SDE and Filtration Prob.

(1) Existence and Uniqueness:

① Consider: $dX_t/dt = b(t, X_t) + \sigma(t, X_t) W_t$

where W_t is white noise.

Write in Itô interpretation:

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dB_t \quad \cdots \quad (*)$$

Thm. For $T > 0$. $b(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$. and $\sigma(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$. measurable.

s.t. i) $|b(t,x)| + |\sigma(t,x)| \lesssim 1 + |x|$

ii) $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \lesssim |x-y|$

If Z is a r.v. indept of $\mathcal{F}_\infty^{(m)}$ generated by $\vec{B}_t$. s.t. $Z \in L^2$. Then:

I.V.P: $(*)$ with $X_0 = Z$. has a strong solution X_t (i.e. X_t adapted to $\mathcal{F}_t^{(m)}$ $\cap Z$. given $(\vec{B}_t)$ in advance).

s.t. i) X_t is t-conti

ii) $\mathbb{E}(\int_0^T |X_t|^2 dt) < \infty$.

iii) It's strongly unique. (pointwise)

Rmk: Condition i) ensures X_t won't explore. While Condition ii) is for the unique solution.

Pf: 1) Unique:

Suppose $X_1(t,w)$, $X_2(t,w)$ are solutions with initial values Z, $\hat{Z}$ respectively.

Set $a = b(s, X_1(s)) - b(s, X_2(s))$

$\gamma = \sigma(s, X_1(s)) - \sigma(s, X_2(s))$

$$\Rightarrow E|X_1(t,w) - X_2(t,w)|^2 = E(|Z - \hat{Z} + \int_0^t a + \int_0^t \gamma \, dB_s|^2)$$

$$\leq E|Z-\hat{Z}|^2 + E(\int_0^t a^2) + E(\int_0^t \gamma^2)$$

$$\overset{(\text{cond.})}{\lesssim} E(|Z-\hat{Z}|^2) + (1+t)\int_0^t E|X_1 - X_2|^2$$

follows from Hölder, Itô isometry.

Set $Z = \hat{Z}$. By Gronwall's inequi.

2) Existence:

By Picard seq:

Set $Y_t^{(0)} = X_0$, $Y_t^{(k+1)} = X_0 + \int_0^t b(s, Y_s^{(k)}) ds + \int_0^t \sigma(s, Y_s^{(k)}) dB_s$

Note: $\begin{cases} E|Y_t^{(1)} - Y_t^{(0)}|^2 \leq A_1 t \\ E|Y_t^{(k)} - Y_t^{(k-1)}|^2 \lesssim_T \int_0^t E|Y_s^{(k-1)} - Y_s^{(k-2)}|^2 ds \end{cases}$

$\Rightarrow$ inductively : $E|Y_t^{(k+1)} - Y_t^{(k)}|^2 \le A_T^{k+1} t^{k+1} / (k+1)!$

$\Rightarrow (Y^{(k)})_k$ is Cauchy in $L^2(m \times \mathbb{P})$

Define the limit of $Y_t^{(k)}$ is X_t.

set $k \to \infty$ in Picard seq. it's the solution.

3') Note Itô integral has a conti. modification.

③ <u>Weak Solution</u>:

<u>Def</u>: Weak solution for $(*)$ is pair of process $((\tilde{X}_t, \tilde{B}_t), \mathcal{H}_t)$ on $(\Omega, \mathcal{H}, \mathbb{P})$. s.t. given only $b(t, X_t)$, $\sigma(t, X_t)$, $(*)$ holds.

<u>Lemma</u>. If b, σ satisfies conditions of Thm above. Then $\forall$ solution is weakly unique. (i.e. have same finite-dimension dist.)

<u>Pf</u>: If $((\tilde{X}_t, \tilde{B}_t), \tilde{\mathcal{H}}_t)$, $((\hat{X}_t, \hat{B}_t), \hat{\mathcal{H}}_t)$ are two weak solutions.

suppose $\tilde{Y}_t$, $\hat{Y}_t$ are two strong solutions from $\tilde{B}_t$, $\hat{B}_t$, respectively.

$\Leftarrow$) prove : $\tilde{Y}_t$ has same law as $\hat{Y}_t$.

It's easy to see from Picard seq:

$(\tilde{Y}_t^{(k)}, \tilde{B}_t) \overset{d}{\sim} (\hat{Y}_t^{(k)}, \hat{B}_t)$, $\forall k$. set $k \to \infty$.

Rmk: $\exists$ SDE. s.t. Weak solution exists but no strong solution.

e.g. $dX_t = \mathrm{Sgn}(X_t) dB_t$. (Tanaka Equation)

(2) Filtering Problem:

Consider: $dX_t = b(t, X_t) dt + \sigma(t, X_t) dU_t$.

where $X_t \in \mathbb{R}^n$. $b: \mathbb{R}^{n+1} \to \mathbb{R}^n$. $\sigma: \mathbb{R}^{n+1} \to \mathbb{R}^{n \times p}$.

U_t is p-dim BM indept of X_0. (dist is known)

Assume b. σ satisfies Exist and Unique Thm.

With observation: $dZ_t = c(t, X_t) dt + \gamma(t, X_t) dV_t$.

s.t. $Z_0 = 0$. V_t is r-dim BM. indept of U_t. X_0.

$c: \mathbb{R}^{n+1} \to \mathbb{R}^m$. $\gamma: \mathbb{R}^{n+1} \to \mathbb{R}^{m \times r}$. Satisfies E & U Thm.

Q: What's the best estimate $\hat{X}_t$ of X_t based on the observation Z_t ?

Rmk: $(\hat{X}_t)$ should satisfy:

i) $\hat{X}_t \in \mathcal{G}_t = \sigma(Z_s, 0 \le s \le t)$.

ii) $E|X_t - \hat{X}_t|^2 = \inf\{E|X_t - Y_t|^2 \mid Y \in K_t\}$

$K_t =: \{Y: \Omega \to \mathbb{R}^n \mid Y \in \mathcal{G}_t, Y \in L^2(P)\}$.

Lemma $\mathcal{H} \subset \mathcal{F}$. sub-$\sigma$-algebra. $X \in L^2(\mathbb{P})$. Set

$\mathcal{N} = \{Y \in L^2(\mathbb{P}),\ Y \in \mathcal{H}\}$. $P_{\mathcal{N}}$ is ortho. proj. from $L^2(\mathbb{P})$. to $\mathcal{N}$. Then:

$P_{\mathcal{N}}(X) = E(X|\mathcal{H})$.

Pf: $\int_{\Omega} (X - P_{\mathcal{N}}(X)) Y = 0$. $\forall Y \in \mathcal{N}$.

Set $Y = I_A$. $A \in \mathcal{H}$. By def of $E(X|\mathcal{H})$.

Cor. $\hat{X}_t = P_{K_t}(X_t) = E(X_t | \mathcal{G}_t)$.

① One dimension Linear Problem:

Consider one-dim linear system:

$dX_t = F(t) X_t dt + C(t) dU_t$. $F. C \in \mathbb{R}^1$.

$dZ_t = G(t) X_t dt + D(t) dV_t$. $G. D \in \mathbb{R}^1$. $Z_0 = 0$

Assume: i) $F. G. C. D.$ bdd on bdd intervals.

ii) $X_0 \sim$ normal dist. indept of $U. V$.

iii) $D(t) > 0$.

Step. 1: Z-liner and Z-measurable estimates.

Lemma. $X.\ Z_s \in L^2(\mathbb{P})$. $\forall s \le t$. If $(X. Z_{s_1}, \dots Z_{s_n})$ $\sim$ Normal dist. $\forall s_1 \dots s_n \le t$. Then:

$P_{\mathcal{L}_t}(X) = P_{K_t}(X)$. $\mathcal{L}_t = CLS\{Z_s : s \le t\}$.

Pf: i) $\tilde{X} = X - P_{\mathcal{L}_t}(X)$ indept with Z_s. $\forall\, 0 \le s \le t$

ii) $E(\chi_A \tilde{X}) = 0$. $\forall A \in \mathcal{G}_t$.

$\Rightarrow P_{\mathcal{L}_t}(X) = E(X \mid \mathcal{G}_t)$

Rmk: Estimate of normal dist. will be the "worst". (Only by LFs)

Lemma $M_t = \begin{pmatrix} X_t \\ Z_t \end{pmatrix} \in \mathbb{R}^2$ is Gaussian process.

Pf: $dM_t = H_t M_t dt + K_t d\vec{B}_t$. $M_0 = \begin{pmatrix} X_0 \\ 0 \end{pmatrix}$

$H_t = \begin{pmatrix} F(t) & 0 \\ G(t) & 0 \end{pmatrix}$. $K_t = \begin{pmatrix} C(t) & 0 \\ 0 & D(t) \end{pmatrix}$

By Picard Iteration:

$M_{n+1}(t) = \int_0^t H(t) M_n(t) dt + \int_0^t K_t d\vec{B}_t + M(0)$

$M_n(t)$ is Gaussian $\forall n \longrightarrow M(t)$.

Step. 2: Innovative Process

Def: $\mathcal{L}(Z, T)$ = closure of all linear combination: $c_0 + c_1 Z_{t_1} + \cdots c_n Z_{t_n}$. $0 \le t_i \le T$. in $L^2(P)$

Lemma. $\int_0^T f^2 \lesssim \mathbb{E}((\int_0^T f(t) dZ_t)^2) \lesssim \int_0^T f^2$

for $\forall f \in L^2[0, T]$

Pf: $\mathbb{E}(\int_0^T f(t)G(t)X_t dt \int_0^T f(t)D(t)dV_t) = 0$ (indept)

$\mathbb{E}((\int_0^T f(t)G(t)X_t dt)^2) \underset{\text{Hölder}}{\lesssim} \int_0^T f^2(t)dt$

$\mathbb{E}((\int_0^T f(t)D(t)dV_t)^2) = \int_0^T f^2(t)D^2(t)dt$

Lemma. $\mathcal{L}(Z,T) = \{c_0 + \int_0^T f(t)dZ_t \mid f \in L^2[0,T],\ c_0 \in \mathbb{R}^1\}$

Pf: 1°) RHS $\subset$ LHS:

$$\int_0^T f(t)dZ_t = \lim_n \sum_i^{p_n} f(t_i^n)\Delta Z_{t_i}$$

2°) LHS $\subset$ RHS:

$$\sum c_i Z_{t_i} = \sum c_i' \Delta Z_i = \sum c_i' \int_{t_{i-1}}^{t_i} dZ_s$$

Besides, by Itô isometry, RHS is closed.

Def: N_t is innovation process if $N_t = Z_t - \int_0^t (G_t X_t)^\wedge$

where $(G_t X_t)^\wedge = P_{\mathcal{L}(Z,t)}(G(t)X_t) \hat{=} G(t)\hat{X}(t)$

i.e. $dN_t = G(t)(X_t - \hat{X}_t)dt + D(t)dV_t$.

Lemma. i) N_t has orthogonal increments.

ii) $\mathbb{E}(N_t^2) = \int_0^t D^2(s)ds$ iii) $\mathcal{L}(N,t) = \mathcal{L}(Z,t)$.

iv) N_t is Gaussian process.

Rmk: We want to replace Z_t by N_t.

Pf: i) It follows from $X_t - \hat{X}_t \perp \mathcal{L}(Z,t)$

and V_t has indept increment

ii) By Ito Formula:

$dN_t^2 = 2N_t\, dN_t + 2\cdot\frac{1}{2} D^2(t)\, dt$

$\mathbb{E}(\int_0^t N_s\, dN_s) = 0$ follows from N_t has indept increments.

iii) $\mathcal{L}(N,t) \subset \mathcal{L}(Z,t)$ is trivial.

Conversely, we want to express Z_t by N_t:

$\int_0^t f(s)\, dN_s = \int_0^t f(s)\, dZ_s - \int_0^t f(r) h(r) \hat{X}_r\, dr$

Note: $h(r)\hat{X}_r = c(r) + \int_0^r g(r,s)\, dZ_s$ for some $g(r,\cdot) \in L^2[0,r]$ since $\hat{X}_r \in \mathcal{L}(Z,r)$.

$\Rightarrow \int_0^t (f(s) - \int_s^t f(r) g(r,s)\, dr)\, dZ_s - \int_0^t f(r)c(r)\, dr$

$= \int_0^t f(s)\, dN_s.$

By Volterra Integral Equation:

$\forall h \in L^2[0,t].\ \exists f \in L^2[0,t].\ s.t.\ s \le t.$

$f(s) - \int_s^t f(r) g(r,s)\, dr = h(s)$

Set $h(s) = \chi_{[0,t_1]}(s)$.

iv) Z_t is Gaussian $\Rightarrow$ $\hat{X}_t$ is (limit of ...)

$\Rightarrow N_t$ is.

<u>Step. 3</u>: Construct BM by (N_t).

Def: $dR_t = \frac{1}{D(t)} d N_t(\omega)$. $R_0 = 0$.

Lemma. (R_t) is 1-dimension BM.

Pf: i) conti. ii) orthogonal increment

iii) Gaussian follows from prop. of (N_t).

iv) $\mathbb{E}(R_t) = 0$. $\mathbb{E}(R_t R_s) = \min\{t, s\}$.

Note that $dR_t^2 = 2R_t dR_t + dt$

$\Rightarrow \mathbb{E}(R_t^2) = t$.

So $\mathbb{E}(R_t R_s) = \min\{t, s\}$. by ii).

Note that $\mathcal{L}(N, t) = \mathcal{L}(R, t)$. $\Rightarrow \hat{X}_t = P_{\mathcal{L}(R,t)} X_t$

In $\mathcal{L}(R,t)$. $\hat{X}_t$ can be described well:

Lemma. $\hat{X}_t = \mathbb{E}(X_t) + \int_0^t \frac{\partial}{\partial s} \mathbb{E}(X_t R_s) dR_s$.

Pf: Suppose $\hat{X}_t = C_0(t) + \int_0^t g(s) dR_s \in \mathcal{L}(R,t)$

$C_0(t) = \mathbb{E}(\hat{X}_t) = \mathbb{E}(P_{K(R,t)}(X_t)) = \mathbb{E}(X_t)$

Combined with $X_t - \hat{X}_t \perp \int_0^t f(s) dR_s$

$\Rightarrow \mathbb{E}(X_t \int_0^t f(s) dR_s) = \mathbb{E}(\int_0^t g(s) dR_s \int_0^t f(s) dR_s)$

$= \mathbb{E}(\int_0^t f(s) g(s))$ (by Itô isometry)

Set $f(s) = \chi_{[0,r]}$. $\therefore \int_0^r g(s) ds = \mathbb{E}(X_t R_r)$

Step. 4: Explicit formular for (X_t)

As in ODE, it's easy to obtain:

$$X_t = e^{\int_0^t F(s)ds}\left(X_0 + \int_0^t e^{-\int_0^s F(u)du} C(s)\,dU_s\right)$$

generally, $X_t = e^{\int_r^t F(s)ds} X_r + \int_r^t e^{\int_s^t F du} C(s)\,dU_s$

if we start at time $r < t$.

<u>Rmk:</u> $\mathbb{E}(X_t) = \mathbb{E}(X_0)\, e^{\int_0^t F(s)ds}$

<u>Step. 5:</u> SDE for $\hat{X}_t$

First we have: $\hat{X}_t = \mathbb{E}(X_t) + \int_0^t \frac{\partial}{\partial s}\mathbb{E}(X_t R_s)\,dR_s$

Note $R_s = \int_0^s \frac{G(r)}{D(r)}(X_r - \hat{X}_r)\,dr + V_s$. $\tilde{X}_r =: X_r - \hat{X}_r$.

$\Rightarrow \mathbb{E}(X_t R_s) = \int_0^s \frac{G(r)}{D(r)}\mathbb{E}(X_t \tilde{X}_r)\,dr$

By explicit formula of X_t:

$\mathbb{E}(X_t \tilde{X}_r) = e^{\int_r^t F(s)ds}\,\mathbb{E}(X_r \tilde{X}_r) = e^{\int_r^t F} S(r)$.

where $S(r) = \mathbb{E}((\tilde{X}_r)^2)$, MSE of X_r.

Second. claim: $\frac{dS}{dt} = 2F(t)S(t) - \frac{G^2(t)}{D^2(t)}S^2(t) + C^2(t)$.

(The Riccati Equation)

Note: $S(t) = T(t) - \int_0^t \left(\frac{\partial}{\partial s}\mathbb{E}(X_t R_s)\right)^2 ds - \mathbb{E}(X_t)^2$

$T(t) = \mathbb{E}(X_t^2)$, satisfies $\frac{dT}{dt} = 2F(t)T(t) + C^2(t)$.

Finally, we can obtain:

$$d\hat{X}_t = \left(F(t) - \frac{G^2(t)S(t)}{D(t)}\right)\hat{X}_t\, dt + \frac{S(t)}{D^2(t)} \cdot G(t)\, dZ_t.$$

Rmk: We can see how the error $S(t)$ influences the estimate $\hat{X}_t$.

② Multidimensional Case:

Thm (Kalman - Bucy Filter)

Solution $\hat{X}_t = \mathbb{E}(X_t \mid \mathcal{G}_t)$ of filtering problem:

$$\begin{cases} dX_t = F(t)X_t\, dt + C(t)\, dU_t. & F \in \mathbb{R}^{n\times n}.\ C \in \mathbb{R}^{n\times p}. \\ dZ_t = G(t)X_t\, dt + D(t)\, dV_t. & G \in \mathbb{R}^{m\times n}\ D \in \mathbb{R}^{m\times r} \end{cases}$$

satisfies SDE:

$$d\hat{X}_t = (F - SG^T(DD^T)^{-1}G)X_t\, dt + SG^T(DD^T)^{-1}\, dZ_t$$

$\hat{X}_0 = \mathbb{E}(X_0)$. where $S(t) = \mathbb{E}((X_t - \hat{X}_t)(X_t - \hat{X}_t)^T)$

satisfies Riccati equation:

$$\frac{dS}{dt} = FS - SF^T - SG^T(DD^T)^{-1}GS + CC^T. \quad \text{s.t.}$$

$$S(0) = \mathbb{E}((X_0 - \mathbb{E}(X_0))(X_0 - \mathbb{E}(X_0))^T).$$

Under condition: i) $D_t \in \mathbb{R}^{m\times r}$. $D_t D_t^T$ is invertible. $\forall t$

ii) $(D_t D_t^T)^{-1}$ is bdd on $\forall$ bdd t-intervals.