Hausdorff Measure

. The dimension of a set plays a crucial role in geometry, which can be understood in terms of how the set replicates under scalings.

(1) <u>Metric Exterior Measures:</u>

. X is a metric space with distance func. $d(\cdot,\cdot)$.

<u>Def:</u> An exterior measure m^* on X is metric exterior measure if it satisfies:

$m^*(A \cup B) = m^*(A) + m^*(B)$, whenever $d(A,B) > 0$.

<u>Thm.</u> The Borel sets in X is m^*-measurable.

So $m^*|_{B_X}$ is a measure.

<u>Pf:</u> It suffices to prove: closed sets in X are m^*-measurable (generates B_X).

For $F \underset{\text{closed}}{\subseteq} X$. $\forall A \in P(X)$.

Denote $A_n = \{x \in F^c \cap A \mid d(x,F) \geq \frac{1}{n}\}$. By prop. of m^*

$\therefore m^*(A) \geq m^*(F \cap A) + m^*(A_n) = m^*(A_n \cup (F \cap A))$

prove: $\lim\limits_{n\to\infty} m^*(A_n) = m^*(F^c \cap A)$

Denote $B_n = A_n^c \cap A_{n+1} (\neq \emptyset)$ $\because F^c \cap A = A_n \cup (\bigcup\limits_{n}^{\infty} B_k)$

$\therefore m^*(A_n) \leq m^*(F^c \cap A) \leq m^*(A_n) + \sum\limits_k m^*(B_k)$

prove: $\sum m^*(B_n)$ converges

It follows from: $d(B_{n+1}, A_n) \geq \frac{1}{n} - \frac{1}{n+1}$. $A_n \supseteq B_{n+1} \cup A_{n+2}$

$$\therefore \begin{cases} \mu^*(A_{2k+1}) \geq \mu^*(B_{2k}) + \mu^*(A_{2k+1}) \\ \mu^*(A_{2k}) \geq \mu^*(B_{2k+1}) + \mu^*(A_{2k+2}) \end{cases} \quad \because \mu^*(A_n) \leq \mu^*(A).$$

$$\Rightarrow \sum_1^{2n} \mu^*(B_k) \leq \mu^*(A_{2n}) + \mu^*(A_{2n+1}).$$ Bounded, monotone.

<u>prop.</u> If Borel measure μ is finite on all balls in metric space X. Then μ is "regular."

<u>Pf</u>: Denote $B_n = \{x \mid d(x, x_0) < n\}$. Fix $x_0 \in X$. $\therefore X = \cup B_n$.

C is the collection of sets satisfies the conclusion.

1) C is a σ-algebra.

i) $E \in C \Rightarrow E^c \in C$ is trivial.

ii) $\{E_k\} \subseteq C \Rightarrow \cup E_k \in C$.

For out regular: Choose $O_k \supseteq E_k$. $\mu(O_k / E_k) < \frac{\varepsilon}{2^k}$

Then $O = \cup O_k \overset{\varepsilon}{\supseteq} \cup E_k$.

For inner regular: Choose $E_k \supseteq F_k$. $\mu(E_k/F_k) < \frac{\varepsilon}{2^k}$.

$F = \cup F_k$ may not be closed!

<u>Claim</u>: $\exists F^* \subseteq F$. closed. s.t. $\mu(F/F^*) < \varepsilon$.

<u>Pf</u>: WLOG. Suppose $F_k \uparrow F = \bigcup_n (F \cap (\bar{B}_n / B_{n-1}))$

By $F_k \cap (\bar{B}_n / B_{n-1}) \to F \cap (\bar{B}_n / B_{n-1})$. $(k \to \infty)$

$\Rightarrow \exists N(n)$. s.t. $\mu(F / (F_{N(n)} \cap (\bar{B}_n / B_{n-1}))) < \frac{\varepsilon}{2^n}$

set $F^* = \cup (F_{N(n)} \cap (\bar{B}_n / B_{n-1}))$

Check F^* is closed followed from:

$\forall n$. $\bar{B}_n \cap F^*$ is closed.

For $\forall x_n \to x$. $\{x_n\}$ bounded $\subseteq F^*$

$\Rightarrow \exists N$. $\{x_n\} \subseteq \bar{B}_N \cap F^*$

We have $x \in F^*$. □

<u>Rmk</u>: $\forall n$. $\bar{B}_n \cap F$ is closed $\Rightarrow F$ is closed.

2°) $\forall\, O$ is open. $O \in \mathcal{C}$.

Since $F_k = \{x \in \bar{B}_k \mid d(x, O^c) \geq \frac{1}{k}\}$.

$\therefore F_k \uparrow O$.

(2) Hausdorff Measure:

① Def: $m_\alpha^*: P(\mathbb{R}^d) \longrightarrow \overline{\mathbb{R}^+}$. For $\forall\, E \subseteq \mathbb{R}^d$. $\operatorname{diam} S = \sup_{x,y \in S} |x-y|$

$$m_\alpha^*(E) = \lim_{\delta \to 0} \inf \{ \sum_k (\operatorname{diam} F_k)^\alpha \mid E \subseteq \cup F_k,\ \operatorname{diam} F_k \leq \delta,\ \forall k \}.$$

is exterior α-dimensional Hausdorff measure.

Denote: $\mathcal{H}_\alpha^\delta(E) = \inf \{ \sum_k (\operatorname{diam} F_k)^\alpha \mid E \subseteq \cup F_k,\ \operatorname{diam} F_k \leq \delta,\ \forall k \}$.

$\therefore m_\alpha^*(E) = \lim_{\delta \to 0} \mathcal{H}_\alpha^\delta(E)$. exists. Since $\mathcal{H}_\alpha^\delta(E) \uparrow$ as $\delta \downarrow$

Remark: i) The key ideal in it is scaling

if set F is scaled by r. then $m_\alpha^*(F)$

is scaled by r^α.

ii) Intuitively. if E is β dimension.

when $\alpha < \beta$. then $m_\alpha^*(E) = \infty$

when $\alpha = \beta$. then $m_\alpha^*(E) \in (0, \infty)$

when $\alpha > \beta$. then $m_\alpha^*(E) = 0$.

② Properties:

i) If $E_1 \subseteq E_2$ Then $m_\alpha^*(E_1) \leq m_\alpha^*(E_2)$

ii) If $E = \cup E_i$. Then $m_\alpha^*(E) \leq \sum m_\alpha^*(E_i)$

iii) If $d(E_1, E_2) > 0$. Then $m_\alpha^*(E_1 \cup E_2) = m_\alpha^*(E_1) + m_\alpha^*(E_2)$

Remark: Note that m_α^* is metric outer measure

$\therefore m_\alpha^*|_{B_X}$ is a measure denoted m_α.

iv) $E \in B_{\mathbb{R}^d}$. Then $c_d m_d(E) = m(E)$. where m is Lebesgue measure on $\mathbb{R}^d$. $c_d = \alpha(d)/2^d$

Pf: We only prove a weaker one: $m_d(E) \approx m(E)$ in sense that: $c_d m_d(E) \le m(E) \le 2^d c_d m_d(E)$.

Consider cover by balls.

$\mathcal{H}_d^\delta(E) \le \sum (\text{diam} B_i)^d \le c_d^{-1} \sum m(B_i) \sim c_d^{-1} m(E)$.

$\therefore m_d(E) \le c_d^{-1} m(E)$.

Conversely. $E \subseteq \cup F_n$. Let B_n centers at one point of F_n. $\text{diam} B_n = 2 \,\text{diam} F_n$.

$\therefore m(E) \sim \sum m(B_n) = c_d \sum (\text{diam} B_n)^d = 2^d c_d \sum (\text{diam} F_n)^d$

v) If $m_\alpha^*(E) < \infty$. Then $\forall \beta > \alpha$. $m_\beta^*(E) = 0$. and $\forall \beta < \alpha$. $m_\beta^*(E) = \infty$. whenever $m_\alpha^*(E) > 0$.

Pf: Note that $\mathcal{H}_\beta^\delta(E) \le \delta^{\beta-\alpha} m_\alpha^*(E)$.

(3) Hausdroff Dimension:

. Note that for $E \in B_{\mathbb{R}^d}$. there exists a unique α. st. $m_\beta(E) = \begin{cases} \infty, & \beta < \alpha \\ 0, & \beta > \alpha. \end{cases}$

i.e. $\alpha = \sup\{\beta \mid m_\beta(E) = \infty\} = \inf\{\beta \mid m_\beta(E) = 0\}$.

we say E has Hausdorff dimension α.

If $m_\alpha(E) \in (0, \infty)$. Then α is strict.

① Examples:

i) Cantor set:

Thm. The Cantor set $C_{\frac{1}{3}}$ has strict Hausdorff dimension $\alpha = \log 2 / \log 3$

1°) $m_\alpha(C_{\frac{1}{3}}) \leq 1$

Pf: $C_{\frac{1}{3}} = \cap C_K$. where C_K is collection of 2^k intervals of diameter 3^{-k}.

$\therefore \mathcal{H}_\alpha^\delta(C_{\frac{1}{3}}) \leq 2^k \cdot (3^{-k})^\alpha = 1$. where $3^{-k} < \delta$.

$\therefore m_\alpha(C_{\frac{1}{3}}) \leq 1$.

2°) $m_\alpha(C_{\frac{1}{3}}) > 0$

Pf: Lemma. Suppose f defined on cpt set E. satisfies γ-Hölder condition. Then

$$\begin{cases} m_\beta(f(E)) \leq M^\beta m_\alpha(E). \ \beta = \frac{\alpha}{\gamma} \\ \dim f(E) \leq \frac{1}{\gamma} \dim E. \end{cases} \text{ holds.}$$

Pf: $\{F_k\}$ covers E. Then

$$\operatorname{diam}(f(E \cap F_k))^{\frac{\alpha}{\gamma}} \leq (M \operatorname{diam}(F_k)^\gamma)^{\frac{\alpha}{\gamma}}$$

$\Rightarrow$ It suffices to prove: Cantor-Lebesgue function F satisfies γ-Hölder condition. $\gamma = \frac{\log 2}{\log 3}$

Note: $F_n \to F$. where F_n increases at most 2^{-n} on each interval of length 3^{-n}.

$\therefore |F_n(x) - F_n(\eta)| \leq (\frac{1}{2})^n \cdot \frac{|x-\eta|}{3^{-n}} = (\frac{3}{2})^n |x-\eta|$.

And: $|F_n(x) - F(x)| \leq 2^{-n}$. we obtain:

$|F(x) - F(\eta)| \leq 2^{-n+1} + (\frac{3}{2})^n |x-\eta|$.

Choose n. st. $3^n |x-\eta| \in (1,3)$

$\therefore |F(x) - F(\eta)| \leq c \cdot 2^{-n} = c \cdot (3^{-n})^\gamma \leq M |x-\eta|^\gamma$.

$\therefore m_1 [0,1] \leq M^\gamma m_\gamma (C_{\frac{1}{3}})$.

ii) Rectifiable Curve:

Thm. Suppose γ is conti and quasi-simple. Then γ is rectifiable $\Leftrightarrow$ $I = \{\gamma(t) \mid t \in [a,b]\}$ has strict Hausdorff dimension one. $L(I) = m_1(I)$.

Pf: Consider arc-length parametrization $\tilde{\gamma}(s)$.

Then $\tilde{\gamma}(s)$ satisfies Lipschitz condition.

i.e. $|\tilde{\gamma}(s_1) - \tilde{\gamma}(s_2)| \leq |s_1 - s_2|$.

$\therefore m_1(I) \leq L(I)$

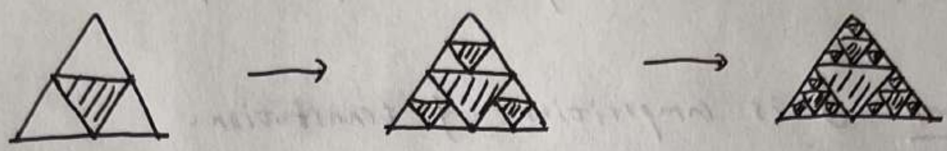
For the reverse:

Partition I with point $\{s_k\}$. $I = \cup I_k$.

Then $m_1(I) = \sum m_1(I_k) \geq \sum l_k \to L(I)$

where $l_k = |\gamma(s_k) - \gamma(s_{k+1})|$

iii) Sierpinsky Triangles:



S_1 S_2 S_3

· we begin from a closed equilateral triangle S_0. Then we remove the shaded triangle whose vertexs lie in the middle of laterals of S_0, obtain S_1. S_1 is the three closed equilateral triangles called the first generation.

Repeat the process on S_1. we obtain $S_2 \subseteq S_1$.

Then i) S_k is union of 3^k disjoint closed equilateral triangles of length 2^{-k}.

ii) S_k is cpt. $S_{k+1} \subseteq S_k$.

Let $S = \bigcap_0^\infty S_k$. cpt set.

Thm. S has strict Hausdroff dimension $\alpha = \log 3/\log 2$.

④ Self-similarity:

· Cantor set $C_{\frac{1}{3}}$, Sierphansky triangles S are the sets containing scales copies of itself.

① **Def:** map $S: \mathbb{R}^n \to \mathbb{R}^n$ is a similarity with ratio $r > 0$. if $|S(x)-S(y)| = r|x-y|$.

Remark: S is composition of translation. rotation and dilation by r.

For many similarities $S_1, S_2 \cdots S_m$ with same ratio r. $F \subseteq \mathbb{R}^n$ is said self-similar if $F = \bigcup_1^m S_k(F)$. e.g. $C_{\frac{1}{3}}$. S...

Thm. For m fixed similarities $\{S_k\}_1^m$ with same ratio r. $0<r<1$. Then exists unique nonempty cpt set F. st. $F = \bigcup_1^m S_k(F)$

② Dimension of self-similar:

Suppose $F = \bigcup_1^m S_k(F)$. $\{S_k(F)\}$ won't overlap. a.e.

Then $m_\alpha(F) = \sum_1^m m_\alpha(S_k(F)) = m r^\alpha m_\alpha(F)$

$\therefore m r^\alpha = 1$. $\alpha = \log m / \log \frac{1}{r}$. $0<r<1$.

The dimension may be $\log m / \log \frac{1}{r}$.

Def: $\{S_k\}_1^m$ are separated. if exists a bounded open O. st. $O \supseteq \bigcup_1^m S_k(O)$. disjoint union

Thm. $\{S_k\}_1^m$ are m separated similarities with common ratio $r \in (0,1)$. Then $F = \bigcup_1^m S_k(F)$ has Hausdorff dimension $\log m / \log \frac{1}{r}$