a.s.

Then $\theta_{z_M} g(X_{\tau_M}) = g(X_{\tau_M})$.

Pf: $\tau_M^{z_M} = \tau_M$. Since $\tau_M < \infty$ a.s.

$$\text{Cor. } \mathbb{E}^x (f(X_{\tau_M})) = \int_G \mathbb{E}^y (f(X_{\tau_M})) \alpha^x(X_{\tau_M} \in dy)$$

for $f \in C_b(\mathbb{R}^d)$. $G \subset M$ measurable.

Pf: By cor. above. apply Markov property.

Def: Harmonic measure of X on ∂G is M_h^x .

defined by $M_h^x(F) = \mathbb{E}^x(X_{\tau_h} \in F)$. $F \subset \partial G$.

Prop: By above. $\phi(x) = \mathbb{E}^x(f(X_{\tau_h}))$ satisfies mean value property:

$$\phi(x) = \int_{\partial G} \phi(y) dM_h^x(y). \quad \forall x \in G \subset \mathbb{R}^n \text{ Borel.}$$

(It also holds for all time-homogeneous strong Markov process)

(2) Generator:

Def: generator of X_t is defined by:

$$Af(x) = \lim_{t \rightarrow 0} \frac{\mathbb{E}(f(X_t^x)) - f(x)}{t} \quad \text{with } D(A) =$$

$$\{f: \mathbb{R}^n \rightarrow \mathbb{R} \mid Af \text{ exists } \forall x\}.$$

Lemma. $Y_t = Y_t^x(\omega) = x + \int_0^t u(s, \omega) ds + \int_0^t v(s, \omega) d\vec{B}_s$

is Itô process on $\mathbb{R}^n$. $\vec{B}_s$ is m -dim BM.

If $f \in C_c^2(\mathbb{R}^n)$. Z is stopping time w.r.t

$\mathcal{F}_t^{(n)}$. s.t. $\mathbb{E}^x(Z) < \infty$. u, v are bdd s.t. Y

$\in \text{supp}(f)$. Then:

$$\mathbb{E}^x(f(Y_Z)) = f(x) + \mathbb{E}^x \left(\int_0^Z \left(\sum_i u_i \frac{\partial f}{\partial x_i}(Y_s) + \frac{1}{2} \sum_{i,j} (v_i v_j^T)_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_s) \right) ds \right)$$

Pf: Apply Itô's Formula on $f(Y_t)$.

And replace z by $z_k = z \wedge k$ first.

Then. Let $k \rightarrow \infty$. z converges in L^2 .

Thm. For Itô diffusion $dX_t = b(X_t)dt + \sigma(X_t)dB_t$.

If $f \in C_c^2(\mathbb{R}^n)$. Then $f \in D(A)$. St.

$$Af(x) = \sum b_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Pf: By Lemma. directly check.

cor. $A = \frac{1}{2} \Delta$. for n -dim BM. $(\vec{B}_t)$.

Thm. (Dynkin's Formula)

If $f \in C_c^2(\mathbb{R}^n)$. τ is stopping time. st. $\mathbb{E}^x(\tau) < \infty$.

$$\text{Then: } \mathbb{E}(f(X_\tau^x)) = f(x) + \mathbb{E}\left(\int_0^\tau Af(X_s^x)ds\right)$$

Rmk: If τ is exit time of BAA Borel set.

st. $\mathbb{E}^x(\tau) < \infty$. Then it holds for $f \in C^2$.

ex. i) For n -dim BM: $(\vec{B}_t)$. start at $\vec{n}$.

τ_k is exist time of $K = \{x : |x| < R\}$.

Consider $f(x) = |x|^2$. first set $\sigma_k^\sim = \tau_k \wedge n$

$$\begin{aligned} \text{So: } \mathbb{E}^\sim(f(B_{\sigma_k^\sim})) &= f(n) + \mathbb{E}^\sim\left(\int_0^{\sigma_k^\sim} \frac{1}{2} \Delta f(B_s)ds\right) \\ &= |n|^2 + n \mathbb{E}^\sim(\sigma_k^\sim). \end{aligned}$$

$$\text{Set } n \rightarrow \infty. \quad \text{LHS} \rightarrow \mathbb{E}^a(f(B_{2^k})) = R^2$$

$$\Rightarrow \mathbb{E}^a(z_k) = \frac{1}{r} (R^2 - n^2)$$

ii) For n -dim BM (B_t) , $n \geq 2$, $|b| > R$. Next, we will find prob. of B_t start at b ever hits $K = \{x \mid |x| \leq R\}$.

Denote: τ_k is exit time of $A_k = \{R < |x| < 2^k R\}$.

$$T_k = \inf\{t > 0 \mid B_t \in K\}.$$

$$\text{Set } f(x) = \begin{cases} -\log|x|, & n=2 \\ |x|^{2-n}, & n \geq 3. \end{cases} \quad f_{n,k} \in C_c^\infty(\mathbb{R}^n).$$

Set, $f_{n,k} = f$ on A_k , so $\Delta f_{n,k} = \Delta f = 0$ on A_k .

Denote $p_k = \mathbb{P}^b(|B_{\tau_k}| = R)$, $q_k = \mathbb{P}^b(|B_{\tau_k}| = 2^k R)$

By Dynkin's: $\mathbb{E}^b(f(B_{\tau_k})) = f(b)$, $\forall k$.

(Apply on $f_{n,k}$, Note $|B_{\tau_k}| \in A_k$).

1) $n=2$:

$$-\log R \cdot p_k - \log 2^k R \cdot q_k = -\log |b|.$$

$$\text{So: } q_k \rightarrow 0 \text{ as } k \rightarrow \infty.$$

$$\text{i.e. } \mathbb{P}^b(T_k < \infty) = 1$$

2) $n \geq 3$:

$$p_k \cdot R^{2-n} + q_k (2^k R)^{2-n} = |b|^{2-n}.$$

$$\text{Set } k \rightarrow \infty, \therefore \mathbb{P}^b(T_k < \infty) = \left(\frac{|b|}{R}\right)^{2-n}$$

So: BM is transient $\Leftrightarrow n \geq 3$.

Def: i) Characteristic operator A of I_t diffusion (X_t) is defined by:

$$A f(x) = \lim_{k \rightarrow \infty} \frac{\mathbb{E}^x (f(X_{\tau_{U_k}})) - f(x)}{\mathbb{E}^x (\tau_{U_k})}$$

where $(U_k) \downarrow x$, open set. τ is exist time. If $\mathbb{E}^x (\tau_{U_k}) = \infty \forall k$.

Then define $A f(x) = 0$.

$$D_A = \{f \mid A f(x) \text{ exists } \forall x, \forall (U_k) \downarrow x\}.$$

RMK: We have $D_A \subseteq D_A$ and $A f = A f$ on $\forall f \in D_A$.

Note that C^2 is dense in L^1 . Next, we only need to prove it for C^2 .

ii) $x \in \mathbb{R}^n$ is trap for (X_t) if

$$\mathbb{Q}^x (\{X_t = x, \forall t\}) = 1.$$

Lemma. If x is not a trap for (X_t) . Then:

$$\exists U \ni x, \text{ open set, } \mathbb{E}^x (\tau_U) < \infty.$$

$$\text{pf: } \exists t, \text{ s.t. } \mathbb{Q}^x (\{X_t = x\}) < 1.$$

$\Rightarrow$ By contin. of X , $\exists U$ nbhd of x .

$$\text{s.t. } \mathbb{Q}^x (\{X_t \in U\}) \stackrel{\Delta}{=} p > 0.$$

Discretize τ_n . $\mathbb{E}(\tau_n)$ will be dominated by expectation of $\tau_{\infty}(y)$.

Thm $f \in C^2 \Rightarrow f \in DA$ and satisfies:

$$Af = \sum b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum (\sigma \sigma^T)_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Pf: 1) X is trap for (X_t) .

Then $Af(x) = 0$.

Choose V open, bdd. $V \ni x$. Modify f to f_0 outside V . st. $f_0 \in C_0^2(\mathbb{R}^d)$.

Note: $\mathbb{E}^x(f_0(X_t)) = f(x) \Rightarrow Af(x) = 0$.

2) X is not a trap for (X_t) .

$\exists U$, nbd of x . st. $\mathbb{E}^x(\tau_U) < \infty$.

Wlog. consider $U_k \subset U$. $\tau_k =: \tau_{U_k}$.

$$\left| \frac{\mathbb{E}^x(f(X_{\tau_k})) - f(x)}{\mathbb{E}^x(\tau_k)} - Af(x) \right| =$$

$$\left| \mathbb{E}^x \left(\int_0^{\tau_k} (Af(X_s) - Af(x)) ds \right) \right| / \mathbb{E}^x(\tau_k)$$

$$\leq \sup_{0 \leq s \leq \tau_k} |Af(X_s) - Af(x)| \xrightarrow{k \rightarrow \infty} 0$$

Since $Af \in C(\mathbb{R}^d)$.

Rmk: So we prove: $DA \subset DA \cap C^2$. So:

An $\mathbb{R}^d$ diffusion is a diffusion in sense of Dynkin's. (Defined by anti.

(X_t) with A generator. $\forall f \in C_0^2$. st.

$$\mathbb{E}^x(f(X_t)) = f(x) + \mathbb{E}^x \left(\int_0^t Af(X_s) ds \right)$$

(3) Kolmogorov's Backward Equation:

Thm. For $f \in C^2_c(\mathbb{R}^n)$. Define: $u(t, x) = \mathbb{E}^x(f(X_t))$

Then $u(t, \cdot) \in D_A$, $\forall t$, and satisfies:

$$\begin{cases} \partial u / \partial t = Au, & t > 0, x \in \mathbb{R}^n. \\ u(0, x) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

Moreover if $u(t, x) \in C^{1,2}_b(\mathbb{R}_+ \times \mathbb{R}^n)$ and satisfies equation above. Then $u = W$ n.s.

Pf: $t \mapsto u(t, x)$ is differentiable. Directly check.

For uniqueness:

$$\text{If } \tilde{A}W = AW - \partial W / \partial t = 0, \quad W(0, x) = f(x).$$

$$\text{Define } Y_t(W) = (s-t, X_t^{s,x}), \quad \text{fix } (s, x).$$

which has generator $\tilde{A}$.

$$\text{Apply Dynkin's: } \mathbb{E}^{s,x}(W(Y_t)) = W(s, x), \quad \forall t \geq 0$$

Rmk: Define semigroup $\mathcal{Q}_t = f \mapsto \mathbb{E}^x(f(X_t))$

Then resolvent R_α is: $R_\alpha g(x) =$

$$\int_0^\infty e^{-\alpha t} \mathcal{Q}_t g(x) dt \stackrel{\text{Fubini}}{=} \mathbb{E}^x \left(\int_0^\infty e^{-\alpha t} g(X_t) dt \right)$$

for $\alpha > 0$, $g \in C_b$. (since $\|\mathcal{Q}_t\| \leq 1$).

Lemma: $g \geq 0$ measurable on $\mathbb{R}^n$. Def: $u(x) = \mathcal{Q}_t g(x)$.

i) g is l.s.c $\Rightarrow u$ is l.s.c

ii) $g \in C_b \Rightarrow u$ is conti.

Pf: By Pt of Uniqueness part of E and a Thm:

$$\mathbb{E}(|X_t^x - X_t^y|^2) \leq C(t) |x - y|^2. \quad (\text{From Gronwall})$$

So: We can find seq $(X_n) \rightarrow x$.

$$\text{f.t. } X_t^{x_n} \rightarrow X_t^x \text{ a.s.}$$

$$i) u(x) = \mathbb{E}(g(X_t^x)) \stackrel{\text{f.s.c.}}{\leq} \mathbb{E}(g(\lim_n X_t^{x_n}))$$

$$\stackrel{\text{Fatou's}}{\leq} \liminf_n \mathbb{E}(g(X_t^{x_n})) = \lim u(x_n)$$

ii) Apply i) on g and $-g$.

(4) Feynman - Kac Formula:

Thm. $f \in C^2(\mathbb{R}^n)$, $z \in C^1(\mathbb{R}^n)$, $z \geq 0$. A is generator of (X_t) .

If $v(t, x) = \mathbb{E}^x(e^{-\int_0^t z(X_s) ds} f(X_t))$ Then:

$$\begin{cases} \frac{\partial v}{\partial t} = Av - zv & t > 0, x \in \mathbb{R}^n \\ v(0, x) = f(x) & x \in \mathbb{R}^n. \end{cases}$$

Moreover, if $w(t, x) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n)$ and bad on every $K \times \mathbb{R}^n$, K is cpt. satisfies the equation above. then $w = v$.

Pf: Set $Y_t = f(X_t)$, $Z_t = e^{-\int_0^t z(X_s) ds}$

$$\mathcal{L} Z_t = -Z_t z(X_t) \quad \mathcal{L} Y_t = \dots \quad (\mathcal{L} \hat{0} = 0)$$

$$\mathcal{L}(Y_t Z_t) = Y_t \mathcal{L} Z_t + Z_t \mathcal{L} Y_t \quad (\mathcal{L} Z_t \mathcal{L} Y_t = 0)$$

$\Rightarrow Y_t Z_t$ is $\hat{0}$ process.

5, $V(t, X) = \mathbb{E}^X(Y_t | Z_t)$ is differentiable at t .

$$\frac{1}{r} (\mathbb{E}^X(V(t, X_{t+r})) - V(t, X))$$

$$= \frac{1}{r} (\mathbb{E}^X(\mathbb{E}^{X_r}(Z_t f(X_{t+r}))) - \mathbb{E}^X(Z_t f(X_t)))$$

$$\stackrel{\text{Markov}}{=} \frac{1}{r} (\mathbb{E}^X(\mathbb{E}^{X_r}(f(X_{t+r})) e^{-\int_0^t \gamma(X_s) ds} | \mathcal{F}_t) - \mathbb{E}^X(Z_t f(X_t)))$$

$$= \frac{1}{r} \mathbb{E}^X(Z_{t+r} f(X_{t+r}) e^{\int_0^t \gamma(X_s) ds} - Z_t f(X_t))$$

$$= \frac{1}{r} \mathbb{E}^X(f(X_{t+r}) Z_{t+r} - f(X_t) Z_t) + \frac{1}{r} \mathbb{E}^X(\dots)$$

$$\rightarrow \frac{\partial V}{\partial t} + \gamma(X) V(t, X).$$

For uniqueness:

$$\text{Set } H_t = (s-t, X_t^{s,x}, Z_t), \quad Z_t = Z + \int_0^t \gamma(X_s) ds.$$

$$\text{fix } (s, x, z) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}$$

$$\text{If } w \text{ satisfies: } \hat{A}w = -\frac{\partial w}{\partial t} + Aw - \gamma w = 0$$

$$\text{Note } H \text{ has generator } A_H = -\frac{\partial}{\partial s} + A + \gamma(x) \frac{\partial}{\partial z}$$

Argue as before:

$$\text{Set } \phi(s, x, z) = w(s, x) e^{-z}. \quad (b \text{ and } \gamma). \quad A_H \phi = 0.$$

Apply Dynkin's on $\phi(H_{t \wedge K})$. set $K \rightarrow \infty$.

Rmk: (kill ~ Diffusion)

$$\text{Note for } dX_t = b(X_t) dt + \sigma(X_t) dB_t.$$

(X_t) has generator A :

$$A f(x) = \sum b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum (\sigma \sigma^T)_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}$$

It's natural to ask:

If we can find processes whose generator

has form: $\tilde{A}f = \sum b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} - cf.$

where $c(x) \in C_B(\mathbb{R}^n).$

Def: A killing time kills X_t is a random time τ .

s.t. if set $\tilde{X}_t = X_t$ for $t < \tau$. $\tilde{X}_t = \partial \in \mathbb{R}^n$.

a coffin state. $\Rightarrow (\tilde{X}_t)$ is strong Markov process

and $\mathbb{E}^x(c f(\tilde{X}_t)) =: \mathbb{E}^x(c f(X_t) \chi_{\{c, \tau\}}) = \mathbb{E}^x(c f(X_t)) e^{-\int_0^t c(X_s) ds},$

for $\forall f \in C_B(\mathbb{R}^n)$ for some $c(s) \geq 0$

Remark: i) If $c \geq 0$. Then killing time τ associated with $c(x)$ always exists.

τ : $(\tilde{X}_t)$ is the process with generator

$$\tilde{A} = A - c(x). \text{ by FK formula.}$$

ii) $c(x)$ is interpreted as killing rate:

$$c(x) = \lim_{t \downarrow 0} \frac{1}{t} \mathbb{P}^x(X_0 \text{ is killed in time interval } (0, t]) \quad ("killed" = "in coffin")$$

iii) If $c \geq 0$. Then τ can be constructed explicitly.

(5) Mart. Problem:

① Thm. If X_t is Itô diffusion with generator A .

Then $\forall f \in C_c^2(\mathbb{R}^n)$, $M_t = f(X_t) - \int_0^t A f(X_s) ds$

is a mart. w.r.t $(\mathcal{M}_t)$.

Pf: If $dX_t = b(X_t)dt + \sigma(X_t)dB_t$.

Then: $M_t = f(x) + \int_0^t \nabla f^T(X_r) \sigma(X_r) dB_r$.

by Itô formula, $\langle \text{Ito integral is mart.} \rangle$

Def: L is semi-elliptic differential operator of form: $L = \sum b_i \frac{\partial^2}{\partial x_i^2} + \sum a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$, b_i, a_{ij} are locally bdd Borel. on $\mathbb{R}^n$.

$\tilde{P}^x$ on $(\mathbb{R}^n)^{loc, m}, B$ solves mart problem for L (start at x) if process

$$\begin{cases} M_t = f(X_t^x) - \int_0^t L f(X_r^x) dr \\ M_0 = f(x) \end{cases} \text{ is } \tilde{P}^x\text{-mart}$$

w.r.t $\mathcal{B}_{(X_s^x)^{loc, m}}$ for $\forall f \in C_0^\infty(\mathbb{R}^n)$.

where $X_0^x(\omega) = x, \omega \in \Omega$. Canonical process.

The mart problem is well-posed if $\tilde{P}^x$ is unique p.m. to solve it.

Thm: If $\tilde{Q}^x$ is p.m. on $(\mathcal{L}, \mathcal{M}) = (C(\mathbb{R}^n)^{loc, m}, B)$ induced by the law Q^x of Itô diffusion $(X_t)_{t \geq 0}$ with generator A . Then $\tilde{Q}^x$ solves mart. problem for operator A .

Rmk: It also holds for $(X_t)_t$ is even the

Weak solution of $dX_t = \sigma(X_t)dB_t + b(X_t)dt$

Cor. If $\tilde{P}^x$ solves the mart. problem for

$$L = \sum b_i \frac{\partial}{\partial x_i} + \sum \frac{1}{2} (\sigma \sigma^T)_{ij} \frac{\partial^2}{\partial x_i \partial x_j}. \text{ Then:}$$

there exists a weak solution (X_t) of

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t.$$

Moreover, the mart. problem for L is

well-posed $\Leftrightarrow (X_t)_{t \geq 0}$ is Markov process.

Remark: If b, σ satisfies Lipschitz condition

then, $\tilde{P}^x$ induced by law of $X_t = x + at$

is the unique solution for mart problem.

(But it's not necessary condition)

Thm. (Stroock, Varadhan)

$$L = \sum b_i \frac{\partial}{\partial x_i} + \sum a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \text{ has unique solution}$$

for mart. problem if (a_{ij}) is positive semi-

finite, $a_{ij}(x)$ is conti. $b(x)$ measurable and $\exists D$.

$$\text{s.t. } |b(x)| + \|a(x)\|^{\frac{1}{2}} \leq D(1+|x|).$$

② Note that $(X_t)_{t \geq 0}$ is Itô process $\Rightarrow (\phi(X_t))_{t \geq 0}$ is Itô process as well. for $\phi \in C^2(\mathbb{R}^n)$

Next, we will find conditions s.t. (X_t) is Itô diffusion.

$\Rightarrow \phi(X_t)$ is Itô diffusion as well.

Thm. $dX_t = b(X_t)dt + \sigma(X_t)dB_t$, $b \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^{n \times m}$, $X_0 = x$.
 $dY_t = u(t, \omega)dt + v(t, \omega)dB_t$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^{n \times m}$, $Y_0 = x$.

Then $(X_t) \stackrel{\mathcal{L}}{\sim} (Y_t)$. $(\Leftrightarrow) VV^T(t, \omega) = \sigma\sigma^T(Y_t^x)$.

and $\mathbb{E}^x(u(t, \cdot) | \mathcal{N}_t) = b(Y_t^x)$. $t \times \mathcal{A}_P$ n.s.

where $\mathcal{N}_t = \sigma(Y_s, s \leq t)$.

Pf: $(\Leftarrow)$ Suppose $A = \sum b_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum (\sigma\sigma^T)_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$
 is generator of (X_t) .

Def: $Mf(t, \omega) = \sum u_i(t, \omega) \frac{\partial f}{\partial x_i}(Y_t) + \frac{1}{2}$

$\sum (VV^T)_{ij}(t, \omega) \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_t)$ for $f \in C_c^2$.

Note: $\mathbb{E}^x(f(Y_s) | \mathcal{N}_t) = (\mathcal{I}_t \hat{f})$ (Formula)

$f(Y_t) + \mathbb{E}^x(\int_t^s Mf(r, \omega) dY_r | \mathcal{N}_t)$

$= f(Y_t) + \mathbb{E}^x(\int_t^s \mathbb{E}(Mf(r, \omega) | \mathcal{N}_r) | \mathcal{N}_t)$

$\stackrel{\text{and.}}{=} f(Y_t) + \mathbb{E}^x(\int_t^s Af(Y_r) dY_r | \mathcal{N}_t)$

Where $\mathbb{E}^x$ is expectation under law of (Y_t)

$\Rightarrow M_t = f(Y_t) - \int_0^t Af(Y_s) dY_s$ is mart.

w.r.t. R^x -law of $(Y_t)_{t \geq 0}$

Since (X_t) is Markov $\Rightarrow$ By uniqueness:

$(X_t) \stackrel{\mathcal{L}}{\sim} (Y_t)$. i.e. $R^x = Q^x$.

$(\Rightarrow) \lim_{h \rightarrow 0} \frac{1}{h} (\mathbb{E}^x(f(Y_{t+h}) | \mathcal{N}_t) - f(Y_t)) =$

$\sum \mathbb{E}^x(u_i(t, \omega) | \mathcal{N}_t) \frac{\partial f}{\partial x_i}(Y_t) + \frac{1}{2} \sum \mathbb{E}^x((VV^T)_{ij} | \mathcal{N}_t) \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_t)$

Besides, since $X_t \sim Y_t$, so Y_t is Markov.

$$\begin{aligned} LHS &= \lim_{h \downarrow 0} \frac{1}{h} (\mathbb{E}^{Y_t} (f(Y_{t+h})) - \mathbb{E}^{Y_t} (f(Y_t))) \\ &= \sum \mathbb{E}^{Y_t} (u_i(t, w)) \frac{\partial f}{\partial x_i} (Y_t) + \frac{1}{2} \mathbb{E}^{Y_t} ((VV^T)(t, w)) \frac{\partial^2 f}{\partial x_i \partial x_j} (Y_t) \end{aligned}$$

$$\Rightarrow \begin{cases} \mathbb{E}^X (u(t, w) | N_t) = \mathbb{E}^{Y_t} (u(t, w)) = b(Y_t) \\ \mathbb{E}^X (VV^T(t, w) | N_t) = \mathbb{E}^{Y_t} (VV^T(t, w)) = \sigma \sigma^T(Y_t) \end{cases}$$

The conclusion follows from the next Lemma:

Lemma: There exists an N_t -adapted process $W(t, w)$ s.t. $VV^T(t, w) = W(t, w) \cdot I_n$.

Pf: It's directly from Itô's formula:

$$\begin{aligned} Y_i Y_j(t, w) &= x_i x_j + \int_0^t Y_i \cdot dY_j + \int_0^t Y_j \cdot dY_i + \\ &\quad \int_0^t (VV^T)_{ij}(s, w) ds. \end{aligned}$$

Rmk: $W(t, \cdot)$ and $\sigma(t, \cdot)$ may not be N_t -adapted.

Cor. (Recognize a BM)

A Itô process $dY_t = u(t, w) dt + v(t, w) dB_t$

is BM $\Leftrightarrow \mathbb{E}^X (u(t, \cdot) | N_t) = 0, \quad VV^T(t, w) = I_n$.

Thm: $\phi(X_t)$ image of Itô diffusion X_t by C^2 func.

$\tilde{Z}_t$ (Itô diffusion) $\Leftrightarrow A(f \circ \phi) = \tilde{A}(f) \circ \phi$

$\forall f = \sum a_i x_i + \sum c_{ij} x_i x_j$ (2^n -order poly's), where

$A, \tilde{A}$ are generators of $X_t, \tilde{Z}_t$.

(b) Random Time Change:

Def: i) For $c(t, \omega) \geq 0$, $\mathcal{F}_t$ -adapted.

$\beta_t(\omega) = \int_0^t c(s, \omega) ds$ is said to be a random time change with time change rate $c(t, \omega)$.

ii) Right-inverse of β_t is $\alpha_t = \inf \{s \mid \beta_s \geq t\}$.

Rmk: i) α_t is called right-inverse is from $\beta_t \uparrow$ increasing. So $c(t, \omega)$ is right-contin. Besides, $\beta_{\alpha(t, \omega), \omega} = t$.

ii) If $c(t, \omega) > 0$. Then $\beta_t \uparrow$ strictly. $t \mapsto \alpha_t(\omega)$ is conti. So it's also left-inverse of β_t .

iii) α_t is a $\mathcal{F}_t$ -stopping time:

$$\{\alpha_t < s\} = \{t < \beta_s\} \in \mathcal{F}_s.$$

Question: For X_t , Itô diffusion. Y_t Itô process.

$$\begin{cases} dX_t = \sigma(X_t) dB_t + b(X_t) dt, & X_0 = x, \\ dY_t = \mu(t, \omega) dt + V(t, \omega) dB_t & Y_0 = x. \end{cases}$$

When does there $\exists \beta_t, \alpha_t, Y_{\alpha_t} \stackrel{\mathcal{L}}{\sim} X_t$?

Thm. If $\exists c(t, w) \geq 0$. $\mathcal{F}_t$ -adapted. s.t. $\Lambda_t \times \Lambda_P$ a.s.
 $w(t, w) = c(t, w) b(Y_t)$. $V V^T c(t, w) = c(t, w) \cdot \sigma \sigma^T(Y_t)$.
 Then: $Y_t \stackrel{\sim}{=} X_t$.

Cor. For $c(t, w) \geq 0$. $Y_t = \int_0^t \overline{c(s, w)} \Lambda B_s$.

where B_s is n -dim BM. Then:

Y_t is also an n -dim BM.

Remark i) If $c(t, w) > 0$. Then $Y_t = \hat{B}_{\beta_t}$ where

$\hat{B}_t$ is n -dim BM (since α_t is inverse)

ii) $c(t, w) \equiv 1$. Then it's special case before.

Next, we want to prove: time change of a Itô integral is again a Itô integral. Driven by n different BM. $\tilde{B}_t$:

Lemma. $s \mapsto \alpha(s, w)$ is conti. $\alpha(0, w) = 0$. Fix $t > 0$. s.t.

$\beta_t < \infty$. $\mathbb{E} c(t) < \infty$.

Set $t_j = \begin{cases} j/2^k & \text{if } j/2^k \leq \alpha_t \\ \alpha_t & \text{if } j/2^k > \alpha_t \end{cases}$

Choose r_j s.t. $\tau_{r_j} = t_j$. If $f(s, w) \geq 0$. $\mathcal{F}_s$ -adapted. bdd. s -conti. Then:

$\lim_{k \rightarrow \infty} \sum f(\tau_{r_j}, w) \Delta B_j \stackrel{L^2(\Lambda, P)}{=} \int_0^{\alpha_t} f(s, w) \Lambda B_s$ a.s.

where $\Delta B_j = B_{\alpha_{r_{j+1}}} - B_{\alpha_{r_j}}$.

Thm. (Time change Formula for Itô integral)

$c(s, \omega)$, $\tau(s, \omega)$ are $\mathcal{F}$ -conti. $\tau(0, \omega) = 0$, $\mathbb{E}(\tau_t) < \infty$, $c(s, \omega) > 0$. For B_s m -dim BM and $V(s, \omega) \in V_n^{n \times m}$ kAd. $\mathcal{F}$ -conti. Def:

$$\widetilde{B}_t = \int_0^{\tau_t} \sqrt{c(s, \omega)} \, dB_s \quad \text{as in Lemma. above.}$$

$$\begin{aligned} \text{Then: } \widetilde{B}_t & \text{ is } \mathcal{F}_{\tau_t}^{(m)} \text{-BM and } \int_0^{\tau_t} V(s, \omega) \, dB_s \\ &= \int_0^t V(\tau_r, \omega) \sqrt{c(\tau_r, \omega)} \, d\widetilde{B}_r. \quad \text{P-a.s.} \end{aligned}$$

Remark: Note: $f'(t, \omega) = (B_t^{-1})' = 1/\sqrt{c(\tau_r, \omega)}$.

$$\begin{aligned} \text{pf: } \int_0^{\tau_t} V(s, \omega) \, dB_s & \stackrel{(b.m.)}{=} \lim_{k \rightarrow \infty} \sum V(\tau_j, \omega) \Delta B_{\tau_j} \\ &= \lim_{k \rightarrow \infty} \sum V(\tau_j, \omega) \sqrt{1/c(\tau_j, \omega)} \Delta \widetilde{B}_j \\ &= \int_0^t V(\tau_r, \omega) \sqrt{1/c(\tau_r, \omega)} \, d\widetilde{B}_r \end{aligned}$$

ex. (Brownian Motion on unit sphere)

i) $n=2$:

Consider $g(t, x) = e^{ix} = (\cos x, \sin x)$.

Set $Y(t) = g(t, B_t) = (\cos B_t, \sin B_t)$

where B_t is 1-dim BM.

$Y(t) = (Y_1(t), Y_2(t))$ called BM on unit circle.

satisfies
$$\begin{cases} dY_1 = -\sin(B_t) \, dB_t - \frac{1}{2} \cos(B_t) \, dt \\ dY_2 = \cos(B_t) \, dB_t - \frac{1}{2} \sin(B_t) \, dt \end{cases}$$

$$\text{i.e. } \begin{cases} dY_1 = -Y_2 dB_2 - \frac{1}{2} Y_1 dt \\ dY_2 = Y_1 dB_2 - \frac{1}{2} Y_2 dt \end{cases}$$

ii) $n \geq 3$:

Consider $\phi(x) = x/|x|$, $x \in \mathbb{R}^n \setminus \{0\}$.

Set $Y_t = (Y_1(t), \dots, Y_n(t)) = \phi(\vec{B}_t)$. $\vec{B}_t$ is n -dim BM.

$$\text{By It\^o: } dY = \frac{1}{|B|} \cdot \sigma(Y) dB + \frac{1}{|B|^2} b(Y) dt$$

$$\text{Where } \sigma_{ij}(Y) = \delta_{ij} - Y_i Y_j, \quad b(Y) = -\frac{n-1}{2} \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$$

$$\text{Note } c(s, w) = 1/|Bc(w)|^2, \quad \text{so: } \beta(t, w) = \int_0^t 1/|B_s|^2 ds$$

$$\text{Set } Z_t(w) = Y_{\alpha(t, w)}(w)$$

By Thm above. $dZ_t = \sigma(Z) d\tilde{B} + b(Z) dt$. It's diffusion

We call Z is BM on unit sphere S^{n-1} . Since it's invariant under orthogonal transformation.

Thm.

$\phi: \mathbb{R}^2 \rightarrow \mathbb{C}$. $B = (B_1, B_2)$ 2-dim BM. If $\Delta \phi = 0$.

$|Au|^2 = |Av|^2$, $\nabla u \cdot \nabla v = 0$. Then $\phi(B)$ is 2-dim time change BM in the plane.

Pf: Set $Y = \phi(B) = (u(B_1, B_2), v(B_1, B_2))$

$$dY = \frac{1}{2} \begin{pmatrix} A_u \\ A_v \end{pmatrix} dt + \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} dB$$

So: Y is a mart $\Leftrightarrow \Delta \phi = 0$

By Gr. above: $\phi(B_1, B_2) = (\tilde{B}_{\rho_1}^{(1)}, \tilde{B}_{\rho_2}^{(1)})$

Where $\tilde{B}^{(1)}, \tilde{B}^{(2)}$ are two 1-dim BM.

$$\text{and } \beta_1(t, \omega) = \int_0^t |\nabla u|^2(B_1, B_2) ds$$

$$\beta_2(t, \omega) = \int_0^t |\nabla v|^2(B_1, B_2) ds$$

If $|\nabla u|^2 = |\nabla v|^2$, $\nabla u \cdot \nabla v = 0$. Then:

$$\sigma \sigma^T = |\nabla u|^2(B_1, B_2) I \Rightarrow Y = \vec{B}_{B_+}$$

Remark: L'evy have proved:

$\phi(B_1, B_2)$ is time change 2-lim
 $B_m \Leftrightarrow \Delta \phi = 0$.

(7) Girsanov Thm for Diffusion:

It claims: i) If we change drift coefficient of a given Itô process. Then: the law won't change dramatically

ii) Moreover, the law of new process will be absolutely conti. w.r.t the original process.

Lemma (Bayes' Rule)

For p.m. μ, ν on $(\mathcal{X}, \mathcal{G})$. st. $\nu(\omega) = f(\omega) \mu(\omega)$ for some $f \in L^1$. If X is r.v. st. $\mathbb{E}_\nu(|X|) = \int |X| f(\omega) \mu < \infty$.

Then $\mathbb{E}_\nu(X | \mathcal{H}) \mathbb{E}_\mu(f | \mathcal{H}) = \mathbb{E}_\mu(X f | \mathcal{H})$

for $\forall \mathcal{H} \subset \mathcal{G}$, sub σ -algebra.

Pf: 1) $\forall \eta \in \mathcal{H}$. $\int_{\mathcal{H}} \mathbb{E}_\nu(x|\mathcal{H}) f d\mu = \int_{\mathcal{H}} \mathbb{E}_m(fx|\mathcal{H}) d\mu$

2) On the other hand:

$$\begin{aligned} LHS &= \mathbb{E}_m(\mathbb{E}_\nu(x|\mathcal{H}) f \cdot \chi_\eta) \\ &= \mathbb{E}_m(\mathbb{E}_m(\mathbb{E}_\nu(x|\mathcal{H}) f | \mathcal{H}) \chi_\eta) \\ &= \mathbb{E}_m(\mathbb{E}_\nu(x|\mathcal{H}) \mathbb{E}_m(f|\mathcal{H}) \chi_\eta) \end{aligned}$$

Thm. (Girsanov Thm. Version I)

Y_t is Itô process: $dY_t(\omega) = \mu(t, \omega) dt + \sigma dB_t$

and $Y_1 = 0$. for $T(\text{fix}) \leq \infty$. B_t is n -dim BM.

If $M_t = e^{-\int_0^t \mu(s, \omega) dB_s - \frac{1}{2} \int_0^t \sigma^2(s, \omega) ds}$ is P -mart.

w.r.t. $(\mathcal{F}_t^{(n)})_{t \leq T}$ Def: $\mu(\omega) = M_T(\omega) \mu_P(\omega)$.

Then: μ is p.m. on $\mathcal{F}_T^{(n)}$, st. Y_t is n -dim BM. w.r.t. $\mathcal{Q}$, $0 \leq t \leq T$.

Rmk: i) $P \rightarrow \mu$ is called Girsanov trans. of measure.

ii) Recall Novikov condition: $\mathbb{E}_P(e^{\frac{1}{2} \int_0^T \sigma^2 ds}) < \infty$ guarantee $(M_t)_{t \leq T}$ is P -mart.

iii) M_t is P -mart $\Rightarrow M_T \mu_P = M_t \mu_P$ on $\mathcal{F}_t^{(n)}$. $\forall 0 \leq t \leq T$.

Pf: Check $\mathbb{E}_P(f M_T) = \mathbb{E}_P(f M_t)$.

$\forall f \in \mathcal{F}_t$. b.l.a.

Pf: 1) $\mathbb{Q}(n) = \mathbb{E}_{\mathbb{Q}}(1) = \mathbb{E}_P(M_T) = \mathbb{E}_P(M_0) = 1.$

$\mathbb{Q}_1 = \mathbb{Q}$ is a p.m.

2) Next, check Levy charac. on $\vec{Y}_t$:

First, prove $Y_i(t)$ is $\mathbb{Q}$ -mart. $\forall 1 \leq i \leq n.$

Note set $R_t = - \int_0^t \kappa(t, w) \kappa(t, w) ds - \frac{1}{2} \int_0^t \kappa^2 ds$

$$\begin{aligned} M_t &= M e^{R_t} = e^{R_t} M R_t + \frac{e^{R_t}}{2} (M R_t)^2 \\ &= M_t \sum_i (-\kappa_i(t) \kappa_i(t)) \end{aligned}$$

$$\Rightarrow M(Y_i(t) M(t)) = M_t Y_i^{(i)}(t) \kappa_i(t) \text{ P-mart.}$$

$$Y_i^{(i)} = \begin{cases} -Y_i(t) \kappa_i(t) & j \neq i \\ 1 - Y_i(t) \kappa_i(t) & j = i \end{cases}$$

By Bayes's Lemma:

$$\mathbb{E}_{\mathbb{Q}}(Y_i(t) | \mathcal{F}_s) = \frac{\mathbb{E}_P(M_t Y_i | \mathcal{F}_s)}{\mathbb{E}_P(M_t | \mathcal{F}_s)} = Y_i(s)$$

Analogously, $(Y_i(t) Y_j(t) - \delta_{ij} t) M_t$ is P-mart

Remark: i) Note $M_T > 0$ a.s. So $\mathbb{Q} \sim P.$

ii) Note we shift (B_t) by $\kappa(t, w)$. Then:
it's still BM under $\mathbb{Q}$ if same
mart. condition holds.

Thm. (Girsanov Thm. Version II).

$$Y_t \in \mathbb{R}^n. \quad dY_t = \beta(t, \omega) dt + \theta(t, \omega) dB_t, \quad t \leq T.$$

Itô process. $\beta \in \mathbb{R}^n$. $\theta \in \mathbb{R}^{n \times m}$. B_t is m -dim BM.

If $\exists u(t, \omega) \in W_n^m$. $\gamma(t, \omega) \in W_n^m$. s.t.

$$\theta(t, \omega) u(t, \omega) = \beta(t, \omega) - \gamma(t, \omega).$$

$$\text{and } M_t(\omega) = \exp\left(-\int_0^t u(s, \omega) dB_s - \frac{1}{2} \int_0^t u(s, \omega)^2 ds\right) \\ t \leq T. \quad \text{is } \sim P\text{-mart.}$$

Then for $dQ = M_T dP$ on $\mathcal{F}_T^{(m)}$. Q is p.m. on

$\mathcal{F}_T^{(m)}$. s.t. $\hat{B}(t) = \int_0^t u(s, \omega) ds + B_t$, $t \leq T$ is

BM under Q . And $dY_t = \gamma(t, \omega) dt + \theta(t, \omega) d\hat{B}_t$.

Pf. 1) $\hat{B}(t)$ is BM under Q follows directly from Version I.

2) The representation is easy to check.

Thm. (Girsanov Thm. Version III. for Diffusion)

For X_t^x . $Y_t^x \in \mathbb{R}^n$. Itô Diffusion and process.

$$\text{s.t. } \begin{cases} dX_t = b(X_t) dt + \sigma(X_t) dB_t \\ dY_t = (\gamma(t, \omega) + b(Y_t)) dt + \sigma(Y_t) dB_t. \end{cases}$$

b, σ satisfies Lipschitz cond. $\gamma \in W_n^n$.

If $\exists u(t, \omega) \in W_n^m$. s.t. $\sigma(Y_t) u(t, \omega) = \gamma(t, \omega)$.

Suppose $M_t, Q, \hat{B}(t)$ as defined above.

and M_t is P -mart. w.r.t. $\mathcal{F}_t^{(m)}$. Then:

Q is p.m. on $\mathcal{F}_T^{(m)}$ and $dY_t = b(Y_t)dt + \sigma(Y_t)dB_t$, $t \leq T$.

So: Q -law of $Y_t^x = P$ -law of X_t^x for $t \leq T$.

Pf: It follows from weak uniqueness.

Rmk: It can be used to produce weak solution of SDE:

Suppose Y_t is (weak or strong) solution of $dY_t = b(Y_t)dt + \sigma(Y_t)dB_t$.

We wish to find weak solution for related SDE: $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$.

If $\exists u: \mathbb{R}^n \rightarrow \mathbb{R}^m$ s.t. $\sigma(u(y), y) = b(y) - \mu(y)$,

and satisfies Novikov's condition.

Then: $\exists (\hat{B}_t, Q)$ s.t. $(Y_t, \hat{B}_t)$ is its weak solution under Q .