Abstract Measure

(1) <u>Origin</u> :

① A venerable problem in geometry is to determine the volum of a region $R \subseteq \mathbb{R}^n$

Ideally, we would like a function μ:

$\mu : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}^+}$. satisfies:

$E \subseteq \mathbb{R}^n \longmapsto \mu(E) \in \overline{\mathbb{R}^+}$

i) $\mu(\sum_1^\infty E_i) = \sum_1^\infty \mu(E_i)$. $\{E_i\}$ disjoint.

ii) If E is congruent to F. Then $\mu(E) = \mu(F)$

iii) $Q = \{0 \le X_i \le 1\}$. $\mu(Q) = 1$.

$\Rightarrow$ However, there are not consistent.

<u>e.g.</u> Vitali Set $\mathbb{R}/Q$

② One might consider to weaker is to finite

But it's not wise since we want the limit and continuity theory work smoothly.

$\Rightarrow$ We restrict μ on a kind of special sets

③ Generally, we may consider: $\mu : X \longrightarrow \overline{\mathbb{R}^+}$.

$E \longmapsto \mu(E) \in \mathbb{R}$.

E is a subset of X.

(2) Algebra:

① Elementary Family:

· **Def:** $\mathcal{E} \subseteq \mathcal{P}(X)$ is called elementary family.

if: i) $\emptyset \in \mathcal{E}$. ii) $E, F \in \mathcal{E} \Rightarrow E \cap F \in \mathcal{E}$.

iii) $E \in \mathcal{E} \Rightarrow E^c = \bigcup_1^n E_k$. $E_k \in \mathcal{E}$. disjoint.

prop. If $\mathcal{E}$ is an elementary family. Then the collection $\mathcal{A}$ of finite disjoint unions of members in $\mathcal{E}$ is an algebra.

pf: 1°) Check $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$.

since $A \cup B = A / B \cup B$.

2°) Check $\{A_k\}_1^m \subseteq \mathcal{E} \Rightarrow (\bigcup_1^m A_k)^c \in \mathcal{A}$.

② Product Form:

· For $\{X_\alpha\}_{\alpha \in A}$. consider $X = \prod_{\alpha \in A} X_\alpha$. Denote $\mathcal{M}_\alpha$ is a σ-algebra on X_α. We define the σ-algebra on X is $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha = \{ \pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{M}_\alpha \}$.

Remark: If A is countable. Then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by $\{ \prod_{\alpha \in A} E_\alpha \mid E_\alpha \in \mathcal{M}_\alpha \}$.

prop. Suppose $\mathcal{M}_\alpha$ is generated by $\mathcal{E}_\alpha \subseteq X_\alpha$.

Then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by $\{ \pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{E}_\alpha, \alpha \in A \}$. For A is countable. Then It's generated by $\{ \prod_{\alpha \in A} E_\alpha \mid E_\alpha \in \mathcal{E}_\alpha \}$.

Cor. (Borel Algebra)

$\{X_k\}_1^n$ is set of Metric Space. $X = \prod_1^n X_k$ equipped product metric. Then $\bigotimes_1^n B_{X_k} \subseteq B_X$. If X_j's are separable. Then $\bigotimes_1^n B_{X_k} = B_X$.

Pf: That's because we can find countable basis in X. since it's C_2.

(3) Measure:

· Def: X is a set equipped with σ-algebra M.

A measure μ is: $\mu: M \longrightarrow [\overline{\mathbb{R}}^+]$ st.

i) $\mu(\emptyset) = 0$. ii) $\mu(\sum_1^\infty E_n) = \sum_1^\infty \mu(E_n)$. $\{E_n\}$ disjoint

μ is σ-finite measure if $X = \bigcup_1^\infty E_n$. $\mu(E_n) < \infty$

$E \in M$ is σ-finite set if $E = \bigcup_1^\infty E_n$. $\mu(E_n) < \infty$

Remark: μ has monotone continuity.

· Def: A measure μ is complete if its domain M contains all subsets of μ-null sets.

Prop. For a measure space (X, M, μ). Denote $\mathcal{N} = \{N \in M \mid \mu(N) = 0\}$. $\overline{M} = \{E \cup F \mid E \in M$. and $F \subseteq N$ for some $N \in \mathcal{N}\}$. Then $\overline{M}$ is a σ-algebra. exists a unique extension $\overline{\mu}$ of μ. which is complete on $\overline{M}$.

Pf: Check by disjoint operation

Define : $\bar{\mu}(E \cup F) = \mu(E)$. check it's well-def.

and unique.

(4) Outer Measure:

① Def: An outer measure μ^* on set X is.

$\mu^*: P(X) \longrightarrow \overline{\mathbb{R}^+}$. s.t.

i) $\mu^*(\emptyset) = 0$ ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

iii) $\mu^*(\sum A_n) \leq \sum \mu^*(A_n)$

prop. (The way to obtain outer measure)

$\mathcal{E} \subseteq P(X)$. $\ell: \mathcal{E} \to \overline{\mathbb{R}^+}$. where $\emptyset, X \in \mathcal{E}$.

$\ell(\emptyset) = 0$. Then $\forall A \in P(X)$. define:

$\mu^*(A) = \inf \{ \sum_1^\infty \ell(E_n) \mid E_n \in \mathcal{E}, A \subseteq \bigcup_1^\infty E_n \}$.

Then μ^* is an outer measure

Remark: μ^* is generated from the family of elementary sets. with ℓ.

Pf: Check by definition.

Since $\mu^*(A) \overset{\epsilon}{\curvearrowright} \sum_1^\infty \ell(E_n)$, $\exists \{E_n\} \subseteq \mathcal{E}$.

Def: $A \in P(X)$ is μ^*-measurable, if for $\forall E \in P(X)$.

we have: $\mu^*(E) = \mu^*(A \cap E) + \mu^*(A^c \cap E)$

Remark: It means A behaves well which can separate the outer measure of arbitary set $E \in P(X)$.

② Carathéodory Thm:

If μ^* is outer measure on X. Then the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra. $\mu^*|_{\mathcal{M}}$ is complete.

Pf: 1°) First check $A, B \in \mathcal{M} \Rightarrow A \cup B \in \mathcal{M}$.

2°) For $\forall E \in P(X)$. Denote $B = \bigcup_1^\infty A_n$, $B_n = \bigcup_1^n A_k$

prove: $\mu^*(E \cap B_n) = \sum_1^n \mu^*(E \cap A_k)$. A_n disjoint.

$\therefore \mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \sum_1^n \mu^*(E \cap A_k) + \mu^*(E \cap B^c)$

3°) Check if $\mu^*(A) = 0$. Then A is μ^*-measurable set!

Remark: It can be applied in extending the measures from algebra to σ-algebra.

Def: $\mathcal{A} \subset P(X)$ is an algebra. $\mu_0 : \mathcal{A} \to \bar{\mathbb{R}}_+$ is called a premeasure if:

i) $\mu_0(\emptyset) = 0$

ii) $\mu_0(\sum_1^\infty A_j) = \sum_1^\infty \mu_0(A_j)$. if $\bigcup_1^\infty A_j \in \mathcal{A}$. A_j disjoint.

$\Rightarrow$ Using premeasure μ_0 and $\mathcal{A}$ to define an outer measure: $\forall E \in P(X)$.

$$\mu^*(E) = \inf \{ \sum_1^\infty \mu_0(A_j) \mid A_j \in \mathcal{A}. \ E \subseteq \bigcup_1^\infty A_n \}.$$

Prop. i) $\mu^*|_{\mathcal{A}} = \mu_0$

ii) $\forall E \in \mathcal{A}$. E is μ^*-measurable.

Pf: i) Check by definition: for $E \in \mathcal{A}$

We have: $\mu_0(E) \le \mu^*(E)$. $\mu_0(E) \ge \mu^*(E)$

ii) For $A \in \mathcal{A}$. $E \in P(X)$. $\mu^*(E) \overset{\varepsilon}{\sim} \sum \mu_0(B_n)$

$\therefore \mu^*(E) + \varepsilon \ge \sum \mu_0(B_n) = \sum \mu_0(B_n \cap A) + \sum \mu_0(B_n \cap A^c)$

$\ge \mu^*(E \cap A) + \mu^*(E \cap A^c)$. $\forall \varepsilon > 0$. Let $\varepsilon \to 0$.

Thm. $\mathcal{M}$ is the σ-algebra generated by $\mathcal{A}$. Then exists a measure μ. $\mu = \mu^*|_{\mathcal{M}}$. $\mu|_{\mathcal{A}} = \mu_0$. satisfying:

i) If ν is another measure extending μ_0 on $\mathcal{M}$. Then $\forall E \in \mathcal{M}$. $\nu(E) \le \mu(E)$

ii) If μ_0 is σ-finite. Then μ is the unique extension of μ_0 on $\mathcal{M}$.

Pf: Note firstly that the family of μ^*-measurable sets contains $\mathcal{A}$ so $\mathcal{M}$. $\therefore \mu^*|_{\mathcal{M}}$ is a measure.

i) $\forall E \in \mathcal{M}$. $E \subseteq \bigcup_1^\infty A_j$. $A_j \in \mathcal{A}$. Then

$\nu(E) \le \nu(\bigcup_1^\infty A_j) \le \sum_1^\infty \nu(A_j) = \sum_1^\infty \mu_0(A_j)$

$\therefore \nu(E) \le \mu^*|_{\mathcal{M}}(E) = \mu(E)$

ii) If $E \in \mathcal{M}$. $\mu_0(E) < \infty$. Then exist $A = \bigcup_1^\infty A_j$

$E \subseteq A$. $\mu(A/E) < \varepsilon$.

Note that: $\nu(A) = \lim_{n\to\infty} \nu(\bigcup_1^n A_k) = \lim_{n\to\infty} \mu(\bigcup_1^n A_k) = \mu(A)$

we can obtain $\mu(E) \le \nu(E) + \varepsilon$. $\forall \varepsilon > 0$

(5) Borel measure on $\mathbb{R}'$:

· Next, we're constructing theory of measuring $E \in P(\mathbb{R})$ base on its length

Suppose $\mathcal{A}$ is the collection of finite disjoint unions of h-intervals (i.e. $(a,b]$ or (a,∞) or $\mathbb{R}$, $-\infty \leq a < b < \infty$)

$\Rightarrow$ $\mathcal{A}$ is an algebra on $\mathbb{R}'$. since the family of h-intervals is elementary family.

Def: $B_{\mathbb{R}}$ is σ-algebra generated by $\mathcal{A}$.

prop. For $F: \mathbb{R}' \to \mathbb{R}'$. increasing, right-conti

$\mu_0 : \mathcal{A} \to \overline{\mathbb{R}^+}$. st. i) $\mu_0(\emptyset) = 0$.

ii) $\mu_0(\bigcup_1^n (a_i, b_i]) = \sum_1^n F(b_i) - F(a_i)$

Then $\mu_0|_{\mathcal{A}}$ is a premeasure.

Pf: 1°) Check μ_0 is well-def

2°) Firstly prove: for $I = (a,b] = \bigcup_1^\infty I_j$. union of h-intervals, disjoint

$\mu_0(I) \geq \mu_0(\bigcup_1^n I_i)$, $\forall n$. let $n \to \infty$.

3°) For the converse:

By partition and refinement

e.g. $(a,b] \xrightarrow[\text{by}]{\text{replaced}} [a+\delta, b]$. where

$F(a+\delta) - F(a) < \varepsilon$. by right-conti.

since all operations are "countable".

We can dominate $\mu_0(I)$ by finite ε.

For $a=-\infty$. Approxi. by $I-\mu.b]$.

4°) Consider finite disjoint union of I.

<u>Thm.</u> $F: \mathbb{R}' \to \mathbb{R}'$. any increasing. right-conti function

Then there exists a unique Borel measure μ_F. st.

$\mu_F(a,b] = F(b) - F(a)$. $\forall -\infty \le a < b < \infty$

i) If G is another such function, then

$\mu_F = \mu_G \Leftrightarrow F-G$ is const.

ii) Conversely. if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets

define: $F(x) = \begin{cases} \mu(0,x]. & x>0 \\ 0. & x=0 \\ -\mu(x,0]. & x<0 \end{cases}$. increasing

and right-conti. Besides $\mu = \mu_F$.

<u>Remark:</u> Increasing. Right-conti Functions $\xLeftrightarrow{\text{correspond}}$ Borel measures on $\mathbb{R}'$.

<u>Pf:</u> Note that $\mathbb{R}'$ is σ-finite.

So we obtain the uniqueness and existence from the procedure of constructing μ.

<u>Def:</u> $\overline{\mu}_F$ is completion of μ_F. Its domain is $\mathcal{M}_{\overline{\mu}_F}$, which contains $\mathcal{B}_{\mathbb{R}}$.

Remark: Note that $\forall E \in \mathcal{M}_\mu$:

$$\mu(E) = \inf \{ \sum F(b_i) - F(a_i) \mid E \subseteq \cup (a_i, b_i] \}$$
$$= \inf \{ \sum \mu(a_i, b_i] \mid E \subseteq \cup (a_i, b_i] \}$$

Lemma. $\forall E \in \mathcal{M}_\mu$, $\mu(E) = \inf \{ \sum \mu(a_i, b_i) \mid E \subseteq \cup (a_i, b_i) \}$

Pf: Note that $(.] \longrightarrow (.)$.

We can introduce one more index.

Thm. For $E \in \mathcal{M}_\mu$. Then

$$\mu(E) = \inf_{u \text{ open}} \{ \mu(u) \mid E \subseteq u \} = \sup_{k \text{ cpt}} \{ \mu(k) \mid k \subseteq E \}.$$

Pf: It's directly application of Lemma.

When E is unbounded. truncate it!

Cor. If $E \in \mathcal{M}_\mu$. Then $E = V / N_1 = H \cup N_2$.

where V is G_δ set. H is F_σ set. N_1. N_2 are μ-null sets.

prop. If $E \in \mathcal{M}_\mu$. $\mu(E) < \infty$. then $\forall \varepsilon > 0$

exist $A = \bigcup_1^n U_k$. union of open intervals

s.t. $\mu(E \Delta A) < \varepsilon$.

Remark: For $F(x) = x$. Denote $\mu_x = m$ is called

Lebesgue measure. Its domain is $\mathcal{L}$. It

measures the length of sets.

Integration

(1) Measurable functions:

· Measurable mappings are morphisms in the category of measurable spaces.

<u>Def</u>: $f: (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ is measurable function if $\forall E \in \mathcal{N}$. then $f^{-1}(E) \in \mathcal{M}$.

<u>Remark</u>: It has properties of morphism in category.

<u>prop</u>. (Criteria)

$\mathcal{N} = \mathcal{M}(\mathcal{E})$. then $f: X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable

$\Leftrightarrow \forall E \in \mathcal{E}.\ f^{-1}(E) \in \mathcal{M}$.

<u>Pf</u>: $\mathcal{N} = \{E \subseteq Y \mid f^{-1}(E) \in \mathcal{M}\} \supseteq \mathcal{E}$. σ-algebra.

<u>Cor</u>. X, Y topo/metric space. For $f: X \longrightarrow Y$ conti. Then f is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

<u>Remark</u>: We say "$f: X \to \mathbb{R}$ is $\mathcal{M}$-measurable" means f is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable.

<u>Prop</u>. $(X, \mathcal{M})$. $(Y_\alpha, \mathcal{N}_\alpha)$ $\alpha \in A$ are measurable spaces.

Suppose $Y = \prod_{\alpha \in A} Y_\alpha$. $\mathcal{N} = \bigotimes_{\alpha \in A} \mathcal{N}_\alpha$. $\pi_\alpha: Y \longrightarrow Y_\alpha$.

Then $f: X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable $\Leftrightarrow$

$f_\alpha = \pi_\alpha \circ f$ is $(\mathcal{M}, \mathcal{N}_\alpha)$-measurable.

<u>Thm.</u> (Approximation)

i) $f: (X, m) \to \overline{\mathbb{R}}^+$ measurable. Then there exists $\{\phi_n\}$ seq of simple functions. $0 \le \phi_1 \le \cdots \le \phi_n \cdots \le f$. $\phi_n \to f$ pointwise. Besides. $\phi_n \xrightarrow{u} f$ on any set $\subseteq \{f < \infty\}$.

ii) $f: (X, m) \to \mathbb{C}$. measurable. Then exists $\{\phi_n\}$ seq of simple functions. $0 \le |\phi_1| \le \cdots \le |\phi_n| \cdots \le |f|$. $\phi_n \to f$ pointwise. $\phi_n \xrightarrow{u} f$ on any set $\subseteq \{f < \infty\}$.

<u>Pf:</u> Partition its range $\{\frac{n}{2^k} \le f < \frac{n+1}{2^k}\}_{n \in \mathbb{Z}}$

<u>Prop.</u> i) f is measurable. $f = g$. μ-a.e. then g is measurable

ii) f_n is measurable. $\forall n \in \mathbb{N}$. $f_n \to f$ μ-a.e. then f is measurable

$\iff$ μ is complete measure.

<u>Pf:</u> ($\Leftarrow$) It's easy to check.

($\Rightarrow$) Note that $f^{-1}(-\infty, n] \,\Delta\, g^{-1}(-\infty, n] \uparrow \{f \ne g\}$.

$\bigcap_n f_n^{-1}(-\infty, m] \uparrow \{f_n \nrightarrow f\}$. $(m \to \infty)$

We can find subset of arbitrary μ-null set which won't measurable.

Let f. g. f_n are simple functions.

<u>prop.</u> For $(X, \overline{m}, \overline{\mu})$ is completion of (X, m, μ). If f is $\overline{m}$-measurable on X. Then exists m-measurable $g(x)$. st. $f = g$. $\overline{\mu}$-a.e.

Pf: $\exists\ \phi_n$, seq of $\bar{m}$-measurable functions

$\phi_n \to f$. Let $\psi_n = \phi_n$, except on E_n.

where $\bar{m}(E_n) = 0$. Let $E = \bigcup E_n$. $\bar{m}$-null

Then $\exists\ N$, m-null set: $N \supseteq E$.

$\therefore\ \psi_n \chi_{X/N} \to g = f$, $\bar{m}$-a.e.

(2) Integration of nonnegative:

Fix $(X, \mathcal{M}, m)$. Define L^+ = space of all measurable functions $: X \to \overline{\mathbb{R}}^+$

$\Rightarrow$ For simple function $\phi \in L^+$. $\phi = \sum_1^n a_i \chi_{E_i}$

Def: $\int_X \phi\, dm = \sum_1^n a_i\, m(E_i)$

For general $f \in L^+$. Def: $\int f\, dm = \sup_{\phi \text{ is simple}} \{ \int \phi\, dm \mid 0 \le \phi \le f \}$.

Thm (Monotone Convergence)

If $\{f_n\}$ seq $\subseteq L^+$. s.t. $f_n \le f_{n+1}$. $f = \lim_n f_n$. Then

$\int f\, dm = \lim_n \int f_n\, dm$.

Pf: By definition, return to the integral of simple Func.

$0 \le \phi \le f$. $E_n = \{ f_n > \alpha\phi \} \uparrow X$. $\alpha \in (0,1)$

$\int f_n\, dm \ge \int_{E_n} f_n\, dm > \alpha \int_{E_n} \phi\, dm$

check $\int_{E_n} \phi \uparrow \int_X \phi\, dm$ $\therefore \int f_n \ge \int f$.

The converse is trivial!

Fatou's Lemma.

$\{f_n\}$ seq $\subseteq L^+$. Then $\int \underline{\lim} f_n \leq \underline{\lim} \int f_n$

Pf: Apply Monotone Converge. Or by simple function.

(3) Integration of complex functions:

$$f = f^+ - f^-. \quad \int f \, d\mu = \int f^+ d\mu - \int f^- d\mu.$$

① Modes of Convergence:

· Def: i) $\{f_n\}$ measurable on $(X, \mathcal{M}, \mu)$. Cauchy in measure. if $\mu(\{|f_n - f_m| > \varepsilon\}) \to 0$. $(m, n \to \infty)$

ii) $\{f_n\}$ measurable on $(X, \mathcal{M}, \mu)$. $f_n \xrightarrow{a.un} f$. if $\forall \varepsilon > 0$. $\exists E \subseteq X$. s.t. $\mu(E) < \varepsilon$. $f_n \xrightarrow{u} f$ on E^c.

Remark: Cauchy in measure / a.e. / L^p

$\Leftrightarrow$ $f_n \to f$ in measure / a.e. / L^p

If X is complete metric space.

Relation:

$$\begin{tikzcd}
 & f_n \to f \text{ in } L^p \arrow[dl, "\exists\{f_{n_i}\}"'] \arrow[dr, shift left] & \\
f_n \to f \text{ a.e.} \arrow[rr, shift left, "\mu(X)<\infty"] \arrow[d, shift right, "\mu(X)<\infty"'] & & f_n \xrightarrow{m} f \arrow[ll, shift left, "\exists\{f_{n_i}\}"] \arrow[ul, shift left, "\mu(X)<\infty,\ \exists g \in L^p,\ g \geq |f_n|"] \\
f_n \to f \text{ a.un} \arrow[u, shift right] & &
\end{tikzcd}$$

Thm. $\{f_n\}$ Cauchy in measure. Then $\exists f$ is measurable st. $f_n \xrightarrow{m} f$ and $\exists \{f_{n_i}\} \longrightarrow f$. a.e. (Riesz)

moreover for $f_n \xrightarrow{m} g$. then $g = f$. a.e.

Pf: $\exists \{g_i\} = \{f_{n_i}\} \subseteq \{f_n\}$. st.

$m(\{|g_i - g_{i+1}| \geq \frac{1}{2^i}\}) \leq \frac{1}{2^i}$. Denote it by E_i

Then $\overline{\lim}\, E_i = F$ has m-measure zero.

$\lim g_i = \lim (\sum_2^i g_k - g_{k-1}) + g_1$ exists on F^c.

Set $f = \lim g_n$ on F^c Let $f = 0$ on F.

$\therefore f_{n_i} \longrightarrow f$. a.e. Check: $f_n \xrightarrow{m} f$

And $\{|g - f| \geq \varepsilon\} \subseteq \{|f_n - f| \geq \frac{\varepsilon}{2}\} \cup \{|f_n - g| \geq \frac{\varepsilon}{2}\}$

Thm.
i) $m(X) < \infty$. $f_n \longrightarrow f$. a.e $\Rightarrow f_n \longrightarrow f$. a.un.

ii) $f_n \longrightarrow f$. a.un $\Rightarrow f_n \longrightarrow f$ a.e.

Pf: i) is Egorov Thm

ii) If exist a set F. $m(F) > 0$

st. $f_n \not\longrightarrow f$. a.e on F.

But $\exists \delta > 0$. $0 < \delta < m(F)$. st.

$\exists E_\delta$. $m(E_\delta) = \delta$. $f_n \xrightarrow{u} f$ on F/E_δ

Contradict with $f_n \not\to f$. a.e. on F/E_δ

Remark: If $m(X) = \infty$. i) won't hold:

e.g. $X = [0, +\infty)$ with Lebesgue measure m.

$$f_n = \begin{cases} 1, & x \in [n, n+\frac{1}{n}] \\ 0, & \text{otherwise} \end{cases}$$

Thm. (Lusin)

For f measurable on $(X, \mathcal{M}, \mu)$. $\mu(\operatorname{supp} f) < \infty$. For $\forall \varepsilon > 0$

Then $\exists E$. s.t. $\mu(E) < \varepsilon$. s.t. $f|_{E^c}$ is conti.

Remark: The converse is true:

For $\forall \delta > 0$. $\exists F_\delta$ is closed set. s.t. $\mu(\operatorname{supp} f / F_\delta) < \delta$

$f|_{F_\delta}$ is conti. Then f is measurable.

Pf: $\forall k$. $\exists F_k$. s.t. $\mu(\operatorname{supp} f / F_k) \leq \frac{1}{2^k}$

Let $F = \bigcup\limits_{1}^{\infty} F_k$. $\therefore \mu(\operatorname{supp} f / F) \leq \frac{1}{2^n}$. $\forall n$.

$\therefore \mu(\operatorname{supp} f / F) = 0$. (Let $n \to \infty$.)

$\therefore \operatorname{supp} f \cap [f \geq a] = (\operatorname{supp} f / F \cap [f \geq a])$

$\cup \left(\bigcup\limits_{1}^{\infty} (F_n \cap [f \geq a])\right)$ is μ-measurable

(It means we only need to consider each conti part. Since $\mu(\operatorname{supp} f / F \cap [f \geq a]) = 0$)

② Dominated Convergence Thm:

$\{f_n\} \subseteq L^1(\mu)$. s.t. i) $f_n \to f$. a.e or in measure with $\mu(X) < \infty$

ii) $\exists g \in L^1(\mu)$. s.t. $|f_n| \leq g$. $\forall n$. a.e.

Then $f \in L^1(\mu)$. $\lim \int f_n \, d\mu = \int \lim f_n \, d\mu = \int f \, d\mu$.

Pf: For a.e. part. apply Fatou's Lemma on

$g + f_n$ and $g - f_n$ which 're not negative.

For converge in measure:

Otherwise, $\exists \{f_{n_j}\} \subseteq \{f_n\}$. s.t. $\int |f_{n_j} - f_{n_j}| \geq \varepsilon$.

But $f_{n_j} \xrightarrow{m} f$. $\therefore \exists \{f_{n_j}'\} \subseteq \{f_{n_j}\}$. s.t.

$f_{n_j}' \longrightarrow f$. a.e. which is a contradiction

<u>Cor.</u> $\exists \phi \in L^1(m)$. Simple function. $\phi \xrightarrow{L^1} f$.

where $f \in L^1(m)$.

$\exists g_n \in L^1(m) \cap C$. supports on bounded set

s.t. $g_n \xrightarrow{L^1} f$.

<u>Cor.</u> $f: X \times [a,b] \longrightarrow \mathbb{C}$. $f(\cdot, t)$ is integrable for each $t \in [a,b]$. $F(t) = \int_X f(x,t)\, d\mu$. Then.

i) If $\exists g \in L^1(m)$. s.t. $|f(x,t)| \leq g(x)$. $\forall x, t$

$\lim\limits_{t \to t_0} f(x,t) = f(x,t_0)$. $\forall x \in X$. Then $\lim\limits_{t \to t_0} F(t) = F(t_0)$

So if $f(x,t)$ is conti for $\forall x \in X$. then $F(t)$ is also conti

ii) If $\frac{\partial f}{\partial t}$ exists. $\exists g \in L^1(m)$. $|\frac{\partial f}{\partial t}| \leq g$. $\forall x, t$.

Then $F(t) \in C^1[a,b]$. $F'(t) = \int_X \frac{\partial f}{\partial t}(x,t)\, d\mu$.

③ <u>With Riemann Integral:</u>

f is bounded on $[a,b]$. Then.

i) f $\mathcal{R}$-integrable $\Rightarrow$ f $\mathcal{L}$-integrable. $\int_a^b f\, dx = \int_{[a,b]} f\, dm$.

ii) f is $\mathcal{R}$-integrable. $\Leftrightarrow$ $m(\{f \text{ isn't conti}\}) = 0$.

(4) <u>Product Measures</u>:

· <u>Def</u>: Rectangle in $X \times Y$ is the form: $A \times B$. $A \in X$. $B \in Y$. For $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ two measure space. The collection $\mathcal{A}$ of finite disjoint unions of $A \times B$. $A \in \mathcal{M}$. $B \in \mathcal{N}$. is an algebra $\Rightarrow$ it generate $\mathcal{M} \otimes \mathcal{N}$.

<u>Def</u>: $\mu \times \nu : \mathcal{A} \longrightarrow [0, +\infty]$ is a premeasure.

$\mu \times \nu (A \times B) = \sum_{1}^{n} \mu(A_i)\nu(B_i)$. if $A \times B = \bigcup_{1}^{n} A_i \times B_i$

Check it's well-def.

<u>Remark</u>: It can extend to $\bigotimes_{1}^{n} \mathcal{M}_k$. Define:

$$\prod_{1}^{n} \mu_k \left(\prod_{1}^{n} A_k \right) = \prod_{1}^{n} \mu_i(A_i)$$

<u>prop</u>. i) If $E \in \mathcal{M} \otimes \mathcal{N}$. Then $E_x \in \mathcal{N}$. $E_y \in \mathcal{M}$. for $\forall x, y$

ii) If f is $\mathcal{M} \otimes \mathcal{N}$-measurable. Then $f_x \cdot f^y$ is $\mathcal{M}$-measurable. $\mathcal{N}$-measurable. $\forall x, y$.

<u>Pf</u>. Denote $\mathcal{R}$ is the collection of E satisfies the condition. $\therefore$ $\mathcal{R}$ is σ-algebra. Besides. $\mathcal{R}$ contains all rectangles.

<u>Lemma</u>. (Monotone Class)

$\mathcal{A} \subseteq P(X)$ an algebra. Then $\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$

Pf: It suffices to prove: $C(A)$ is an algebra.

So $C(A)$ is a σ-algebra. $M(A) \subseteq C(A)$.

Fix $E \in C(A)$. Denote $C^E_{(A)} = \{F \in C(A) |$
$F/E.\ E/F.\ E \cap F \in C(A)\}$.

1°) $E \in A$. Then $\forall F \in A$. $F \in C^E(A)$. $\therefore C(A) = C^E_{(A)}$

Fix $F \in C(A)$ $\therefore F \in C^E(A)$. $\forall E \in A$. $\therefore E \in C^F_{(A)}$

2°) $E \in C(A)$. By 1°) $A \subseteq C^E(A)$. also.

since $C^E(A)$ is monotone class. $\therefore C^E(A) = C(A)$.

Thm. (X, M, μ). (Y, N, ν) are σ-finite. For $E \in M \otimes N$.

Then $\nu(E_x)$. $\mu(E^y)$ are measurable. $\forall x, y$. Besides,

$\mu \times \nu(E) = \int_X \nu(E_x) d\mu = \int_Y \mu(E^y) d\nu$

Pf: Denote $C = \{E \in M \otimes N \mid E$ satisfies the condition$\}$.

A the collection of finite disjoint unions of rectangles $\subseteq C$

Check C is monotone class by mono Converge Thm.

Thm. (Fubini - Tonelli)

(X, M, μ) and (Y, N, ν) are σ-finite measure space.

i) If $f \in L^+(\mu \times \nu)$. Then $\int f_x d\nu \in L^+(\mu)$. $\int f^y \in L^+(\nu)$.

Besides. $\int f d\mu \times \nu = \int [\int f d\nu] d\mu = \int (\int f d\mu) d\nu$.

ii) If $f \in L^1(\mu \times \nu)$. Then $f_x \in L^1(\nu)$. a.e. x. $f^y \in L^1(\mu)$.

a.e. y. $\int f_x d\nu \in L^1(\mu)$. $\int f^y d\mu \in L^1(\nu)$.

Besides. $\int f d\mu \times \nu = \int (\int f d\nu) d\mu = \int (\int f d\mu) d\nu$.

Pf: Approxi. by nonnegative simples function. for its f^+. f^-. use monotone converge.

Thm. (Complete Form)

$(X, \mathcal{M}, \mu)$. $(Y, \mathcal{N}, \nu)$ are σ-finite. complete. space.

Let $(X \times Y, \mathcal{L}, \lambda)$ is completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$

If f is $\mathcal{L}$-measurable. Then f_x is $\mathcal{N}$-measurable for a.e. x. f^y is $\mathcal{M}$-measurable for. a.e. y. $\int f_x$ is $\mathcal{M}$-measurable. $\int f^y d\mu$ is $\mathcal{N}$-measurable

If $f \in L^1(\lambda)$. Then f_x. $f^y \in L^1(\nu)$. $L^1(\mu)$ for a.e. x and y.

$\int f_x d\nu \in L^1(\mu)$. $\int f^y d\mu \in L^1(\nu)$. Besides.

$\int f d\lambda = \int (\int f d\mu) d\nu = \int (\int f d\nu) d\mu$.

Remark: $f \in \mathcal{M} \otimes \mathcal{N}$-measurable $\Rightarrow$ $f \in \mathcal{L}$-measurable

$\Downarrow$ $\Downarrow$

f^x. f_y are $\mathcal{M}$-measurable $\mathcal{N}$-measurable. $\forall x, y$

f_x. f^y are $\mathcal{M}$. $\mathcal{N}$-measurable. a.e. x and y.

consider $f = \chi_{N \times \{0\}}$. N is Vitali set.

(5) n-Dimensional Lebesgue Integration:

· Lebesgue measure m^n on $\mathbb{R}^n$ is the completion of $\prod_1^n m$ on $\bigotimes_1^n \mathcal{B}_{\mathbb{R}}$. Denote it simply by m.

And its domain is $\mathcal{L}^n$.

① Jordan Content:

We consider another method of measure comparing to Lebesgue measure:

1°) $\forall k \in \mathbb{Z}$. Let $\mathcal{Q}_k$ is the collection of cubes with length 2^{-k} and vertices are on $(2^{-k}\mathbb{Z})^n$.

2°) For $E \subseteq \mathbb{R}^n$. $\underline{A}(E,k) = \bigcup\limits_{\substack{Q_r \in \mathcal{Q}_k \\ Q_r \subseteq E}} \{Q_r\}$, $\bar{A}(E,k) = \bigcup\limits_{\substack{Q_r \in \mathcal{Q}_k \\ Q_r \cap E \neq \emptyset}} \{Q_r\}$ → Finite Union!

Then $m(\underline{A}(E,k)) \uparrow$. $m(\bar{A}(E,k)) \downarrow$. limits exist.

Denote: $\underline{k}(E) = \lim\limits_{k \to \infty} m(\underline{A}(E,k))$. $\bar{k}(E) = \lim\limits_{k \to \infty} m(\bar{A}(E,k))$

called inner content and outer content.

If $\underline{k}(E) = \bar{k}(E)$. Then denote it by $k(E)$. Jordan content.

3°) If $k(E)$ exists. Then $E \in \mathcal{L}$. $m(E) = k(E)$

Actually $\underline{k}(E) = m(\underline{A}(E))$. $\bar{k}(E) = m(\bar{A}(E))$.

$\underline{A}(E) = \bigcup\limits_k \underline{A}(E,k)$. $\bar{A}(E) = \bigcap\limits_k \bar{A}(E,k)$. $\underline{A}(E) \subseteq E \subseteq \bar{A}(E)$

Jordan content exists $\Leftrightarrow$ $m(\bar{A}(E)/\underline{A}(E)) = 0$.

$\therefore E \in \mathcal{L}$ and $m(E) = k(E)$.

Prop. i) U is open $\subseteq \mathbb{R}^n$. Then $\underline{A}(U) = U$. U is countable disjoint union of interiors of cubes.

ii) K is cpt $\subseteq \mathbb{R}^n$. Then $\bar{A}(K) = K$

Remark: i) Actually the collection K of sets where Jordan content exists isn't an σ-algebra. It's not a true measure.

ii) Jordan inner/outer content is approxi. by finite cubes, but Lebesgue's is by countable cubes. There will cause a huge difference. e.g. $\bar{k}(E) = \bar{k}(\bar{E})$.

So for $\mathbb{Q}$. $\bar{k}(\mathbb{Q}) = 1$. But $m^*(\mathbb{Q}) = 0$.

since $\underline{k}(\mathbb{Q}) = 0$. $\therefore$ $\mathbb{Q}$ has no Jordan content.

But $\mathbb{Q} \in \mathcal{L}$.

② Transformation:

Thm. $T \in GL(n, \mathbb{R})$

i) If f is $\mathcal{L}$-measurable on $\mathbb{R}^n$. Then so $f \circ T$. for $f \in L^1(m)$. we have:

$\int f(x)\, dx = |T| \int f \circ T(x)\, dm$.

ii) For $E \in \mathcal{L}^n$, $m(E) = |T|^{-1} m(T(E))$

Pf: Consider from simple functions and when T is one of three elementary linear transfer.

Thm. $\Omega \underset{\text{open}}{\subseteq} \mathbb{R}^n$. $G: \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism.

i) If f is $\mathcal{L}$-measurable on $G(\Omega)$. Then $f \circ G$ is $\mathcal{L}$-measurable on Ω.

ii) If $f \in L^1(G(\Omega), m)$. Then

$$\int_{G(\Omega)} f\,dx = \int_\Omega f\circ G\,|D_x G|\,dx.$$ in particular.

$$m(G(E)) = \int_E |D_x G|\,dx.$$ when $f = \chi_E$.

Pf: Approxi. from: cubes $\Rightarrow$ open sets $\Rightarrow$

Bound Borel sets $\Rightarrow$ Borel sets $\Rightarrow$

Simple function (Lebesgue measurable)

$\Rightarrow$ general $L^1(m)$ functions.

③ Integration in Polar Coordinates:

Denote: $S^{n-1} = \{\,|x|=1 \mid x \in \mathbb{R}^n\}$. Choose ϕ the transform:

$$\phi: x \longmapsto (|x|, \frac{x}{|x|}) \in (0,\infty)\times S^{n-1}.$$

Thm. There exists a unique Borel measure $\sigma = \sigma_{n-1}$ on S^{n-1}. st. $m_* = \ell \times \sigma$ on $(0,+\infty)\times S^{n-1}$ where $\ell = \ell_n$. satisfies: $\ell(E) = \int_E r^{n-1}\,dr$

If $f \in L^1(m)$. Then $\int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty r^{n-1}\int_{S^{n-1}} f(rx')\,d\sigma(x')\,dr$

Pf: Test by simple functions. Denote $E_n = \phi^{-1}((0,n]\times E)$

The equation holds. It suffices to find σ.

Let $f = \chi_{E_1}$ $\therefore$ $\sigma(E) = n\,m(E_1)$ is a Borel measure.

For m^*: $m^*((a,b]\times E) = \frac{b^n - a^n}{n}\sigma(E)$.

By Caratheodory Thm. $M_E = \{(a,b]\times E \mid a,b \in \mathbb{R}\} = B_{\mathbb{R}^+}\times E$.

$\therefore$ $\bigcup M_E$. $\forall E \in B_{S^{n-1}}$. generates $B_{\mathbb{R}^+}\times B_{S^{n-1}}$

extend m^* on it! Unique is from σ-finite

Differentiation Theory

(1) Sign Measure:

· **Def**: ν is a sign measure on $(X, \mathcal{M})$. if

i) $\nu(\emptyset)=0$. ii) $\nu: \mathcal{M} \longrightarrow [-\infty, +\infty]$. attains at most one of $\pm\infty$.

iii) $\{E_i\}$ disjoint in $\mathcal{M}$. $\nu(\bigcup_1^\infty E_i) = \sum \nu(E_i)$. where the latter sum converge absolutely if $\nu(\bigcup_1^\infty E_i)<\infty$.

Def: A set $E \in \mathcal{M}$ is positive for ν. if $\forall F \subseteq E$. $F \in \mathcal{M}$. $\nu(F) \geq 0$. respective def for negative/null.

Lemma. i) Sign measure ν on $(X, \mathcal{M})$ satisfies the monotone Continuity Thm as usual.

ii) Positive / Negative / null sets are closed under intersection and countable union.

Pf: For i) is same as usual

For ii). $\bigcup P_n = \bigcup (P_n - \bigcup_1^{n-1} P_k) \triangleq \bigcup Q_n$.

Thm. (Hahn - Decomposition)

If ν is sign measure on $(X, \mathcal{M})$. Then exists a positive set P and negative set N. st. $N \cap P = \emptyset$. $N \cup P = X$. If P'. N' is another pair. Then. $P' \Delta P$. $N' \Delta N$ are ν-null sets.

Pf: We want to obtain the "maximal" positive set. Denote $m = \sup_{E \text{ is positive}} \nu(E) < \infty$. (WLOG)

Then $\exists \{P_n\} \longrightarrow m$.

Let $P = \bigcup_1^\infty P_n$. $\therefore \nu(P) = m$

Claim $N = X/P$ is negative.

By contradiction: N can't contain positive set.

Then $\exists A_1 \subseteq N$. $\nu(A_1) > 0$. $\exists A_2 \subseteq A_1$. $\nu(A_2) > \nu(A_1)$.

$\cdots$ $\exists \{A_n\}$. $A_n \subseteq A_{n-1}$. $\nu(A_n) > \nu(A_{n-1}) \cdots > \nu(A_1)$

Since each A_n isn't positive set.

Choose for each j: n_j is the least integer. s.t.

$\exists A_{j+1} \subseteq A_j$ $\nu(A_{j+1}) > \nu(A_j) + n_j^{-1}$.

Consider $A = \bigcap_1^\infty A_n$. $\exists B \subseteq A$. $\nu(B) > \nu(A) + k^{-1}$.

Def: μ, ν are two signed measure on $(X, \mathcal{M})$. they're mutually singular. if $\exists E, F \in \mathcal{M}$. $E \cup F = X$. $E \cap F = \emptyset$. $\mu(E) = \nu(F) = 0$. Denote it by $\mu \perp \nu$.

Thm. (Jordan Decomposition)

If ν is a signed measure. Then there exists unique positive measures ν^+, ν^- st. $\nu = \nu^+ - \nu^-$. $\nu^+ \perp \nu^-$.

Pf: Def: $\begin{cases} \nu^+(E) = \nu(E \cap P) \\ \nu^-(E) = -\nu(E \cap N) \end{cases}$ $N \cup P$ is Hahn decomposition w.r.t ν.

Remark: Total variation of ν: $|\nu| = \nu^+ + \nu^-$

Def: $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-) \Rightarrow L^1(\nu) = L^1(|\nu|)$

(2) The Lebesgue - Radon - Nikodym Thm :

① Def: ν is signed measure. μ is positive measure on (X, m). ν is absolutely conti w.r.t μ. if. $\nu(E)=0$ whenever $\mu(E)=0$. Denote $\nu \ll \mu$.

Remark: $\nu \ll \mu \Leftrightarrow |\nu| \ll \mu \Leftrightarrow \nu^+ \ll \mu. \nu^- \ll \mu$.

Pf: ($\Rightarrow$) For $\mu(E)=0$ $\therefore \mu(E \cap N)=\mu(E \cap P)=0$.

Thm. ν is finite signed. μ is positive measure on (X, m).

Then $\nu \ll \mu \Leftrightarrow \forall \varepsilon>0. \exists \delta>0$. St. $|\nu(E)|<\varepsilon$. whenever $\mu(E)<\delta$.

Pf: By contradiction. $\exists \varepsilon_0$. $E_n \in m$. st. $\mu(E_n)<2^{-n}$. $\nu(E_n) \geqslant \varepsilon_0>0$. Consider $\overline{\lim} E_n=F$.

Remark: Note $\nu(E)=\int_E f d\mu$. $\nu \ll \mu$.

We may express $\nu \ll \mu$ by $d\nu=f d\mu$.

Lemma. (Relation of two measures)

$\nu. \mu$ are two finite measures on $(X. m)$.

Then i) $\nu \perp \mu$

ii) $\exists \varepsilon>0$. $E \in m$. st. $\mu(E)>0$. $\nu>\varepsilon \mu$ on E

i) or ii) holds.

Pf: Consider signed measure : $\nu - \frac{\mu}{r}$.

with Hahn-Decomposition $X = P_n \cup N_n$.

Let $P = \cup P_n$. $N = \cap N_n$. We have: $\nu(N) = 0$.

consider how μ acts on P.

Thm. (Lebesgue - Radon - Nikodym)

ν is σ-finite signed, μ is σ-finite positive measure on $(X, \mathcal{M})$. Then there exists unique σ-finite signed measure λ, ρ on $(X, \mathcal{M})$. st.

$\lambda \perp \mu$. $\rho \ll \mu$. $\nu = \lambda + \rho$. and exists μ-a.e unique $f \in L(\mu)$. $d\rho = f d\mu$.

Pf: Reduce "ν is signed σ-finite" to "ν is positive finite."

by $\nu = \nu^+ - \nu^-$. $X = \cup A_n$. Separate A_n, $\nu^\pm$.

$\therefore$ Suppose ν, μ are positive, finite.

The ideal is:

Find the Largest part of form $f d\mu$. which is dominated by $\nu(\cdot)$. Then $d\rho = f d\mu$. substitute $d\nu$ by $d\rho$ is the orthogonal part: $d\nu - f d\mu$.

Let $\mathcal{F} = \{ f: X \to \overline{\mathbb{R}^+} \mid \int_E f d\mu \leq \nu(E).\ \forall E \in \mathcal{M} \}$.

1) check $0 \in \mathcal{F}$. $f, g \in \mathcal{F} \Rightarrow \max\{f, g\} \in \mathcal{F}$.

consider $\{ f(x) - g(x) > 0 \} \in \mathcal{M}$.

2) Suppose $a = \sup_{f \in \mathcal{F}} \int f d\mu$. $a \leq \nu(X) < \infty$.

$\exists f_n$. st $\int_X f_n d\mu \to a$.

To guarantee the convergence. Let $g_n = \max\{f_1 \cdots f_n\}$

$\lim_n g_n = g$ exists. By MCT. $\int_X g d\mu = a$.

3°) Check $d\lambda = d\nu - g\, d\mu$ is singular w.r.t $d\mu$

Apply Lemma on λ, μ.

4°) Uniqueness is from: $\lambda \ll \mu$, $\lambda \perp \mu \Rightarrow \lambda = 0$.

Remark: f is called the Radon-Nikodym derivate of ν w.r.t μ if $d\nu = f d\mu$

Denote $f \triangleq \frac{d\nu}{d\mu}$.

③ Properties:

prop. ν is σ-finite sign. μ and λ are σ-finite on $(X, \mathcal{M})$. st. $\nu \ll \mu \ll \lambda$. Then.

i) If $g \in L^1(\nu)$. Then $g \frac{d\nu}{d\mu} \in L(\mu)$. and

$$\int g\, d\nu = \int g \frac{d\nu}{d\mu} d\mu.$$

ii) We have: $\nu \ll \lambda$. and $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda}$. λ. a.e.

Pf. Consider ν^+, ν^- separately by $X = P \cup N$. restrict on N or P.

i) Test by simple functions χ_E. Then by MCT.

Linearity: $\frac{d(\nu_1+\nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}$. μ.a.e is from:

$d(\nu_1+\nu_2) = \frac{d(\nu_1+\nu_2)}{d\mu} d\mu = d\nu_1 + d\nu_2 = (\frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}) d\mu$.

Consider $E_+ = \{ \frac{d(\nu_1+\nu_2)}{d\mu} > \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu} \}$. is μ-null set!

ii) Let $g = \chi_E \frac{d\nu}{d\mu}$. $\therefore \int g\, d\mu = \int g \frac{d\mu}{d\lambda} d\lambda$.

Cor. $\mu \ll \lambda$, $\lambda \ll \mu$. Then $\frac{d\lambda}{d\mu} \frac{d\mu}{d\lambda} = 1$. a.e.

prop. (Product Case)

μ_1, ν_1 σ-finite on (X_1, m_1). μ_2, ν_2 σ-finite on (X_2, m_2). If $\nu_1 \ll \mu_1$, $\nu_2 \ll \mu_2$. Then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$.

and $\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2)$

Pf: Let $M = \{E \in m_1 \otimes m_2 \mid \text{If } \mu_1 \times \mu_2(E) = 0 \text{ Then } \nu_1 \times \nu_2(E) = 0\}$

Then A the collection of finite disjoint rectangles.

$A \subseteq M$. Check M is σ-algebra. $\therefore M = m_1 \otimes m_2$.

Since $d(\nu_1 \times \nu_2) = \frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) \, d\mu_1 \times \mu_2$.

$= d\nu_1 \, d\nu_2 = \frac{d\nu_1}{d\mu_1}(x_1) \, d\mu_1 \, \frac{d\nu_2}{d\mu_2}(x_2) \, d\mu_2 = \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2) \, d\mu_1 \times \mu_2$.

$\therefore \frac{d\nu_1 \times \nu_2}{d\mu_1 \times \mu_2}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2)$. a.e. The last equation by Fubini

Cor. It can be extend to $d\mu_1 \times \cdots \mu_n$!

Prop. (Complex case)

ν is complex measure on (X, m). i.e. $\nu : m \longrightarrow \mathbb{C}$.

(Lebesgue - Radon - Nikodym still holds for its imaginary and real parts separately!)

i) $|\nu(E)| \le |\nu|(E)$, $\forall E \in m$.

ii) $\nu \ll |\nu|$, $\left|\frac{d\nu}{d|\nu|}\right| = 1$. $|\nu|$-a.e.

iii) $f \in L^1(\nu)$. Then $\left|\int f d\nu\right| \le \int |f| \, d|\nu|$

Cor. $|\nu_1 + \nu_2(E)| \le |\nu_1|(E) + |\nu_2|(E)$. $\forall E \in m$.

Pf: Consider the form: $d\nu = f d\mu$.

Use $d\mu$ as an intermedium.

③ Lebesgue Differentiation

Theory on Nikodym Derivates:

- m is Lebesgue measure on $\mathbb{R}^n$
- ν is signed, finite on cpt set, outer-regular Borel measure on $\mathbb{R}^n$. $\quad \Big| \; f \in L^1_{loc}(m)$

<u>Thm.</u> Suppose $d\nu = d\lambda + f\,dm$ is its L-R-N representation. Then $\lim\limits_{r\to 0} \dfrac{\nu(E_r)}{m(E_r)} = f$. m-a.e. x

where $\{E_r\}_{r>0}$ is shrinking regularly to x.

<u>Pf:</u> $\dfrac{1}{m(E_r)} \int_{E_r} d\nu = \dfrac{\nu(E_r)}{m(E_r)} = \int_{E_r} d\lambda / m(E_r)$

$+ \dfrac{1}{m(E_r)} \int_{E_r} f\,dm = \dfrac{\lambda(E_r)}{m(E_r)} + \dfrac{1}{m(E_r)} \int_{E_r} f\,dm$.

By LDT. $\dfrac{1}{m(E_r)} \int_{E_r} f\,dm \to f(x)$. m-a.e. x.

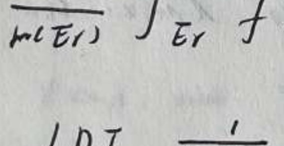
It suffices to prove: $\dfrac{\lambda(E_r)}{m(E_r)} \to 0$. a.e. x. $r \to 0$

Since $\lambda \perp m$. Let A be: $\lambda(A) = m(A^c) = 0$.

$F_k = \{x \in A \mid \overline{\lim\limits_{r\to 0}} \dfrac{\lambda(B(r,x))}{m(B(r,x))} > \frac{1}{k}\}$. Show $m(F_k) = 0$. $\forall k$.

1°) λ is also regular, by $d|\nu| = d|\lambda| + |f|\,dm$

2°)



Let $U_\varepsilon \supseteq A$. s.t. $\lambda(U_\varepsilon) < \varepsilon$.

$V_\varepsilon = \bigcup B_x = \bigcup \{B_x \mid \dfrac{\lambda(B_x)}{m(B_x)} > \frac{1}{k}\}$

cover F_k.

By Vitali Thm: $\forall c < m(F_k)$

$c \leq 3^n \sum_1^r m(B_x) \leq 3^n k \sum_1^r \lambda(B_x) \leq 3^n k\, \lambda(U_\varepsilon)$

$\therefore m(F_k) \leq 3^n k \lambda \varepsilon$. $\forall \varepsilon > 0$ $\therefore \dfrac{d\lambda}{dm} = 0$. a.e. x.

(3) Bounded Variation:

① Thm (Correspondence)

If μ is complex. Borel on $\mathbb{R}$. Then $F(x) = \mu(-\infty, x]$

$\in NBV = \{F \in BV \mid F(-\infty) = 0,\ F \text{ is right-conti}\}$.

Conversely. for $F \in NBV$. there exists unique Borel

measure μ_F. st. $F(x) = \mu_F(-\infty, x]$. $|\mu_F| = \mu_{T_F}$

Pf: Easy to check by separating Re. Im. $\pm$ part.

② Correspondence with decomposition of measures:

prop. If $F \in NBV$. Then $F' \in L^1(m)$.

i) $\mu_F \perp m \iff F' = 0$, m-a.e.

ii) $\mu_F \ll m \iff F = \int_{-\infty}^{x} F'(t)dt \iff F \in AC(\mathbb{R}')$

Pf: Since F is d.f. of μ_F $\therefore$ μ_F is regular.

$\therefore F'(x) = \lim_{r \to 0} \frac{\mu_F(E_r)}{m(E_r)}$ exists for a.e. x.

where $E_r = (x, x+r]$ or $(x-r, x]$.

For ii) $\mu_F \ll m \iff F \in AC(\mathbb{R}')$:

($\Rightarrow$) By "ε-δ" def of AC of two measures.

($\Leftarrow$) If $m(E) = 0$. $\exists U_n$. union of disjoint open intervals. $U_1 \supset U_2 \cdots \supseteq E$. $m(U_1) < \delta$.

$\mu_F(U_n) \to \mu_F(E)$.

$\therefore |\mu_F(E)| \le \Sigma |\mu_F(a_j^n, b_j^n)| \le \Sigma |F(b_j^n) - F(a_j^n)| < \varepsilon$.

Let $\varepsilon \to 0$. $\therefore |\mu_F(E)| = 0$.

Thm: (Fundamental Thm for Lebesgue Integrals)

For $a<b \in \mathbb{R}$. $F:[a,b] \to \mathbb{C}$. Then the following are equivalent:

i) $F \in AC[a,b]$

ii) $F(x) = \int_a^x f(t)\,dm + F(a)$ for some $f \in L^1([a,b],m)$

iii) F' exists a.e. on $[a,b]$. $F' \in L^1([a,b],m)$ and

$F(x) - F(a) = \int_a^x F'(t)\,dt$.

Pf: i) $\Rightarrow$ iii) Let $G = F - F(a)$ and $F(x) = F(b)$. $\forall x > b$.

Then $G \in NBV$.

From a Lemma. follows from above:

If $f \in L^1(m)$. Then $F = \int_{-\infty}^x f\,dt \in NBV \cap AC[a,b]$. $F' = f$. a.e.

If $F \in NBV \cap AC[a,b]$. Then $F' \in L^1(m)$. $F = \int_{-\infty}^x F'(t)\,dt$.

iii) $\Rightarrow$ ii) is trival. ii) $\Rightarrow$ i) Let $f(t) = 0$. $\forall t \notin [a,b]$.

③ Decomposition of measure:

· For complex Borel μ. $\mu = \mu_d + \mu_c$. where μ_d is discrete part. μ_c is conti measure (has no atoms)

$\mu_c = \mu_{ac} + \mu_{sc}$. μ_{ac} is AC w.r.t m. $\mu_{sc} \perp m$. From Nikodym Decompose.

$\Rightarrow \mu = \mu_d + \mu_{ac} + \mu_{sc} \xrightarrow{\text{Correspond}} F(x) = F_J + \int_{-\infty}^x f(t)\,dt + F_c - \int_{-\infty}^x f(t)\,dt$.

④ Integration by Part:

$F, G \in NBV$. at least one of them is conti. Then $[a,b] \subseteq \mathbb{R}$.

$$\int_{[a,b]} F\,dG + \int_{[a,b]} G\,dF = F(b)G(b) - F(a)G(a)$$

Pf: By Fubini on $\mu_F \times \mu_G(\{a<x\le y\le b\})$