Percolation of Vacant Set.

Def: i) For $u > 0$. $\text{Perco}(u) = \{ w \in \Omega \mid \mathcal{V}^u(w)$ contains a infinite connected component$\} \in \mathcal{A}$.

ii) $u_* = \sup\{ u \geq 0 \mid \mathbb{P}(\text{perco}(u)) > 0 \}$.

iii) $\eta(u) = \mathbb{P}(0 \overset{\mathcal{V}^u}{\longleftrightarrow} \infty)$.

Thm: i) $\mathbb{P}(\text{perco}(u)) \in \{0, 1\}$.

ii) $\forall u > 0$. Then: $\eta(u) > 0 \iff \mathbb{P}(\text{perco}(u)) = 1$

i.e. $u_* = \sup\{ u \geq 0 \mid \eta(u) > 0 \}$.

Pf: i) By ergodic of $\mathbb{P}$ and $\text{perco}(u)$ is translation - invariant.

ii) ($\Rightarrow$) $\mathbb{P}(\text{perco}(u)) \geq \eta(u) > 0$

($\Leftarrow$) If $\eta(u) = 0$. Note $\text{Perco}(u) = \bigcup_{x \in \mathbb{Z}^d} \{ x \overset{\mathcal{V}^u}{\longleftrightarrow} \infty \}$.

$\Rightarrow \mathbb{P}(\text{perco}(u)) \leq \sum_{\mathbb{Z}^d} \eta(u) = 0$.

Thm. For $\forall d \geq 3$. $u_* \in (0, \infty)$.

Next. We will only prove the weaker one:

Thm. $\exists\, d_0 \in \mathbb{N}$. s.t. $\forall d \geq d_0 \Rightarrow u_*(d) > 0$.

Rmk: Actually we will show $0 \in$ infinite connected component of $\mathcal{V}^u \cap \mathbb{Z}^2 \times \{0\}^{d-2}$ has positive prob. for $u > 0$ when $d \geq d_0$

Denote: i) For $d \geq 3$. $F = \mathbb{Z}^2 \times \{0\}^{d-2}$. ii) $x \overset{*}{\sim} y$ if $|x-y|_\infty = 1$.

prop. For all d large enough, $\exists u_1(d) > 0$. st.

$\forall u \in [0, u_1(d)] \Rightarrow \mathbb{P}(\mathcal{I}^u \supseteq K) \leq 14^{-|K|}$

for $\forall K \subset\subset F$. (Exponential decay)

$\Rightarrow$ Using the prop. we can prove the Thm:

Pf: Fix $u \leq u_1$. prove: $\mathbb{P}(0 \overset{\mathcal{V}^u \cap F}{\longleftrightarrow} \infty) > 0$ when d large.

Set $\mathcal{C}$ is connected compont of 0 in $\mathcal{V}^u \cap F$.

$\pi = (y_1 \cdots y_k)$ is $*$-path in F. $y_k \overset{*}{\sim} y_{k+1}$

if $y_1 = y_k$. then π is $*$-circuit.

Note: $\{\mathcal{C}$ is finite$\} = \{\exists *$-circuit around 0 in $F\}$. a.s.

$\Rightarrow \mathbb{P}(|\mathcal{C}| < \infty) \leq \sum_{n \geq 0} \mathbb{P}\left(\begin{array}{c}\mathcal{I}^u \cap F \text{ contains } *\text{-circuit} \\ \text{around } 0 \text{ passing } (n, 0 \cdots 0)\end{array}\right)$

$\leq \sum_{n \geq 0} \sum_{\pi_n} \mathbb{P}(\pi_n \subseteq \mathcal{I}^u)$

Sum over $\pi_n = (y_1 \cdots y_{n+1})$ starts from $(n, 0 \cdots 0)$

$\pi_n \subset F$. is a $*$-path.

$\Rightarrow$ has cardinality $\leq 8 \cdot 7^{n-1}$. So: RHS $\overset{\text{(prop.)}}{<} \infty$.

Next. We prove the prop. used :

Denote: i) $\varphi(w) = \sum_{n\geq 0} I_{\{X_n(w) \in F\}}$ for $w \in W^+$.

number of w visits F.

ii) $q = \mathbb{P}_0^{\mathbb{Z}^{d-2}}(\widetilde{H}_0 = \infty) = 1/g_{d-2}(0)$.

where $\mathbb{P}_0^{\mathbb{Z}^{d-2}}$ is law of $(d-2)$-dim

SRW starts at 0.

i) Prop. $q = q(d) \longrightarrow 1$ as $d \to \infty$.

Rmk: It's crucial in choosing the base

of exponential decay large enough.

ii) Lemma If $\lambda > 0$. $d \geq 5$. $\chi(\lambda) = e^{\lambda}\left(\frac{2}{d} + (1-\frac{2}{d})(1-q)\right) < 1$. Then :

$\mathbb{E}_x(e^{\lambda\varphi}) = \mathbb{E}_0(e^{\lambda\varphi}) = q e^{\lambda}(1-\frac{2}{d})/(1-\chi(\lambda)) < \infty$.

Pf: i) First prove $d = 3k$. case.

decompose X in k-tuple of i.i.d.

3-dim SRW.

Then use the monotonicity of $q(d)$

to interpolation.

ii) Set $R_0 = \inf\{n \geq 1 \mid X_n \in F\}$.

$D_0 = \inf\{n \geq R_0 \mid X_n \notin F\}$.

$$D_i = \infty \cdot I_{\{R_i=\infty\}} + (R_i + T_F \circ \theta_{R_i}) I_{\{R_i<\infty\}}$$

$$R_i = \infty \cdot I_{\{D_{i-1}=\infty\}} + (D_{i-1} + H_F \circ \theta_{D_{i-1}}) I_{\{D_{i-1}<\infty\}}.$$

$$z = \inf\{n \mid R_n = \infty\}. \Rightarrow \varphi(w) = \sum_0^{z(w)-1} T_F \circ \theta_{R_i(w)}$$

1) Claim: $\mathbb{P}_x(T_F = n) = (1-\frac{2}{\lambda})(\frac{2}{\lambda})^{n-1}$. $x \in F$

$\mathbb{P}_x(z = n) = q(1-q)^{n-1}$. $x \in F$

T_F, z are memoryless $\Rightarrow \sim$ Geo(λ)

Combine: $q \overset{\text{smp}}{=} \mathbb{P}_x(R_{i+1} = \infty \mid R_i < \infty)$. $x \in F$.)

2) $\mathbb{E}_x(e^{\lambda\varphi}) = \sum_{n\geq 1} \mathbb{E}_x(e^{\lambda \sum_0^{n-1} T_F \circ \theta_{R_i}} I_{\{z=n\}})$

$\overset{\text{i.i.d of }(R_i)\mid z}{=} \sum \mathbb{E}_x(\mathbb{E}_x^n(e^{\lambda T_F} \mid z) I_{\{z=n\}})$

$= \dfrac{q\,\mathbb{E}_0(e^{\lambda T_F})}{1-(1-q)\,\mathbb{E}_0(e^{\lambda T_F})} = q(1-\frac{2}{\lambda})e^{\lambda}/(1-\chi(\lambda))$

Return to pf of prop.:

Pf: $\mathbb{P}(J^n \leq k) = \mathbb{P}(J^n \wedge k = k) \overset{\widehat{\mu}_{k,n}}{=} \widetilde{\mathbb{P}}(\bigcup_1^{N_k} \text{range}(w_i) \supseteq k)$

$\leq \widetilde{\mathbb{P}}(\sum_1^{N_k} \varphi(Y_i) \geq |k|.)$.

Y_i are i.i.d SRW. Since $k \subset\subset F$. $\widetilde{J}_k^n$ must return to F at least $|k|$ times if $\widetilde{J}_k^n \supseteq k$.

$\text{RHS} \overset{\text{chebysh.}}{\leq} e^{-\widetilde{\lambda}|k|} \cdot \mathbb{E}(e^{\widetilde{\lambda} \sum_1^{N_k} \varphi(Y_i)})$

$\overset{N_k \sim \text{poi}}{=} e^{-\widetilde{\lambda}|k|} \cdot e^{u\,\text{cap}(k)(\mathbb{E}_0(e^{\widetilde{\lambda}\varphi})-1)}$

using Lemma above. Set $u_1 = (\widetilde{\lambda} - \lambda) \cdot g_{(0)}$. $\dfrac{1-\chi(\widetilde{\lambda})}{e^{\widetilde{\lambda}}-1} > 0$

For λ large. $q(\lambda) \geq \frac{1}{2}$. Set $\lambda = \log 4$.