Haar measure on LCH Topo group.

consider on $G = (\underline{G}, \cdot)$, a LCH topological group.

<u>Def</u>: i) A σ-algebra $\mathcal{A}$ preserved by left/right translations if $\forall g \in G$. $A \in \mathcal{A}$. $gA \in \mathcal{A} / Ag \in \mathcal{A}$.

ii) A positive measure μ on $(G, \mathcal{A})$ is called left/right invariant measure if $\forall g \in G$. $A \in \mathcal{A}$. $\mu(A) = \mu(gA) / = \mu(Ag)$

<u>Rmk</u>: μ is left invariant $\iff \forall f : G \to \mathbb{R}_{\geq 0}$ measurable and $g \in G$. $\int_G f \, d\mu = \int_G f \circ l_g \, d\mu$.

iii) For topo space X. $\Lambda_0 : C_c(X, \mathbb{R}_{\geq 0}) \to \mathbb{R}_{\geq 0}$ is a cone linear functional (CLF) if

(a) $\forall f_1, f_2 \in C_c(X)_{\geq 0}$. $\Lambda_0(f_1 + f_2) = \Lambda_0(f_1) + \Lambda_0(f_2)$

(b) $\forall c > 0$. $f \in C_c(X)_{\geq 0}$. $\Lambda_0(cf) = c\Lambda_0(f)$.

<u>Rmk</u>: $\{ PLF\text{'s} : C_c(X) \to \mathbb{C} \} \longrightarrow \{ CLF : C_c(X)_{\geq 0} \to \mathbb{R}_{\geq 0} \}$

$(C_c(X) \xrightarrow{\Lambda} \mathbb{C}) \longmapsto (C_c(X)_{\geq 0} \xrightarrow{\Lambda|_{C_c(X)_{\geq 0}}} \mathbb{R}_{\geq 0})$

is a bijection. (separate $f \in C_c(X)$ into $f = f^+ - f^-$. injective is trivial. Surjective: set $\Lambda = \Lambda_0(f^+) - \Lambda_0(f^-)$ for Λ_0 CLF)

$\Rightarrow$ By Riesz Represent : Radon measures $\overset{\text{corr.}}{\sim}$ CLFs

prop. For Λ CLF ~ μ radon measure. Then:

μ is left invariant $\Leftrightarrow$ $\forall g \in G$. $f \in C_c(G)_{\geq 0}$

we have: $\Lambda(f) = \Lambda(f \circ l_g)$

Pf: ($\Leftarrow$) trivial ($\Rightarrow$) Approxi. $C_c(G)$ by simple Func.

(1) **Existence**:

Thm. (Weil. Cartan)

On LCH topo group. There $\exists$ left / right

invariant radon measure $\neq 0$.

By prop. above. It suffices to find a CLF

$\Lambda : C_c(G)_{\geq 0} \to \mathbb{R}_{\geq 0}$. $\Lambda \neq 0$. st. $\Lambda(f) = \Lambda(f \circ l_g)$. $\forall g$.

① **Approxi. family of conc**

Sublinear left. inv. Functions:

Def: For $f \in C_c(G)_{\geq 0}$. $\varphi \in C_c(G)_{>0}$.

Let $S_\varphi(f) = \{ \sum_1^n c_i \mid n \in \mathbb{N}.\ c_k \geq 0.\ \exists g_k \in G.$

st. $f \leq \sum_1^n c_i \varphi \circ l_{g_i} \}$.

prop. $S_\varphi(f)$ is nonempty.

Pf: Set $U_\varphi = \{ x \in G \mid \varphi(x) > \max_G \varphi / 2 \} \underset{\text{open}}{\subseteq} G$.

Note $\mathrm{supp}(f) \subset G = \bigcup_{g \in G} (g^{-1} U_\varphi)$.

follows from $\bigcup g^{-1}U_\varphi = \bigcup U_{\varphi\circ l_g}$, which's

union of translation of open set.

$\Rightarrow \exists (g_k)_1^n$. $\operatorname{supp} f \subseteq \bigcup_1^n (g_i^{-1} U_\varphi)$.

So: $f \leq \sum_1^n \left(\frac{2\max f}{\max \varphi\circ l_{g_i}} \right) \varphi\circ l_{g_i}$.

<u>Denote</u>: For convenience, $\varphi_{g_i} =: \varphi\circ l_{g_i}$.

<u>Def</u>: For $f \in C_c(G)_{\geq 0}$, $\varphi \in C_c(G)_{>0}$, $(f:\varphi) := \inf S_\varphi(f)$.

<u>Lemma.</u> For $f, f', \varphi, \varphi' \in C_c(G)_{>0}$, Then:

i) $0 < (f:\varphi) < \infty$. ii). $\forall g \in G$, $(f:\varphi) = (f_g:\varphi)$.

iii) $(f+f':\varphi) \leq (f:\varphi) + (f',\varphi)$.

iv) $\forall c \geq 0$. $(cf:\varphi) = c(f:\varphi)$

v) $f \leq f' \Rightarrow (f:\varphi) \leq (f':\varphi)$.

vi) $(f:\varphi) \leq (f:\varphi')(\varphi':\varphi)$.

<u>Pf</u>: i) $S_\varphi(f) \subset [\frac{f(p)}{\sup \varphi}, \infty)$ for some p. s.t. $f(p)>0$

ii) is trivial. For iii). $\exists (c_k), (c_k')$.

$f \leq \sum_1^n c_k \varphi_{g_k}$, $\sum_1^n c_k \in ((f:\varphi), (f:\varphi)+\varepsilon)$

$f' \leq \sum_1^{m'} c_k' \varphi_{g_k'}$. $\sum_1^{m'} c_k' \in ((f':\varphi), (f':\varphi)+\varepsilon)$

$\Rightarrow (f+f',\varphi) \leq \sum_1^n c_k + \sum_1^{m'} c_k' \leq (f:\varphi) + (f':\varphi) + 2\varepsilon$

iv) v) are trivial. vi) is smilar as ii):

Find (c_k). $(f:\varphi') \overset{\varepsilon}{\sim} \sum_1^m c_k$. $(\varphi',\varphi) \overset{\varepsilon}{\sim} \sum_{m+1}^n c_k$.

Fix $\varphi_0 \in C_c(G)_{>0}$ in the following:

Def: For $f \in C_c(G)_{\geq 0}$. $\Lambda_\varphi(f) =: \dfrac{(f:\varphi)}{(\varphi_0:\varphi)}$

Cor. $\forall f, f'$ and $\varphi \in C_c(G)_{>0}$. We have:

i) $0 < \Lambda_\varphi(f) < \infty$. ii) $\forall g \in G$. $\Lambda_\varphi(f) = \Lambda_\varphi(f_g)$

iii) $\Lambda_\varphi(f+f') \leq \Lambda_\varphi(f) + \Lambda_\varphi(f')$.

iv) $\forall c \geq 0$. $\Lambda_\varphi(cf) = c\Lambda_\varphi(f)$.

v) $f \leq f' \Rightarrow \Lambda_\varphi(f) \leq \Lambda_\varphi(f')$.

vi) $1/(\varphi_0:f) \leq \Lambda_\varphi(f) \leq (f:\varphi_0)$.

Pf of Thm:

Lemma $\forall \varepsilon > 0$, and $f_1, f_2 \in C_c(G)_{\geq 0}$. $\exists V$, nbd of e in G. s.t. $\forall \varphi \in C_c(G)_{>0}$. If $\operatorname{supp}(\varphi) \subset V$. Then: $\Lambda_\varphi(f_1) + \Lambda_\varphi(f_2) \leq \Lambda_\varphi(f_1+f_2) + \varepsilon$.

Pf: Set $h_k = f_k / f_1 + f_2 + \delta f =: f_k/f_\delta$.

where $\operatorname{supp}(f_1+f_2) \subset f$. $\Rightarrow h_k \in (0,1)$.

$\forall \varepsilon' > 0$. $\exists$ nbd V of e in G. s.t.

$y^{-1}x \in V \Rightarrow |h_k(x) - h_k(y)| < \varepsilon'$. $k = 1, 2$.

Find (c_k), (g_k). s.t. $f_\delta \leq \sum c_k \varphi_{g_k}$

$\Rightarrow f_k = f_\delta h_k \leq \sum c_j h_k(x) \varphi_{g_j}(x)$

$\leq \sum c_j (h_k(g_j^{-1}) + \varepsilon') \varphi_{g_j}(x)$

if $\operatorname{supp}\varphi \subset V$. So $x \in \operatorname{supp}\varphi_{g_i} \Rightarrow g_i^{-1}x \in V$.

So: $(f_1 : \varphi) + (f_2 : \varphi) \leq \Sigma\, c_j\, (h_1(g_j^{-1}) + h_2(g_j^{-1}) + 2\varepsilon')$

$\leq \Sigma\, c_j\, (1 + 2\varepsilon')$

$\Rightarrow \Lambda_\varphi(f_1) + \Lambda_\varphi(f_2) \leq \Lambda_\varphi(f_\delta)(1+2\varepsilon')$

$\leq (\Lambda_\varphi(f_1+f_2) + \delta\Lambda_\varphi(f))(1+2\varepsilon')$

Take $\delta, \varepsilon' > 0$ small enough. Obtain conclusion.

Return to pf: (Weil)

For $f \in C_c(G)_{>0}$. Set $I_f := [\, 1/(\varphi_0 : f),\ (f : \varphi_0)]$

$I = \prod_{f \in C_c(G)_{>0}} I_f$ cpt. by Tychonoff's Thm.

$\forall\, \varphi \in C_c(G)_{>0}. \longmapsto P_\varphi = (\Lambda_\varphi(f))_f \in I$. by Cor.

1') $\forall$ open nbd V. of e. Set $A_V = \{ P_\varphi \mid \varphi \in C_c(G)_{>0},\ \operatorname{supp}\varphi \subset V\}$.

Note: $V \subset V_1 \cap V_2$. nbd's of e $\Rightarrow A_V \subseteq A_{V_1} \cap A_{V_2}$.

$\Rightarrow (\bar{A}_V)_{V \text{ is nbd of } e}$ is family of closed sets

in cpt space I. st. has FIP property.

So: $\exists\, P = (P_f)_f \in \bigcap_V \bar{A}_V$.

Def: $\Lambda : C_c(G)_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$. $\Lambda(f) := \begin{cases} 0. & \text{if } f = 0 \\ P_f. & \text{if } f \in C_c(G)_{>0} \end{cases}$

2') Claim: Λ has desired properties:

For $\varepsilon > 0$. $f_1, f_2 \in C_c(G)_{>0}$. $c > 0$. $g \in G$.

Set: $\Omega = \{ Q = (Q_f)_{f \in I} \mid |Q_f - P_f| < \varepsilon,\ \text{if } f = f_1, f_2, f_1+f_2, cf_1, cf_2, f_{1q}, f_{2q} \}$.

which is open in I, and $P \in \Omega$, by def.

Denote V is nbd of e in Lemma above.

Then $\Omega \cap A_V \ni \{P\}$, open.

Choose $P_\varphi \neq P \in \Omega \cap A_V \Rightarrow |\Lambda_\varphi(g) - \Lambda(g)| < \varepsilon$

So: $|\Lambda(f_1) + \Lambda(f_2) - \Lambda(f_1+f_2)| \leq |\Lambda(f_1) - \Lambda_\varphi(f_1)|$

$+ |\Lambda(f_2) - \Lambda_\varphi(f_2)| + |\Lambda_\varphi(f_1+f_2) - \Lambda(f_1+f_2)| +$

$|\Lambda_\varphi(f_1) + \Lambda_\varphi(f_2) - \Lambda_\varphi(f_1+f_2)| \overset{\text{(Lem)}}{\lesssim} \varepsilon$

For other properties: $|\Lambda(f) - \Lambda(cf)/c| < \varepsilon$ ...

similarly argue.

(2) <u>Uniqueness</u>:

<u>Thm.</u> For $\mu, \tilde{\mu}$ two left/right invariant Haar measure on G, LCH topo group. Then $\exists$ $\alpha > 0$, s.t. $\alpha\mu = \tilde{\mu}$.

<u>e.g.</u> $\mathcal{L}$, Lebesgue measure on $(\mathbb{R}^n, \cdot)$