Random Interlacement Point Process.

(1) Measure Construction:

Def: i) $W = \{ w : \mathbb{Z} \to \mathbb{Z}^d \mid |w(n) - w(n+1)| = 1.\ \forall n.$

and $|w(n)| \to \infty\ (n \to \infty) \}$.

$W^+ = \{ w : \mathbb{N} \to \mathbb{Z}^d \mid \cdots$ as above $\}$.

ii) "$\sim$" is equi relation on W:

$w \sim w' \iff \exists k \in \mathbb{Z}.\ \theta_k(w) = w'$.

Set $W^* = W/\sim$. $\pi^* : W \to W^*$ is the canonical projection.

iii) For $(X_n)_{\mathbb{Z}}$ canonical coordinates on $W.\ W^+$.

$\mathcal{W} = \sigma(X_n, n \in \mathbb{Z})$, $\mathcal{W}^+ = \sigma(X_n, n \geq 0)$

σ-algebra defined on $W.\ W^+$.

Set $\mathcal{W}^* = \{ A \subset W^* \mid \pi^{*-1}(A) \in \mathcal{W} \}$.

natural σ-algebra on W^*.

Denote: i) $\forall K \subset\subset \mathbb{Z}^d$. $W_K = \{ w \in W \mid X_n(w) \in K$ for some $n \in \mathbb{Z} \} \subset W$. $W_K^* \hat{=} \pi^*(W_K)$.

ii) Set $W_K^n = \{ w \in W \mid H_K(w) = n \}$ where $H_K(w) =: \inf\{ n \in \mathbb{Z} \mid w(n) \in K \}$.

Rmk: i) $W_K = \sum_{n \in \mathbb{Z}} W_K^n$

ii) $W_K^* = \pi^*(W_K^0) = \pi^*(W_K)$.

iii) $\mathbb{P}_x$, $\mathbb{E}_x$ are law and expectation of SRW

<u>Rmk:</u> Note $d \geq 3$, SRW^d is transient. $\Rightarrow \mathbb{P}_x(W^+) = 1$.

Next, we construct a σ-finite measure on $(W^*, \mathcal{W}^*)$:

<u>Def:</u> For $K \subset\subset \mathbb{Z}^d$, $\widetilde{H}_K(w) =: \inf\{n \geq 1 \mid w(n) \in K\}$.

Set Q_K on $(W, \mathcal{W})$ by formula:

$Q_K((X_{-n})_{n\geq 0} \in A, X_0 = x, (X_n)_{n\geq 0} \in B)$

$= \mathbb{P}_x(A \mid \widetilde{H}_K = \infty)\, e_K(x)\, \mathbb{P}_x(B)$ for $A, B \in \mathcal{W}^+$, $x \in K$.

<u>Prop.</u> $\forall K \subset\subset \mathbb{Z}^d$, $Q_K(W) = \mathrm{cap}(K)$

<u>Pf:</u> $Q_K(W) = Q_K(W_K^0) = \sum_{x\in K} Q_K(X_0 = x)$

$= \sum_{x \in K} e_K(x) = \mathrm{cap}(K)$

<u>Rmk:</u> $\overline{Q}_K = Q_K/\mathrm{cap}(K)$ is p.m. on $(W, \mathcal{W})$, supports on W_K^0, which can be defined by:

i) $X_0 \sim \widetilde{e}_K$ (normalized e.m.) on K under Q_K.

ii) Condition on X_0, $(X_{-n})_{n\geq 0} \mid \widetilde{H}_K = \infty$ is indept of $(X_n)_{n\geq 0}$

iii) Condition on X_0, $(X_n)_{n\geq 0}$ is SRW starts at X_0.

iv) Condition on X_0, $(X_{-n})_{n\geq 0}$ is SRW starts at X_0 conditioned on $(X_{-n})_{n\geq 0}$ never returns to K after first step.

Thm. There exists unique σ-finite measure ν on $(W^*, \mathcal{W}^*)$ satisfying: $\forall K \subset\subset \mathbb{Z}^d$,

$\nu(A) = Q_K(\pi^{*-1}(A))$, $\forall A \in W_K^*$.

Rmk: i) Restate: $I_{W_K^*}\nu = \pi^* \circ Q_K$.

ii) If $\mathbb{Z}^d = \bigcup K_n$ s.t. $K_n \subset K_{n+1}$, $\forall n$, K_n are finite sets. Def $Q_{\mathcal{K}}$ on W:

$Q_{\mathcal{K}} = \sum_{n \geq 1} (1 - I_{W_{K_{n-1}}}) Q_{K_n}$. $\Rightarrow \nu = \pi^* \circ Q_{\mathcal{K}}$.

Pf: 1°) Uniqueness:

Consider $K_n \uparrow$, finite, $\bigcup K_n = \mathbb{Z}^d$. Then:

$W^* = \bigcup W_{K_n}^*$. So for $\forall A \in W^*$,

$\Rightarrow \nu(A) = \lim_{n\to\infty} \nu(A \cap W_{K_n}^*) = \lim_{n\to\infty} Q_{K_n}(\pi^{*-1}(A \cap W_{K_n}^*))$

it's uniquely determined.

2°) Existence:

Check ν is well-def in the condition:

$K \subseteq K' \subset\subset \mathbb{Z}^d$, $A \in W^*$, $A \subset W_K^* \subset W_{K'}^*$.

$\Rightarrow Q_{K'}(\pi^{*-1}(A)) = Q_K(\pi^{*-1}(A))$.

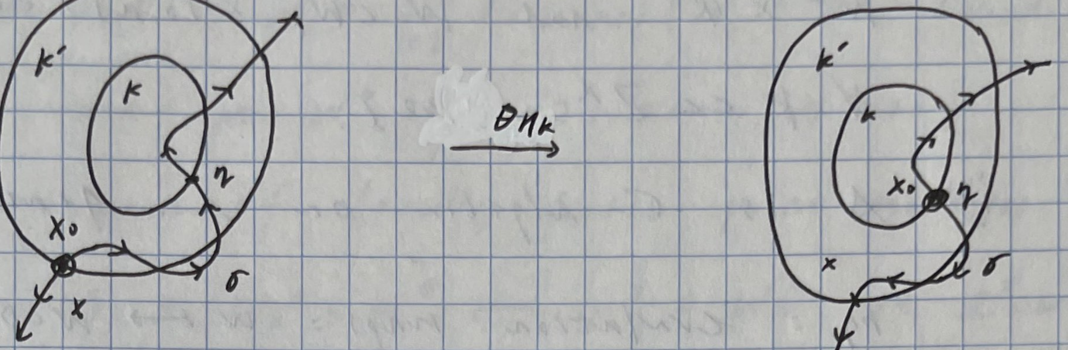
Rmk: So define $\nu(A) = \sum_{n \geq 1} Q_{K_n}(\pi^{*-1}(A \cap (W_{K_n}^* \setminus W_{K_{n-1}}^*)))$

satisfying the condition. Besides,

$Q_{K_n}(\pi^{*-1}(W_{K_n}^* \setminus W_{K_{n-1}}^*)) = cap(K_n) - cap(K_{n-1}) < \infty$

$\Rightarrow \nu(\cdot)$ is σ-finite on W.

Note: $\theta_{H_K} = W_K \cap W_{K'}^0 \to W_K^0$ is bijection:



(i.e. θ_{H_K} only reverses σ-path)

It suffices to prove: $\forall B \in \mathcal{W}$. $B \subseteq W_K$

$Q_{K'}(w \in W_K \cap W_{K'}^0 \mid \theta_{H_K}(w) \in B) = Q_K(w \in W_K^0 \mid w \in B)$

$\Leftarrow$ check for $B = \{w \mid X_m(w) \in A_m, m \in \mathbb{Z}\}$

__Cor.__ (Sweeping identity)

$\forall K \subseteq K' \subset\subset \mathbb{Z}^d$. $e_K(y) = \mathbb{P}_{e_{K'}}(H_K < \infty, X_{H_K} = y)$

__Pf:__ Set $B = \{X_0 = y\}$ in the equation above.

__Cor.__ $\forall K \subseteq K' \subset\subset \mathbb{Z}^d$. $\mathbb{P}_{\tilde{e}_{K'}}(H_K < \infty) = \dfrac{\mathrm{cap}(K)}{\mathrm{cap}(K')}$

__Pf:__ Sum up $y \in K$. in cor. above.

(2) __Random Interlacement PP:__

Consider $W^* \times \mathbb{R}^+$. the labeled trajectories space. equipped with σ-algebra: $\mathcal{W}^* \otimes \mathcal{B}_{\mathbb{R}^+}$. For λ the Lebesgue measure on $\mathbb{R}$. Set σ-finite measure: $\nu \otimes \lambda$ on it (Note: $\nu \otimes \lambda(W_K^* \times [0,u]) = u\,\mathrm{cap}(K) < \infty$ $K \subset\subset \mathbb{Z}^d$)

Def: i) $\Omega = \{ w =: \sum_{n\geq 0} \delta_{(w_n^*, u_n)} \mid (w_n^*, u_n) \in W^* \times \mathbb{R}^+$. and $w(W_K^* \times [0,u]) < \infty$ for $\forall K \subset\subset \mathbb{Z}^d$. $u \geq 0 \}$.

ii) $\mathcal{A}$ is σ-algebra on Ω. generated by evaluation maps: $w \longmapsto w(D) = \sum_{n\geq 0} \mathbb{1}\{(w_n^*, u_n) \in D\}$ for $D \in \mathcal{W}^* \otimes \mathcal{B}_{\mathbb{R}^+}$.

iii) Set $\mathbb{P}$ on $(\Omega, \mathcal{A})$. s.t.

$$w = \sum_{n\geq 0} \delta_{(w_n^*, u_n)} \overset{\mathbb{P}}{\sim} PPP(\nu \otimes \lambda) \text{ on } W^* \times \mathbb{R}^+.$$

iv) Random element in $(\Omega, \mathcal{A}, \mathbb{P})$ is called a random interlacement point process.

Def: (Second def of RI)

Random interlacement at level u: $\mathcal{I}^u$ is random subset of $\mathbb{Z}^d$. s.t.

$$\mathcal{I}^u(w) =: \bigcup_{u_n \leq u} \text{range}(w_n^*). \text{ for } w = \sum_{n\geq 0} \delta_{(w_n^*, u_n)}$$

where $\text{range}(w^*) = \{ X_n(w) \mid w \in \pi^{*-1}(w^*), n \in \mathbb{Z} \}$

Set Vacant set $\mathcal{V}^u = \mathbb{Z}^d / \mathcal{I}^u$.

Rmk: For $K \subset\subset \mathbb{Z}^d$. $u \geq 0$.

$$\mathbb{P}(\mathcal{I}^u \cap K = \emptyset) = \mathbb{P}(w(W_K^* \times [0,u]) = 0)$$

$$= \exp(-u\,\text{cap}(K))$$

Def: (RI PP on subset)

i) $M =: \{ \mu = \sum_{i \in I} \delta_{(w_i, u_i)} \mid I \subset \mathbb{N},\ (w_i, u_i) \in W_+ \times \mathbb{R}^+,$

and $\mu(W^+ \times [0,u]) < \infty,\ \forall u \geq 0 \}$

Space of locally finite point measures on $W_+ \times \mathbb{R}^+$.

ii) $S_K = W_K^* \ni w^* \longmapsto w^0 \in W_K^0$. s.t. $\pi^*(w^0) = w^*$

iii) For $w = (w(n))_{\mathbb{Z}} \in W$. Set $w_+ = (w(n))_{\mathbb{N}} \in W^+$.

For $K \subset\subset \mathbb{Z}^d$. Set $\mu_K : \Omega \to M$. def by:

$$\int f \, d\mu_K(\omega) = \int_{W_K^* \times \mathbb{R}^+} f(S_K(w^*)_+, u) \, \omega(dw^*, du)$$

for $\omega \in \Omega$. $f: W_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ measurable.

Rmk: i) It means we only care about the trajectories after hitting K.

ii) Alternatively definition of μ_K:

$$\mu_K(\omega) =: \sum_{n \geq 0} I\{w_n^* \in W_K^*\} \, \delta_{(S_K(w_n^*)_+, u_n)}$$

iv) For $u > 0$. Set $\mu_{K,u}$ on $\Omega \to$ finite point measure on W_+:

$\mu_{K,u}(\omega)(dw) = \mu_K(\omega)(dw \times [0,u])$ $\omega \in \Omega$.

Rmk: i) Alternative definition of $\mu_{K,u}$:

$$\mu_{K,u}(\omega) = \sum_{n \geq 0} I\{w_n^* \in W_K^*,\ u_n \leq u\} \, \delta_{S_K(w_n^*)_+}$$

for $\omega = \sum_{n \geq 0} \delta_{(w_n^*, u_n)} \in \Omega$.

ii) It means we collect trajectories from μ_K with label less than u.

Prop. $\mu_{K,u}$ is PPP on W^+ with intensity measure $u \cdot \mathrm{cap}(K) \cdot \mathbb{P}_{\tilde{e}_K}$

Pf: $\omega \sim \mathrm{PPP}(\nu \otimes \lambda)$. For $\varphi : W_K^* \times \mathbb{R}^+ \to W^+$:

$$\mu_{K,u}(B) = \varphi \circ \omega(B)$$

$$\overset{\mathrm{RHS}}{=} \sum \delta_{\{(w_n^*, u_n) \in \varphi^{-1}(B)\}}$$

$$\overset{\mathrm{LHS}}{=} \sum \delta_{s_K(w_n^*)_+}(B)\, \mathbb{I}[w_n^* \in W_K^*,\, u_n \le u]$$

$$\varphi \circ (\nu \otimes \lambda)(B) = (\nu \otimes \lambda)(\varphi^{-1}(B))$$

$$= (\nu \otimes \lambda)\{w_n^* \mid w_n^* \in W_K^*,\ s_K(w_n^*)_+ \in B\} \times [0,u]$$

$$= u \cdot Q_K(\pi_*^{-1}(w_n^* \in W_K^*,\ s_K(w_n^*)_+ \cdots))$$

$$= u\, Q_K((X_{-n}) \notin K,\ X_0 \in K,\ (X_n) \in B)$$

$$= u\, \mathrm{cap}(K)\, \mathbb{P}_{\tilde{e}_K}(B)$$

Apply transf. property of PPP.

Cor. $\mu_{K,u} - \mu_{K,u'}$ is PPP on W^+ with intensity $(u-u')\,\mathrm{cap}(K)\, \mathbb{P}_{\tilde{e}_K}$, indept from $\mu_{K,u'}$ for $u > u'$.

Def: Restriction of RI at level u on $K \subset\subset \mathbb{Z}^d$ as $\mathcal{I}^u \cap K = K \cap \left(\bigcup_{w \in \mathrm{supp}(\mu_{K,u})} \mathrm{range}(w) \right)$

prop. For $N_K \sim \mathrm{POI}(u \cdot \mathrm{cap}(K))$. $(W^i) \overset{\text{i.i.d}}{\sim}$ $\mathbb{P}_{\tilde{e}_K}$. random walk indept of $\mathbb{P}_{\tilde{e}_K}$.

$\Rightarrow$ mixed binomial process $\tilde{\mu}_{K,u} = \sum_{1}^{N_K} \delta_{W_i}$.

$\sim \mathrm{PPP}(u \cdot \mathrm{cap}(K)\, \mathbb{P}_{\tilde{e}_K})$ on W_+.

cor. $\tilde{\mathcal{J}}^u_K = \bigcup_{1}^{N_K} (\mathrm{range}(W^i) \cap K) \overset{d}{\sim} \mathcal{J}^u \cap K$.

Thm. For $d \geq 3$. $u > 0$. There exists $R_0(d,u) < \infty$.

s.t. $\forall R \geq R_0 \Rightarrow \mathbb{P}(B(R) \subset \mathcal{J}^u) \geq \frac{1}{2} e^{-\ln^2(R) \cdot R^{d-2}}$

Pf: Suppose $\tilde{\mathbb{P}}$ is law of the mixed binomial process $\tilde{\mu}_{K,u}$. $K = B(R)$.

$$\mathrm{LHS} \underset{\text{dist.}}{\overset{\text{i.d}}{=}} \tilde{\mathbb{P}}(B(R) \subseteq \tilde{\mathcal{J}}^u_K)$$

$$= \sum_{n \geq 0} \tilde{\mathbb{P}}(B(R) \subseteq \tilde{\mathcal{J}}^u_K \mid N_K = n)\, \tilde{\mathbb{P}}(N_K = n)$$

$$\tilde{\mathbb{P}}(B(R) \subseteq \tilde{\mathcal{J}}^u_K \mid N_K = n) = 1 - \tilde{\mathbb{P}}\Big(\bigcup_{B(R)} \{x \notin \tilde{\mathcal{J}}^u_K\} \,\Big|\, \square\Big)$$

$$\geq 1 - \sum_{x \in B(R)} \mathbb{P}\Big(\bigcap_{1}^{n} H_x(W^i) = \infty\Big)$$

$$= 1 - |B(R)| \big(1 - \mathrm{cap}(0)/\mathrm{cap}(B(R))\big)^n$$

1) $\exists\, n_0(R) = [C_0 \ln(R) \cdot R^{d-2}]$. s.t. $\forall n \geq n_0$.

$$\Rightarrow \tilde{\mathbb{P}}(B(R) \subseteq \tilde{\mathcal{J}}^u_K \mid N_K = n) \geq \frac{1}{2}$$

2) By Stirling's approxi. $\exists R_0$. s.t. $\forall R \geq R_0$.

$$\tilde{\mathbb{P}}(N_K = n_0(R)) \geq e^{-\ln^2(R) \cdot R^{d-2}}.$$