Random Interlacement.

(1) Definition:

Def: i) For $d \geq 3$. $\{0,1\}^{\mathbb{Z}^d} \ni \xi$. Set $\mathcal{J}(\xi) =:$

$\{ x \in \mathbb{Z}^d \mid \xi_x = 1 \} \subseteq \mathbb{Z}^d$.

Set coordinate maps $(\Psi_x)_{\mathbb{Z}^d}$. s.t.

$\Psi_x : \{0,1\}^{\mathbb{Z}^d} \longrightarrow \{0,1\}$. $\Psi_x(\xi) = \xi_x$.

<u>Rmk:</u> We can see $\{0,1\}^{\mathbb{Z}^d}$ as a subset of $\mathbb{Z}^d$.

ii) Set $\mathcal{G} = \sigma(\Psi_x, x \in \mathbb{Z}^d)$.

We say A is local event with the support K if $A \in \sigma(\Psi_x, x \in K)$.

iii) For $K_0 \subset K \subset\subset \mathbb{Z}^d$. $K_1 = K/K_0$. We call

$\{ \xi \in \{0,1\}^{\mathbb{Z}^d} \mid \Psi_x(\xi) = 0, \forall K_0. \ \Psi_x(\xi) = 1, \forall K_1 \}$

$= \{ \mathcal{J}(\xi) \cap K = K_1 \}$ by cylinder set of K

<u>Rmk:</u> Every local event is disjoint union of cylinder sets.

Actually $\sigma(\Psi_x, x \in K)$ has $2^{|K|}$ atoms with form of cylinder sets.

iv) For $u > 0$. $\mathbb{P}^u$ is p.m. on $(\{0,1\}^{\mathbb{Z}^d}, \mathcal{G})$

satisfies: $\mathbb{P}^u(\mathcal{J} \cap K = \emptyset) = e^{-u\,\mathrm{cap}(K)}$.

Rmk: Note $\{\mathcal{I} \cap K = \emptyset\} = \{\varphi_x = 0,\ \forall x \in K\}$ forms a π-system, determine a p.m. uniquely.

v) Random subset $\mathcal{I}$ of $\mathbb{Z}^d$ in $([0,1]^{\mathbb{Z}^d}, \mathcal{F}, \mathbb{P}^u)$ is called random interlacement at level u.

prop. (FKG inequal.)

For $A, B \in \mathcal{F}$, increasing $\Rightarrow \mathbb{P}^u(A \cap B) \geq \mathbb{P}^u(A)\mathbb{P}^u(B)$

prop. For $K_0 \leq K \subset\subset \mathbb{Z}^d$, $K_1 = K / K_0$. Then we have:

$$\mathbb{P}^u(\mathcal{I} \cap K = K_1) = \mathbb{P}^u(\varphi|_{K_0} = 0,\ \varphi|_{K_1} = 1)$$

$$= \sum_{K' \leq K_1} (-1)^{|K'|} e^{-u\,\mathrm{cap}(K_0 \cup K')}$$

Pf: Set $\tilde{\varphi}$ satisfying: $\tilde{\varphi}|_{K_0} = 0$

By inclusion - exclusion formula:

$$\mathbb{P}^u\Big(\bigcup_{x \in K_1} \{\tilde{\varphi}(x) = 0\}\Big) = -\sum_{K' \leq K_1,\ K' \neq \emptyset} (-1)^{|K'|}\ \mathbb{P}^u(\tilde{\varphi}|_{K'} = 0)$$

$$= \mathbb{P}^u(\tilde{\varphi}|_{K_0} = 0) - \sum_{K' \leq K_1} (-1)^{|K'|}\ \mathbb{P}^u(\tilde{\varphi}|_{K'} = 0)$$

$$= \mathbb{P}^u(\tilde{\varphi}|_{K_0} = 0,\ \exists x \in K_1,\ \tilde{\varphi}(x) = 0)$$

(2) Properties:

① Correlation:

prop. For $u > 0$, $\mathrm{Cov}_{\mathbb{P}^u}(\varphi_x, \varphi_y) \sim \frac{2u}{g^2(0)}\, g(y-x)\, e^{-\frac{2u}{g(0)}}$

as $|x-y| \to \infty$.

Rmk: Note $g(y-x) \sim (1+|x-y|)^{2-d}$.

$\Rightarrow$ it has polynomial decay correlation.

Pf: $Cov_{\mathbb{P}^u}(\varphi_x, \varphi_y) = Cov_{\mathbb{P}^u}(1-\varphi_x, 1-\varphi_y)$

$$= \mathbb{P}^u(\varphi|_{\{x,y\}} = 0) - \mathbb{P}^u(\varphi_x = 0)^2$$

$$= e^{-u\,cap\{x,y\}} - e^{-u(cap\{x\}+cap\{y\})}$$

$$= e^{-\frac{2u}{g(0)+g(y-x)}} - e^{-\frac{2u}{g(0)}}$$

② Shift-invariance:

Lemma. $\forall x \in \mathbb{Z}^d$, $u > 0$. $\mathbb{P}^u$ is preserved under transformation θ_x.

Pf: Note for $K \subset\subset \mathbb{Z}^d$. $\{\mathcal{I} \cap K = \emptyset\}$.

$$\theta_x \circ \mathbb{P}^u(\mathcal{I} \cap K = \emptyset) = \mathbb{P}^u(\mathcal{I} \cap (K-x) = \emptyset)$$

$$= e^{-u\,cap(K-x)}$$

$$= \mathbb{P}^u(\mathcal{I} \cap K = \emptyset).$$

$\Rightarrow$ So also holds for $A \in \sigma(\varphi_x, x \in K)$.

Note for general $B \in \mathcal{F}$. $\forall \varepsilon > 0$. $\exists K \subset\subset \mathbb{Z}^d$. s.t. $\exists B^\varepsilon \in \sigma(\varphi_x, x \in K)$. have : $\mathbb{P}^u(B^\varepsilon \Delta B) \le \varepsilon$. (By MCT argument)

Thm. $\forall u \geq 0$. $0 \neq x \in \mathbb{Z}^d$. θ_x is ergodic on space $(\{0,1\}^{\mathbb{Z}^d}, \mathcal{F}, \mathbb{P}^u)$.

Pf: It suffices to prove: $\forall K \subset\subset \mathbb{Z}^d$.

$$\lim_{n\to\infty} \mathbb{P}^u(B_\varepsilon \cap \theta_x^n(B_\varepsilon)) = \mathbb{P}^u(B_\varepsilon)^2. \quad \forall B_\varepsilon \in \sigma(\Psi_x, K)$$

(With $\mathbb{P}^u(B \Delta B_\varepsilon) \le \varepsilon$. Set $\varepsilon \to 0$)

1) Claim: for $\forall K_1, K_2 \subset\subset \mathbb{Z}^d$. we have:

$$\lim_{|\eta|\to\infty} \operatorname{cap}(K_1 \cup (K_2+\eta)) = \operatorname{cap}(K_1) + \operatorname{cap}(K_2)$$

$$\Leftarrow \begin{cases} \forall z \in K_1, & \lim_{|\eta|\to\infty} e_{K_\eta}(z) = e_{K_1}(z) \\ \forall z \in K_2, & \lim_{|\eta|\to\infty} e_{K_\eta}(z) = e_{K_2}(z) \end{cases}$$

where $K_1 \cup (K_2+\eta) =: K_\eta$

Note: $0 \le e_{K_1}(z) - e_{K_\eta}(z) =$

$$\mathbb{P}_z(\widetilde{H}_{K_1} = \infty, \widetilde{H}_{K_\eta} < \infty) \le \mathbb{P}_z(H_{K_2+\eta} < \infty)$$

$$\lesssim \sum_{K_2} \mathbb{P}_z(H_{\{v+\eta\}} < \infty) \lesssim \sum_{v \in K_2} g(z, v+\eta)$$

$$\lesssim \sum |v+\eta-z|^{-d+2} \longrightarrow 0 \quad (|\eta| \to \infty)$$

Another equation is similar

2) Write $B_\varepsilon = \sum A$. For A cylinder set:

$A = \{\mathcal{J} \cap K = K_1\}$.

$$\Rightarrow \mathbb{P}^u(A \cap \theta_x^n(A)) = \mathbb{P}^u(\mathcal{J} \cap (K \cup (K+nx)) = K_1 \cup (K_1+nx))$$

$$= \sum_{K'' \subseteq K_1} \sum_{K' \subseteq K_2} (-1)^{|K'|+|K''|} e^{-u \operatorname{cap}((K_0 \cup K'') \cup (K_0 \cup K'+nx))}$$

Apply claim of 1).

Cor. $\mathbb{P}^u(A)\,\mathbb{P}^u(B) = \lim_{n\to\infty} \mathbb{P}^u(A \cap \theta_x^n(B))$ holds

for $A, B \in \mathcal{G}$.

(3) Stochastic Domination:

Def: P, Q are p.m.'s on $(\{0,1\}^{\mathbb{Z}^d}, \mathcal{F})$. We say $P \preceq_{stoch} Q$ if $\forall A \uparrow, A \in \mathcal{F}$

$\Rightarrow P(A) \leq Q(A)$.

Consider $\mathcal{L}^p$ is p.m. on $(\{0,1\}^{\mathbb{Z}^d}, \mathcal{F})$, the law of Bernoulli bond percolation.

prop. i) $\forall u > 0$, $p \in (0,1)$, $\mathbb{P}^u$ doesn't stochastically dominate $\mathcal{L}^p$.

ii) $\forall u > 0$, $p \in (0,1)$, $\mathcal{L}^p$ doesn't stochastically dominate $\mathbb{P}^u$.

Pf: i) Set $A_R = \{ \mathcal{I} \cap B(R) = \emptyset \} \downarrow \in \mathcal{F}$

prove: $\exists R \geq 1$, st. $\mathbb{P}^u(A_R) > \mathcal{L}^p(A_R)$

$LHS = e^{-u\,\mathrm{cap}(B(R))} \sim e^{-R^{d-2}}$

$RHS = (1-p)^{|B(R)|} \sim (1-p)^{R^d}$

ii) Let $A_R' = \{ \mathcal{I} \cap B(R) = B(R) \} \uparrow \in \mathcal{F}$

prove: $\exists R \geq 1$, st. $\mathbb{P}^u(A_R') > \mathcal{L}^p(A_R')$

$LHS \sim p^{R^d}$

$RHS \geq \frac{1}{2} \exp(-\ln^2 R \cdot R^{d-2})$ $(*)$

Rmk: $(*)$ is from another def of RI.

(4) <u>Existence of $\mathbb{P}^u$:</u>

Consider SRW $\tilde{X}$ on $(\mathbb{Z}/N\mathbb{Z})^d = \mathbb{T}_N^d$, $d \geq 3$.

<u>Rmk:</u> Set: $\varphi : \mathbb{Z}^d \to \mathbb{T}_N^d$. canonical projection map of equi. relation mod N on $\mathbb{Z}^d$.

For (X_n) SRW on $\mathbb{Z}^d$. $\Rightarrow \varphi(X_n)$ is SRW on $\mathbb{T}_N^d$. Set $\tilde{X} = \varphi(X)$.

If $X \sim \mathbb{P}_x$. Then $\tilde{X} \sim \varphi \circ \mathbb{P}_x = \tilde{\mathbb{P}}_x$.

<u>Denote:</u> $\mathbb{P} = \frac{1}{N^d} \cdot \sum_{x \in \mathbb{T}_N^d} \tilde{\mathbb{P}}_x$. uniform initial dist. measure of $\tilde{X}$.

<u>Thm.</u> $\lim_{N\to\infty} \mathbb{P}(\{\tilde{X}_0, \cdots \tilde{X}_{\lfloor uN^d \rfloor}\} \cap \varphi(K) = \emptyset) = e^{-u\,\mathrm{cap}(K)}$

for $\forall K \subset\subset \mathbb{Z}^d$.

<u>Cor.</u> For $\mathbb{P}^{u,N}$ on $(\{0,1\}^{\mathbb{Z}^d}, \mathcal{Y})$. $\mathbb{P}^{u,N}(B)$

$= \mathbb{P}((\mathbb{I}_{\{\varphi(x) \in \{\tilde{X}_0, \cdots \tilde{X}_{\lfloor uN^d \rfloor}\}\}})_{x \in \mathbb{Z}^d} \in B)$ for

$\forall B \in \mathcal{Y}$. Then. we have: $\forall K \subset\subset \mathbb{Z}^d$.

$\lim_{N\to\infty} \mathbb{P}^{u,N}(B) = \mathbb{P}^u(B)$. $\forall B \in \sigma(\Psi_x, K)$

<u>Pf:</u> $B = \{A\}$. cylinder sets.

and use the inclusion-exclusion expr.