General Theory of Poisson Process.

1) Point Process:

① Def: For $(X, \mathcal{X})$ measurable space.

i) $N_{<\infty}(X) = \{ \mu \text{ measure on } X \mid \mu(B) \in \mathbb{N}$. for all $B \in \mathcal{X} \}$.

ii) $N(X) = \{ \mu$ measure on $X \mid \mu$ can be written in countable sum of measure in $N_{<\infty}(X) \}$.

iii) measure ν on X is s-finite if ν is countable sum of finite measures.

Rmk: i) σ-finite $\subsetneq$ s-finite.

e.g. $\mu = \sum_1^\infty \delta_0$ isn't σ-finite.

ii) $\forall \nu \in N(X)$ is s-finite

iv) $\mathcal{N}(X) = \sigma(\{ \mu \in N(X) \mid \mu(B) = k \}_{B \in \mathcal{X},\, k \in \mathbb{N}})$

Rmk: We want to measure "$\{ \mu(B) = k \}$" such sets.

v) A point process η on X is a proper point process if $\exists$ r.v.'s (X_k) in X and $\overline{\mathbb{N}}$-valued r.v. k. st. $\eta = \sum_{n=1}^{k} \delta_{X_n}$ a.s.

Rmk: i) We replace $\mathbb{R}$ by (X_n) r.v.'s in X. η can be intepreted as random set of points in X.

ii) class of proper point process is large

① **Def:** i) The intensity measure of a point process η is λ def by: $\lambda(B) = \mathbb{E}(\eta(B))$. $\forall B \in \mathcal{X}$.

ii) A point process on X is r.v. η of $(N(X), N(X))$. s.t. $\eta: \Omega \to N(X)$ is measurable.

Rmk: $\eta = \eta(\omega, B)$ is given randomness. taking values in $\overline{\mathbb{N}}$. satisfying: $\forall B \in \mathcal{X}$. $k \in \overline{\mathbb{N}}$.

$\eta^{-1}\{\mu \in N \mid \mu(B) = k\} = \{\omega \in \Omega \mid \eta(\omega, B) = k\} \in \mathcal{F}$.

iii) We say 2 points process η, η' are a.s. equal if $\exists A \in \mathcal{F}$. s.t. $\mathbb{P}(A) = 1$. $\eta(\omega) = \eta'(\omega)$. $\forall \omega \in A$.

prop. (Campbell's Formula)

For η is point process with intensity λ. $u: X \to \overline{\mathbb{R}}$ measurable. Then:

i) $\int u(x)\, \eta(dx)$ is r.v.

ii) $\mathbb{E}\left(\int u(x)\, \eta(dx)\right) = \int u(x)\, \lambda(dx)$ if $u \geq 0$ or $\int |u|\, d\lambda < \infty$.

Pf: i) Approxi. $u^{\pm}$ by $\sum \lambda_n^{\pm} I_{B_n^{\pm}}(x)$. $u = u^+ - u^-$

ii) Consider $\eta(u) = \eta(u^+) - \eta(u^-)$. Approxi. $u^{\pm}$.

③ Characterization:

Def: Laplace functional of point process η on X is map: $L_\eta : \{u : X \to \mathbb{R}^+ \text{ measurable}\} \to [0,1]$. def by $L_\eta(u) =: \mathbb{E}(\exp(-\int u(x)\eta(dx)))$.

prop. For point processes η, η' on X. Follows are equi:

i) $\eta \overset{d}{\sim} \eta'$. ii) $L_\eta(u) = L_{\eta'}(u)$. $\forall u \in D(L_\eta)$.

iii) $\forall m \in \mathbb{N}$. $(B_k)_1^m \subset \mathcal{X}$. disjoint $(\eta(B_k))_1^m \overset{d}{\sim} (\eta'(B_k))_1^m$

iv) $\forall u : X \to \mathbb{R}^+$ measurable. $\eta(u) =: \int u(x)\eta(dx) \overset{d}{\sim} \eta'(u)$.

prop. i) $(\eta_k)_{\mathbb{N}}$ is seq of point processes $\Rightarrow \sum_{k \geq 0} \eta_k$ is also point process.

ii) $(\eta_k)_{\mathbb{N}}$ is seq of proper point process on $\mathcal{X}$ $\Rightarrow$ $\sum_{k \geq 0} \eta_k$ is also proper point process.

Pf: i) is trivial. For ii):

Suppose: $\eta_k = \sum_i^{\kappa_k} \delta_{X_{ki}}$. Set $\kappa = \sum_1^{\infty} \kappa_n$.

reorder $\bigcup_{i,k} (X_{ki}) = \bigcup (\overline{X}_n)$.

$\Rightarrow \eta =: \sum \eta_k = \sum^{\kappa} \delta_{\overline{X}_n}$ is proper PP.

④ PP on metric space :

Def: X is metric space with metric e.

i) $\mathscr{X}_b = \{A \mid A$ is bdd measurable $\}$.

ii) measure ν is locally finite if $\nu(A) < \infty$ for $\forall A \in \mathscr{X}_b$.

iii) $N_l(X) = \{\mu \in N(X) \mid \mu$ is locally finite $\}$. with $\mathcal{N}_l(X) = \{A \cap N_l(X) \mid A \in \mathcal{N}(X)\}$.

prop. η, η' are point process on X. metric space. If $\eta(u) \sim \eta'(u)$. $\forall u: X \to \mathbb{R}^+$. measurable and $\{u > 0\}$ is bdd. Then $\eta \overset{d}{\sim} \eta'$.

Pf: $\forall v: X \to \mathbb{R}^+$. measurable. $\exists (u_n)$ satisfies condition above. and $u_n \uparrow v$.

Def: i) PP η on metric space X is locally finite if $\mathbb{P}(\eta(B) < \infty) = 1$. for $\forall B \in \mathscr{X}_b$.

ii) $\tilde{\eta}(w, \cdot) = \begin{cases} \eta(w, \cdot) & \text{if } \eta(w, \cdot) \text{ is locally finite} \\ 0 & \text{if otherwise} \end{cases}$

prop. η, η' are locally finite PP on metric space X. If $\eta(u) \sim \eta'(u)$ for $\forall u \in C(X, \mathbb{R}^+)$. $\{u > 0\}$ is bdd. Then: $\eta \sim \eta'$.

(2) Poisson point process:

① Motivation:

On measurable space (M, Σ). We want to find measure M on it. satisfying:

i) $A, A' \subset M$. $A \cap A' = \emptyset \Rightarrow M(A)$ indept of $M(A')$.

ii) Condition on # points in $A = n$. $\Rightarrow$ All points in A are i.i.d. $\sim M|_A(\cdot)/M(A)$.

i.e. We're trying to construct sth. like $\mathrm{Unif}(\mathbb{R}^n)$. (which's impossible)

$\Rightarrow$ So we consider to distribute sets of ∞ points uniformly. (rather than 1 points)

② Definition:

Fix measurable space $(X, \mathcal{X})$.

Def: λ is a s-finite measure on X. Poisson process with intensity λ is PP η

st. i) $\forall B \in \mathcal{X}$. $\eta(B) \sim \mathrm{POI}(\lambda(B))$

ii) $\forall m \in \mathbb{N}$. $(B_k)_1^m \subset \mathcal{X}$. pointwise disjoint $\Rightarrow (\eta(B_k))_1^m$ are indept.

prop. η, η' are 2 poisson process with same s-finite intensity on $X \Rightarrow \eta \overset{d}{\sim} \eta'$.

③ Existence:

Thm (Superposition)

$(\eta_i)_{\mathbb{N}}$ is seq of indept poisson process on X with intensity λ_i. $\Rightarrow \sum_{i\geq 0} \eta_i = \eta$ is poisson process with intensity $\lambda = \Sigma \lambda_i$.

Pf: Check def i), ii) directly.

Def: $\mathbb{V}$, $\mathbb{Q}$ are p.m. on $\mathbb{N}$, and X respectively. If $X_k \overset{i.i.d}{\sim} \mathbb{Q}$, $\forall k \geq 1$, r.v.'s in X. $K \sim \mathbb{V}$ indept of (X_k). Then $\eta = \sum_{k=1}^{K} \delta_{X_k}$ is called mixed binomial process with mixing dist. $\mathbb{V}$ and sampling dist. $\mathbb{Q}$

Prop. For $\gamma \geq 0$, $\mathbb{Q}$ is p.m. on X. If η is mixed binomial process with mixing dist. $POI(\gamma)$ and sampling dist. $\mathbb{Q}$. Then $\eta \sim POI(\gamma \mathbb{Q})$

Pf: For $(k_i)_1^n \subset \mathbb{N}$, $(B_k)_1^n$ pointwise disjoint.

$$P(\eta(B_1) = k_1, \cdots \eta(B_n) = k_n) = (K = \sum_1^n k_i)$$

$$\mathbb{P}(K = \sum_1^n k_i)\, \mathbb{P}(\sum_1^k I_{\{X_i \in B_1\}} = k_1 \cdots \sum_1^k I_{\{X_i \in B_n\}} = k_n)$$

$$= e^{-\gamma} \frac{\gamma^k}{k!} \cdot \frac{k!}{\prod_1^n k_i!} \cdot \prod_1^n (\mathbb{Q}(B_i))^{k_i}$$

$\Rightarrow$ Sum up $k_2 \cdots k_n$. $\eta(B_1) \sim POI(\gamma \mathbb{Q}(B_1))$

Thm. (Existence Thm)

λ is s-finite measure on X. Then there exists a Poisson process on X with intensity λ.

Pf: Set $\eta = \sum_1^{k} \delta_{X_n}$ where $X_n \overset{i.i.d}{\sim} \lambda(\cdot)/\lambda(X)$

and $k \sim POI(\lambda(X))$ if $\lambda(X) \in (0,\infty)$

If $\lambda(X) = \infty$. Note $\exists \lambda_i$. s.t. $\lambda_i(X) < \infty$

$\lambda = \sum \lambda_i$. $\Rightarrow \exists \eta_i$. on $(\bigtimes_1^{\infty} \Omega_n, \bigotimes_1^{\infty} \mathcal{F}_n, Q)$ (*)

s.t. $\eta_i \sim$ Poisson process with λ_i under Q.

Set $\eta = \sum \eta_i$. by superposition Thm.

Lemma (*) $(\Omega_n, \mathcal{F}_n, Q_n)$ $n \geq 1$. are prob. spaces.

Then there exists p.m. Q on space $(\bigtimes_1^{\infty} \Omega_n, \bigotimes_1^{\infty} \mathcal{F}_n)$ s.t. $Q(A \times \bigtimes_{n+1}^{\infty} \Omega_m) = \bigotimes_1^{n} Q_i(A)$. $\forall n$. and $A \in \bigotimes_1^{n} \mathcal{F}_m$.

Cor. λ is s-finite on X. Then there exists prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting r.v.'s (X_n) in X and k in $\overline{\mathbb{N}}$. satisfying:

$\eta =: \sum_1^{k} \delta_{X_k} \sim$ Poisson process with intensity λ.

Rmk: Every PPP with s-finite intensity can has a proper point process dist. But not every PPP is proper point process

Pf: Only consider $\lambda(X)=\infty$. use notations above:

set $Q_i = \lambda_i(\cdot)/\lambda_i(X)$.

Take $(\Omega, \mathcal{F}, P) = \bigotimes_{i\geq 0} ((\mathbb{N}, Poi(\lambda_i(X))) \times \bigotimes_{j\geq 0} (X, \mathcal{X}, Q_i))$. P exists by Lemma.

$\Rightarrow$ On $(\Omega, \mathcal{F}, P)$, $\exists (k_i) \overset{indept}{\sim} Poi(\lambda_i(X))$

and $(X_{ij})_j \overset{i.i.d}{\sim} Q_i$. for $\forall i$.

Set $k = \Sigma k_i$. reorder $\bigcup_{i,j}(X_{ij}) = \bigcup(\tilde{X}_i)$

$\Rightarrow \eta = \sum_1^k \delta_{\tilde{X}_n}$

Rmk: Actually any suitable regular PP on a Borel subset of a complete separable metric space is proper.

Prop. (Converse)

η is a poisson process with intensity λ st. $\lambda(X) \in (0,\infty)$ Then η has dist. of a mixed binomial process with mixing dist. $Poi(\lambda(X))$ and sampling dist. $\lambda(\cdot)/\lambda(X)$. Besides, $\mathbb{P}(\eta(B)=k \mid \eta(X)=n)$

$= \binom{n}{k} \left(\frac{\lambda(B)}{\lambda(X)}\right)^k \left(1-\frac{\lambda(B)}{\lambda(X)}\right)^{n-k}$

Pf: The mixed binomial process η' described above will distribute as η.

Then for $\mathbb{P}(\eta(B)=k \mid \eta(X)=n)$:

it equals to $\mathbb{P}(\eta'(B)=k \mid k=n)$ if $\eta' = \sum_1^k \delta_{X_n}$.

Cor. For η poisson process on X with intensity λ

If $B \in \mathcal{X}$. s.t. $\lambda(B) < \infty$. $B = \sum_1^n B_k$. Then

for $(k_i)_1^n \subset \mathbb{N}$. $m = \sum_1^n k_i$. We have:

$$\mathbb{P}\left(\bigcap_1^n \{\eta(B_i) = k_i\} \mid \eta(B) = m\right) = \frac{m!}{\prod_1^n k_i!} \prod_1^n \left(\frac{\lambda(B_i)}{\lambda(B)}\right)^{k_i}$$

Pf: As above. consider $\eta' \sim \eta$. mixed binomial

Cor. $\mathbb{P}(X_1 = x_1, \cdots X_n = x_n \mid \eta(A) = n) =$

$\prod_1^n \lambda(x_i) \,/\, \lambda(A)^n$. for $x_i \in A$. $\forall 1 \le i \le n$.

Pf: Consider $B = \{x_i\}_1^n$. $\eta(A/B) = 0$.

④ Characterization:

Thm. (Laplace functional)

λ is s-finite measure on X. η is point process

on X. Then η is poisson process with intensity

$\lambda \iff L_\eta(u) = e^{\int (e^{-u(x)} - 1)\lambda(dx)}$.

Pf: Test with simple func. Approxi u by them.

prop. (Transformation)

For η PPP with intensity λ. s-finite. Ref:

$(I_A\lambda)(B) =: \lambda(A \cap B)$. $(\varphi \circ \lambda)(B) =: \lambda(\varphi^{-1}(B))$

$\Rightarrow I_A\eta \sim PPP(I_A\lambda(\cdot))$. on X. $\varphi \circ \eta \sim$

$PPP(\varphi \circ \lambda)$ on X'. where $\varphi: X \to X'$ measurable.