A Problem of Measure

Next, we prove: There exists a finite additive "measure" (set func. actually). satisfies:

i) It defines on $P(\mathbb{R}^d)$.

ii) It agree with Lebesgue measure on measurable sets.

iii) It's translation invariant.

__Remark:__ It can't be σ-additive: Use Vitali set V.

$(q_k) = [0,1] \cap \mathbb{Q}$. $V_k = V + q_k$. Then (V_k) disjoint.

since $V \subset (0,1)$. $\therefore \bigcup_1^\infty V_k \subset (0,2)$

$\hat{m}(\bigcup_1^\infty V_k) = \sum_1^\infty \hat{m}(V_k) \in (0,2)$. It's absurb!

(1) __Thm:__ There's a LF $I: f \mapsto I(f)$. defined on $V = \{f: \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}' \mid |f| < \infty\}$. s.t.

i) $I(f) \geq 0$. if $f \geq 0$.

ii) $I(\alpha f_1 + \beta f_2) = \alpha I(f_1) + \beta I(f_2)$. $\forall \alpha, \beta \in \mathbb{R}'$.

iii) $I(f) = \int_0^1 f(x)\,dx$. if f is $\mathcal{L}$-measurable.

iv) $I(f_h) = I(f)$. $\forall h \in \mathbb{R}'$.

__Pf:__ For applying Hahn-Banach Thm:

consider $V_0 = V \cap \{f \text{ is measurable}\}$.

Define $I_0(f) = \int_0^1 f$. $\forall f \in V_0$.

Next. find p. Minkovskian: $I_0(f) \leq p(f)$.

1°) Set $M_A(f) = \sup_x \left(\frac{1}{N} \sum_1^N f(x+a_k) \right)$.

where $A = \{a_k\}_1^N$. $N = |A|$.

Set $p(f) = \inf \{ M_A(f) \mid |A| < \infty \}$. well-def ($|f| < \infty$)

2°) $p(cf) = c\,p(f)$ is obvious. For $p(f_1+f_2) \leq p(f_1) + p(f_2)$:

$$\text{Find } A, B \text{ s.t. } \begin{cases} M_A(f_1) \leq p(f_1) + \varepsilon, & A = (a_i)_1^N \\ M_B(f_2) \leq p(f_2) + \varepsilon, & B = (b_j)_1^m \end{cases}$$

Set $C = \{a_i + b_j\}_{i,j}$ $|C| = Nm$. $M_C(f_1+f_2) \geq p(f_1+f_2)$

$$\text{Check } \begin{cases} M_C(f_1+f_2) \leq M_C(f_1) + M_C(f_2). \\ M_A(f) = M_{A+h}(f_h). \end{cases} \Rightarrow \begin{cases} M_C(f_1) \leq M_A(f_1) \\ M_C(f_2) \leq M_B(f_2) \end{cases}$$

sum over: $p(f_1+f_2) \leq p(f_1) + p(f_2) + 2\varepsilon$

3°) Note $I_0(f) = \int_0^1 f(x) = \frac{1}{N} \int_0^1 \sum_1^N f(x+a_k)$ (periodic)

$$\leq \int_0^1 M_A(f) = M_A(f).$$

$\Rightarrow I_0(f) \leq p(f)$. Then extend to I on V.

4°) Note that $p(f) \leq 0$, when $f \leq 0$.

$\therefore I(f) \leq 0$. Then I satisfies i)

5°) For iv):

Set $A_N = \{h, 2h \cdots Nh\}$.

Then $|M_{A_N}(f - f_h)| \le 2M/N \to 0$ $(N \to \infty)$.

Since $p(f - f_h) \le M_{A_N}(f - f_h) \to 0$.

$\therefore$ $p(f - f_h) \le 0$. $I(f - f_h) \le 0$.

By symmetry : $I(f) = I(f_h)$

Cor. $\exists\ \hat{m} : P(\mathbb{R}/\mathbb{Z}) \longrightarrow \mathbb{R}^+ \cup \{0\}$ s.t.

i) $\hat{m}(E) = m(E)$. m is Lebesgue measure.

ii) $\hat{m}(E_1 \cup E_2) = \hat{m}(E_1) + \hat{m}(E_2)$. if $E_1 \cap E_2 = \emptyset$.

iii) $\hat{m}(E + h) = \hat{m}(E)$. $\forall h \in \mathbb{R}'$.

pf: Let $\hat{m}(E) = I(\chi_E)$.

(2) $d = 1$. case:

Thm. $\exists\ \hat{m} : P(\mathbb{R}) \longrightarrow \mathbb{R}^+ \cup \{0\}$. satisfies i). ii). iii) above.

pf: Denote $\hat{m}_0$ for the measure in Cor. above.

We wanna to extend it to $\hat{m}$:

Set $I_j = (j, j+1]$. $\mathbb{R} = \bigcup_{j \in \mathbb{Z}} I_j$

Define: $\hat{m}(E) = \hat{m}_0(E)$. if $E \subseteq I_0$

$\hat{m}(E) = \hat{m}_0(E - j)$ if $E \subseteq I_j$

So we can define:

$$\hat{m}(E) = \sum_{-\infty}^{+\infty} \hat{m}(E \cap I_j) = \sum_{-\infty}^{+\infty} \hat{m}_0(E \cap I_j - j)$$

Then i), ii) are from $\hat{m}_0$.

For iii):

If $h = k \in \mathbb{Z}$. It's trivial.

If $h \in (0,1)$. Denote: $E_j' = E \cap (j, j+1-h]$

$E_j'' = E \cap (j+1-h, j+1]$

$\therefore E = (\cup E_j') \cup (\cup E_j'')$. disjoint union.

$$\therefore \hat{m}(E) = \Sigma (\hat{m}(E_j') + \hat{m}(E_j''))$$

$$= \Sigma (\hat{m}(E_j'+h) + \hat{m}(E_j''+h)) = \hat{m}(E+h)$$

Since $E_j'+h \subset I_j$. $E_j''+h \subset I_{j+1}$.

$\Rightarrow$ For $h \in \mathbb{R}^1$. $h = [h] + h - [h]$.

(3) <u>General: $d = k \in \mathbb{Z}'$:</u>

We can parallelly extend the proof of "$d=1$":

$$M_A(f) = \sup_{x \in \mathbb{R}^k} \left(\frac{1}{N} \Sigma f(x + \vec{a}_i) \right).\quad A = \{\vec{a}_i\}_1^N$$

$$f: \mathbb{R}^k / \mathbb{Z}^k \simeq \prod^k [0,1) \longrightarrow \mathbb{R}^1.$$

Decompose $\mathbb{R}^k = \bigcup_{(i_1, i_2, \cdots i_k) \in \mathbb{Z}^k} \left(\prod_j^k I_{i_j} \right)$. We obtain $\hat{m}$.