Lebesgue Measure

(1) Pre:

① An example: Cantor Set:

· Note firstly that:

i) $U \underset{open}{\subseteq} \mathbb{R}^1$: $U = \bigcup_1^\infty I_n$. disjoint interval

ii) $U \underset{open}{\subseteq} \mathbb{R}^d$. $d > 1$: $U = \bigcup_1^\infty Q_j$. almost disjoint cubes

The expression in the first case is unique. But in the last case it's not unique!

· Def: Cantor set C_ξ $(0<\xi<1)$: It means remove the central interval of length ξ. Denote C_ξ^k is the k^{th} stages of the operation:

Then: $C_\xi = \bigcap_{k=1}^{\infty} C_\xi^k$. $m(\bigcap_1^n C_\xi^k) = (1-\xi)^n \to 0$.

$\therefore m(C_\xi) = 0$.

properties:

i) $|C_\xi| = 2^{\aleph}$

ii) C_ξ is totally disconnected (Nowhere is dense)

iii) C_ξ is cpt. No isolated point exists!

iv) C_ξ is perfect set $([0,1]/C_\xi = \bigcup I_n)$

Remark: A more general case:

Cantor-like Set $\hat{C}$: At k^{th} stage. we will remove 2^{k-1} central intervals of length l_k in $\hat{C}_{k-1}$.

Then $\hat{C} = \bigcap_{1}^{\infty} \hat{C}_n$ satisfies prop. i) - iv).

Note that $m(\hat{C}) = 1 - \sum_{k=1}^{\infty} 2^{k-1} l_k$

for l_k is small enough. $m(\hat{C})$ may be > 0!

Cantor-Lebesgue Func:

· Take $C_{\frac{1}{3}}$ for example:

$\forall x \in C_{\frac{1}{3}}$. x has unique representation:

$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, $a_n = 0$ or 2. (Because: $\frac{1}{3^k} = \sum_{k+1}^{\infty} \frac{2}{3^i}$)

$F: C_{\frac{1}{3}} \longrightarrow [0,1]$. It's called the

$\sum_{n=1}^{\infty} \frac{a_n}{3^n} \longmapsto \sum_{n=1}^{\infty} \frac{b_n}{2^n}$. $b_n = \frac{a_n}{2}$ Cantor-Lebesgue Func.

F is conti. surjective. Let $G = \sup_{\eta \le x} F(\eta)$

Then $G: [0,1] \longrightarrow [0,1]$. extend on $[0,1]$

$G(x)$ is monotone. conti.

(2) Exterior Measure:

· Def: $\forall E \subseteq \mathbb{R}^d$. $m^*(E) = \inf\{\sum_1^\infty |Q_n| \mid E \subseteq \bigcup_1^\infty Q_n$

. Q_n are disjoint (almost) closed cubes$\}$

Remark: "Countably infinite" is necessary!

properties:

① Mono: $E_1 \subseteq E_2 \Rightarrow m^*(E_1) \leq m^*(E_2)$

② $m^*(\bigcup_1^\infty E_n) \leq \sum_1^\infty m^*(E_n)$

③ If $d(E_1, E_2) > 0$. Then $m^*(E_1) + m^*(E_2) = m^*(E_1 \cup E_2)$

(3) Lebesgue Measurable:

· E is $\mathcal{L}$-measurable $\Leftrightarrow \forall \varepsilon > 0.\ \exists \mathcal{O} \underset{open}{\subseteq} \mathbb{R}^d.\ E \subseteq \mathcal{O}$.

s.t. $m^*(\mathcal{O} - E) < \varepsilon$. Denote $m^*(E) = m(E)$.

① Properties: i) Open / closed sets are measurable

ii) $m^*(E) = 0 \Rightarrow E$ is measurable

iii) E is measurable $\Rightarrow E^c$ is measurable

iv) $\bigcup_1^\infty E_n$ is measurable if E_n's are measurable

v) For $\{E_n\}_1^\infty$. $E_i \cap E_j = \emptyset$. $i \neq j$. Then:

$m(\bigcup E_n) = \sum m(E_n)$

vi) $E_n \nearrow E$. $m(E) = \lim_n m(E_n)$

$E_n \searrow E$. $\exists N$. s.t. $m(E_N) < \infty$. $m(E) = \lim_n m(E_n)$

Thm. i) If E is $\mathcal{L}$-measurable Then exists $O \underset{open}{\subseteq} \mathbb{R}^d$

$F \underset{close}{\subseteq} \mathbb{R}^d$. s.t. $F \subseteq E \subseteq O$. $m(O/F) \sim \varepsilon$.

ii) In case i). If $m(E) < \infty$. Then exists cpt set K. s.t. $m(E/K) \sim \varepsilon$. $K \subseteq E$. And it exists $F = \bigcup_{1}^{n} Q_j$. closed cubes. $m(E \Delta F) \sim \varepsilon$.

Cor. From i). there exist G_δ set: $\bigcap O_n$. open.

$E \subseteq O_n \therefore \bigcap O_n \supseteq E$. s.t. $m(\bigcap O_n / E) = 0$.

and F_σ set: $\bigcup K_n$. closed. $E \supseteq F_n$. $\therefore E \supseteq \bigcup F_n$

s.t. $m(E/\bigcup K_n) = 0$.

② Techniques for check

$\mathcal{L}$-measurable:

i) Prop. If $m^*(A \Delta B) = 0$ Then $m^*(A) = m^*(B)$

Therefore. A is measurable $\Leftrightarrow$ B is measurable.

ii) Prop. $A \subseteq E \subseteq B$. $m(A). m(B) < \infty$ If $m(A) = m(B)$

Then E is measurable

iii) Prop. $m(E) < \infty$. $E = E_1 \cup E_2$ If $m(E) = m^*(E_1) + m^*(E_2)$

Then E_1, E_2 are measurable.

Pf: Only prove iii)

i). ii) are trivial.

Lemma. A is $\mathcal{L}$-measurable. Then for $\forall B \subseteq \mathbb{R}^n$.

We have: $m(A) + m^*(B) = m^*(A \cup B) + m^*(A \cap B)$

Pf: Apply Caratheodory Thm. on B. $A \cup B$.

It can be extended to general measure.

$\Rightarrow$ Since $\exists H_1, H_2$ with form: $\bigcap_{n=1}^{\infty}(\bigcup_i^{\infty} Q_{ni})$. $H_i \supseteq E_i$

where $m^*(\bigcup_i^{\infty} Q_{ni}) \le m^*(E) + \frac{1}{n}$. $E \subseteq \bigcup_i^{\infty} Q_{ni}$. $\forall n \in \mathbb{Z}^+$

i.e. $m(H) = m^*(E)$. (Denote: $H = \bigcap_n \bigcup_i Q_{ni}$)

$\therefore m(H_1) + m(H_2) \geqslant m(H_1 \cup H_2) \geqslant m^*(E_1 \cup E_2)$

$= m^*(E_1) + m^*(E_2) = m(H_1) + m(H_2)$

$\therefore m(H_1 \cap H_2) = 0$. $m(H_1 - E_1 \cup E_2) = 0$

$\Rightarrow m(H_1 - E_1) = 0$. $\therefore E_1$ is $\mathcal{L}$-measurable. So E_2.

Remark. i) $\forall E \subseteq \mathbb{R}^d$. $\exists H$ is $\mathcal{L}$-measurable. st.

$m(H) = m^*(E)$

ii) If exists H is $\mathcal{L}$-measurable. st.

$m(H/E) = 0$ Then E is $\mathcal{L}$-measurable.

③ Open Interval Problems:

i) $E \subseteq \mathbb{R}^1$. $m^*(E) > 0$. $\forall 0 < \alpha < 1$. $\exists I$ interval.

st. $m^*(E \cap I) \geqslant \alpha m^*(I)$

Pf: $\exists \bigcup Q_n \supseteq E$. $m^*(\bigcup Q_n) \le m^*(E) + \varepsilon$. $\exists I_n$. st.

$m^*(I_n) \le m^*(Q_n) + \frac{\varepsilon}{2^n}$. $Q_n \subseteq I_n$ (refine)

$\therefore \bigcup I_n \supseteq E.\ m^*(\bigcup I_n) \le m^*(E) + 2\varepsilon.$

If $m^*(I_n \cap E) < \alpha m(I_n).\ \forall n.$ Then it will contradict!

ii) <u>Steinhaus Thm:</u>

$m(E)\cdot m(F) > 0.\ E, F \subseteq \mathbb{R}^1.$ Then $E+F$ contains an interval.

<u>Pf:</u> $\chi_E * \chi_F : \mathbb{R} \to \mathbb{R}$. Conti.

$$\int_{\mathbb{R}^2} \chi_E(y)\chi_F(x-y)\,dm = m(E)m(F) > 0.$$

$\therefore m(\{\chi_E * \chi_F > 0\}) > 0.\ \{\chi_E * \chi_F > 0\}$ is open.

$\therefore \exists\ I \subseteq \{\chi_E * \chi_F > 0\}.$

When $x \in I \Rightarrow x-y \in F.\ y \in E.\ \therefore x \in E+F$

i.e. $I \subseteq E+F.$

④ <u>Invariant property of Lebesgue measure:</u>

- For E is $\mathcal{L}$-measurable. $m(E+h) = m(E)$.

$m(\delta E) = \prod_1^n \delta_k m(E).\ E \subseteq \mathbb{R}^n.\ \delta E = \{(\delta_k x_k)_n \mid (x_k)_n \in E\}.$

<u>Pf:</u> Directly from cover of cubes!

(4) <u>$\mathcal{L}$-measurable Functions:</u>

- <u>Def:</u> f is $\mathcal{L}$-measurable $\Leftrightarrow \{f < a\}$ is $\mathcal{L}$-measurable for $\forall a \in \mathbb{R}^1$.

<u>Properties</u>: i) f is $\mathcal{L}$-measurable $\Leftrightarrow$ $\forall$ O open. F close.

$f^{-1}(O)$. $f^{-1}(F)$ is measurable

ii) $\{f_n\}$ measurable. Then $\sup f_n$. $\inf f_n$.

$\overline{\lim} f_n$. $\underline{\lim} f_n$ are measurable.

iii) f. g measurable $\Rightarrow$ f^k. $f+g$. fg are

measurable!

iv) f conti. ϕ is $\mathcal{L}$-measurable $\Rightarrow$

$f \circ \phi$ is $\mathcal{L}$-measurable.

① <u>Approximation</u>:

i) $f \geq 0$. $\mathcal{L}$-measurable.(*) Then exists seq of

$\{\phi_n\}$ simple functions. $\phi_n \geq 0$. $\phi_n \nearrow f$.

$\forall$ $x \in \mathbb{R}^d$. pointwise.

$\Downarrow$ $f = f^+ - f^-$

(*) Simple Func χ_E require E is $\mathcal{L}$-measurable!

ii) f is $\mathcal{L}$-measurable. Then exists seq of

$\{\phi_n\}$ Simple func. $|\phi_n| \leq |\phi_{n+1}|$. $\phi_n \to f$

pointwise for $\forall x \in \mathbb{R}^d$. st. $|\phi_n| \leq |f|$.

$\Downarrow$ Step $\xrightarrow{\text{Approxi}}$ Simple func.

iii) f is $\mathcal{L}$-measurable. Then exists seq of

step functions. $\{\phi_n\}$. $\phi_n \to f$. a.e. x.

<u>Pf</u>: Use simple functions to exhaust f.

And for χ_E : $\exists$ $\{R_j\}$ rectangles.

st. $\cup R_j \overset{\varepsilon}{\simeq} E$. refine R_j to obtain step func.

② Littlewood's Three Principle:

i) Every measurable set is nearly a finite union of intervals.

ii) Every convergent seq of Func. is nearly uniform.

Thm (Egorov)

$\{f_n\}$. seq of measurable func on E.

where $m(E)<\infty$. $f_n \to f$. a.e. on E.

Then for $\forall \varepsilon>0$. $\exists A_\varepsilon \subseteq E$. closed. st.

$f_n \xrightarrow{n} f$ in A_ε. $m(E/A_\varepsilon) < \varepsilon$.

iii) Every measurable function is nearly conti.

Thm. (Lusin)

$f<\infty$. measurable on E. $m(E)<\infty$.

Then for $\forall \varepsilon>0$. $\exists A_\varepsilon \subseteq E$. closed. st.

$f|_{A_\varepsilon}$ is conti. $m(E/A_\varepsilon)<\varepsilon$.

Pf: For ii) The point is:

$E_k^n = \{x \in E \mid |f_j - f| < \frac{1}{n}.\ j \geq k\} \uparrow E\ (k \to \infty)$

exhaust the convergent domain.

For each n. $\exists E_{k_n}^n \overset{\frac{\varepsilon}{2^n}}{\subseteq} E$. Then intersect

them to obtain $A_\varepsilon' = \cap E_{k_n}^n$.

For iii): Since we have $\{\phi_n\}$ step functions.

$\phi_n \to f$. a.e. by ii). Obtain A_ε:

$\phi_n \xrightarrow{u} f$. on A_ε.

Refine each $\operatorname{supp}\phi_n$ to let ϕ_n be conti. on E_n.

$E_n \subseteq \operatorname{supp}\phi_n$. $m(\operatorname{supp}\phi_n / E_n) < \frac{\varepsilon}{2^n}$

Refine $A_\varepsilon - \cup E_n$ to obtain closed set !

Cr. Every measurable function is the limit a.e.

of conti. functions.

Pf. Use $B_n = (-n, n)^d$ to approxi. $\mathbb{R}^d$.

By Lusin's Thm. $\exists E_n \subseteq B_n$. st.

$m(B_n / E_n) < \frac{\varepsilon}{2^n}$. $f|_{E_n}$ is conti.

By Tietze Extension. Let:

F_n is extension of $f|_{E_n}$ on $\mathbb{R}^d$.

$\therefore F_n \longrightarrow f$ outside $\overline{\lim} B_n / E_n$.

$m(\overline{\lim} B_n / E_n) = 0$. $\therefore F_n \longrightarrow f$. a.e.

Integration Theory

(1) Establishment:

① Stage one:

· For simple functions: $\varphi(x) = \sum_1^N C_k \chi_{E_k}$. on $\mathbb{R}^d$

Def: $\int_{\mathbb{R}^d} \varphi \, dm = \sum_1^N C_k \, m(E_k)$

② Stage two:

· For bounded functions support on finite measure set.

f should be $\mathcal{L}$-measurable firstly.

Then $\exists \varphi_n$ simple func. $\to f$. $|\varphi_n| \le |f|$

Def: $\int_{\mathbb{R}^d} f \, dm = \lim_n \int_{\mathbb{R}^d} \varphi_n \, dm$.

(From Egorov Thm. claim $\{\int_{\mathbb{R}^d} \varphi_n\}$ converges)

Remark: It's similar with Rieman integral

Since $\int f \, dx = \lim_{\|T\| \to 0} \sum_1^n f(\xi_i) \Delta x_i$

③ Stage Three:

· For $f \geq 0$. $\mathcal{L}$-measurable. Def:

$\int_{\mathbb{R}^d} f \, dm = \sup \{ \int_{\mathbb{R}^d} g \, dm \mid 0 \le g \le f.\ |g| < \infty.$

$m(\operatorname{supp} g) < \infty \}$.

For $\int_{\mathbb{R}^d} f \, dm < \infty$. We say it's $\mathcal{L}$-intergrable.

④ Stage four:

· For general functions ($\mathcal{L}$-measurable)

Let $f = f^+ - f^-$. Def: $\int f \, dm = \int f^+ dm - \int f^- dm$

$\Rightarrow$ f is $\mathcal{L}$-integrable $\Leftrightarrow$ $\int |f| \, dm$ is finite.

properties:

i) $\exists F$. $m(F) < \infty$. s.t. $\int_{F^c} |f(x)| \, dm < \varepsilon$.

ii) $\forall \varepsilon > 0$. $\exists \delta > 0$. s.t. $\int_E |f| \, dm < \varepsilon$. whenever $m(E) < \delta$.

Pf: By approxi: $f \cdot \chi_{E_n}$'s integral.

(2) Limit Thm:

Fatou's Lemma:

$\{f_n\}$ seq of $\mathcal{L}$-measurable functions. $f_n \geq 0$.

Then $\int \underline{\lim}_n f_n \, dm \leq \underline{\lim}_n \int f_n \, dm$.

Cor. (Monotone Convergence Thm)

$f_n \geq 0$. $f_n \uparrow f$ a.e. Then: $\int \lim f_n = \lim_n \int f_n \, dm$.

Remark: "$f_n \geq 0$" can be removed. Since Let

$F_n = f_n - f_1 \geq 0$. Apply on F_n's.

Cor. (Dominative Convergence Thm)

$g(x)$ is $\mathcal{L}$-integrable. $|f_n| \le g$. $\forall n$.

If $f_n \to f$. a.e. Then $\lim_n \int f_n = \int \lim_n f_n dm$

$f_n \to f$ in L^1.

Pf: Return to the def: operate on g.

$0 \le g \le f$. $|g| < \infty$. $m(\operatorname{supp} g) < \infty$.

Let $g_n = \min\{f_n, g\}$.

For the last one: Separate $E_N = \{g \le N\}$.

By Egorov. Thm.

(3) $L^1(\mathbb{R}^d)$ Space:

· $L^1(\mathbb{R}^d)$ is a linear space. with a complete norm $\|f\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f| dm$.

Pf: For $\varepsilon > 0$. $\exists N$. St. $n, m > N$. $\|f_n - f_m\|_{L^1} < \varepsilon$.

1') Find a subseq has limit function:

Choose $\{n_k\}$. St. $\|f_{n_k} - f_{n_{k+1}}\|_{L^1} \le \frac{1}{2^k}$

$\therefore m(\{f_{n_k} - f_{n_{k+1}} > \frac{1}{2^k}\}) \le \frac{c}{2^k}$

$f = f_1 + \sum_1^\infty (f_{n_k} - f_{n_{k+1}})$ converge. a.e.

2') Since $\|f_n - f\|_{L^1} \le \|f_n - f_{n_k}\|_{L^1} + \|f_{n_k} - f\|_{L^1}$

$\therefore f_n \to f$. in L^1. (Check $f \in L^1(\mathbb{R}^d)$. first)

Remark: $f_n \xrightarrow{L^1} f$. $\Rightarrow$ $\exists f_{n_k} \to f$. a.e.

① Density:

$M = \{ g \mid g$ is simple function on $\mathbb{R}^d \}$.

$N = \{ g \mid g$ is step function on $\mathbb{R}^d \}$.

We have: $\overline{M} \supseteq L^1(\mathbb{R}^d)$. $\overline{N} \supseteq L^1(\mathbb{R}^d)$. $\overline{C_c(\mathbb{R}^d)} \supseteq L^1(\mathbb{R}^d)$

② Invariant properties:

$\| f_h \|_{L^1(\mathbb{R}^d)} = \| f(x+h) \|_{L^1} = \| f \|$.

$\| f(\delta x) \|_{L^1(\mathbb{R}^d)} = \frac{1}{\prod_1^n |\delta_k|} \| f(x) \|_{L^1(\mathbb{R}^d)}$

(Check on simple functions)

Prop. $\| f_h - f \|_{L^1(\mathbb{R}^d)}$, $\| f(\delta x) - f \|_{L^1(\mathbb{R}^d)} \to 0$ $(h \to 0, \delta \to 1)$

whenever $f \in L^1(\mathbb{R}^d)$.

Pf: Approxi by $C_c(\mathbb{R}^d)$.

(4) Tonelli - Fubini Thm.

① Fubini Thm:

$f(\vec{x}, \vec{\eta}) \in L^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. Then:

i) $f^\eta \in L^1(\mathbb{R}^{d_2})$. a.e. η, $f_x \in L^1(\mathbb{R}^{d_1})$. a.e. x.

ii) $\int f \, d(m_1 \times m_2) = \int f \, dm_1 \, dm_2 = \int f \, dm_2 \, dm_1$

Pf: Let $F = \{ f \in L^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \mid f$ satisfies the conclusion $\}$.

1°) F is closed under linear combination and monotone limit. (Monotone Converge Thm)

2°) Check on χ_E. E is G_δ set. $m(E)<\infty$

i) For $E = Q_1 \times Q_2$ bounded open cube in $\mathbb{R}^d$.

ii) For E is subset of boundary of closed cube in $\mathbb{R}^n$.

iii) For $E = \bigcup_1^k Q_i$. finite union of almost disjoint closed cubes in $\mathbb{R}^d$.

Pf: By i). ii) and linearity.

iv) For E is open. $m(E)<\infty$.

Pf: By iii). $\chi_E \longleftarrow \sum_1^N \chi_{Q_i}$ (mono-limit)

$\Rightarrow$ $E = \bigcap_1^\infty O_k$. choose O_0 open. $E \subseteq O_0$ $m(O_0)<\infty$. $O_0 \cap (\bigcap_1^N O_k)$ is decrease seq. From monotone limit $\therefore \chi_E \in F$.

3°) Check on χ_E. $m(E)=0$

Pf: since $\exists$ G. G_δ set. $E \subseteq G$. $m(G)=0$. apply 2°) on G. obtain $\chi_E \in F$.

4°) Check on χ_E. $m(E)<\infty$.

Pf: $E = G_\delta \cup N$. $m(N)=0$. Apply 2°). 3°)

5°) $\forall f \in L^1(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$. $f \in F$.

Pf: $f = f^+ - f^-$. $f^{\pm}$ are increase limit of seq of simple functions.

② Applications:

i) Tonelli Thm:

$f(\vec{x}_1, \vec{x}_2) \geq 0$. belongs to L-measurable func. on $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$

Then: f^y. f_x is L-measurable on $\mathbb{R}^{d_1}$. $\mathbb{R}^{d_2}$. for a.e. y and a.e. x

$\int_{\mathbb{R}^{d_1}} f^y dx$. $\int_{\mathbb{R}^{d_2}} f_x dy$ is L-measurable on $\mathbb{R}^{d_2}$. $\mathbb{R}^{d_1}$. a.e. y and a.e. x.

Pf: Approxi. by L-measurable function:

$$f_N = \begin{cases} f(x,y), & |(x,y)|<N,\ |f|<N \\ 0, & \text{otherwise} \end{cases} \to f$$

Apply Fubini Thm on $\{f_N\}$.

ii) Product measure:

- Note firstly that from i). $\forall E$ is measurable in $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$. so E_x. E^y are measurable in $\mathbb{R}^{d_2}$. $\mathbb{R}^{d_1}$. Next. We can more general case.

Prop. $E = E_1 \times E_2$ is measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$

If $m^*(E_1) > 0$. Then E_2 is measurable

Pf: By Fubini's Thm. E_x is measurable on F.

s.t. $m(F^c) = 0$. Since $m^*(E_1) > 0$ $\therefore E_1 \cap F \neq \emptyset$.

Select $x \in E_1 \cap F$. $E_2 = E_x$ is measurable

Remark: Note $\{0\} \times \mathcal{N} \subseteq \mathbb{R}^1 \times \mathbb{R}^1$. null set.

$m(\{0\}) = 0$. $\mathcal{N}$ is Vitali set. unmeasurable!

Lemma. $m^*(E_1 \times E_2) \leq m^*(E_1) m^*(E_2)$

Pf: easy to check by def of cover.

$\Downarrow$

prop. For. E_1. E_2 are measurable. Then:

$m(E_1) m(E_2) = m(E_1 \times E_2)$

Pf: Only need to check $E_1 \times E_2$ is measurable

Then apply Fubini on it.

Check $m^*(G_1 \times G_2 / E_1 \times E_2) = 0$. where $G_1 . G_2$

is G_δ set of E_1. E_2. s.t. $m(G_1/E_1) . m(G_2/E_2) = 0$.

Apply the lemma!

prop. $f(x)$ is $\mathcal{L}$-measurable on $\mathbb{R}^d$. Then $f(x-y)$

$= \tilde{f}(x, y)$ is $\mathcal{L}$-measurable on $\mathbb{R}^d \times \mathbb{R}^d$

Pf. Check: If $E = \{f < a\}$. measurable set.

Then $E - E = \{x - y \mid x, y \in E\}$. measurable.

Lemma. O is open. Then $O - O$ is open.

Pf: $\forall x \in O - O$. $x = y_1 - y_2$, $y_1, y_2 \in O$

$\therefore \exists U_1, U_2$ open. $y_1 \in U_1$, $y_2 \in U_2$

$\therefore x \in U_1 - y_2 \subseteq O - O$. open neighbour

Lemma. $m(F) = 0$. Then $F - F$ is null set

Pf: Denote $F_k = F \cap B_k$. $B_k = \{|x| < k\}$. check $m(F_k - F_k) = 0$. $\forall k$.

$\exists O_{nk}$ open. $F_k \subseteq O_{nk}$, $m(O_{nk}) < \frac{1}{n}$. Note that:

$m((O_{nk} - O_{nk}) \cap B_k) = \int \chi_{(O_{nk} - O_{nk}) \cap B_k} = \int \chi_{O_{nk}}(x-y)\chi_{B_k}(y)\, dm$

$\therefore LHS = m(O_{nk})\, m(B_k) \to 0$ $\therefore m(F_k - F_k) = 0$. $m(F - F) = 0$.

$\Rightarrow$ Since $E = \cap O_n / N$. $E - E = \cap(O_n - O_n)/N$

$\cap(O_n - O_n)$ is a G_δ set as well!

Cor. If f is measurable on $\mathbb{R}^{d_1}$. Then $\tilde{f}(x, y) = f(x)$

is measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.

Pf: $\{\tilde{f} < a\} = \{f < a\} \times \mathbb{R}^{d_2}$

Remark: The projection is measurable.

Cor. $f \geq 0$ on $\mathbb{R}^d$. $A = \{(x, y) \mid 0 \leq y \leq f(x)\}$. Then:

f is measurable $\Leftrightarrow$ A is measurable.

$\int f\, dm = m(A)$. If one side holds.

Pf: From above: $F(x, y) = y - f(x)$. measurable on $\mathbb{R}^{d+1}$.

$A = \{F \geq 0\} \cap \{y \geq 0\}$. Apply Tonelli Thm.

Differentiation Theory

(1) <u>Differentiation of integral</u>:

· Consider average problem:

$$\lim_{\substack{m(B)\to 0 \\ x\in B}} \frac{1}{m(B)} \int_B f\,dm = f(x),\ a.e.\ x \text{ hold?}$$

① <u>Def</u>: Littlewood maximal function:

$f\in L^1(\mathbb{R}^d)$. $f^*(x) = \sup_{\substack{B \\ x\in B}} \frac{1}{m(B)} \int_B |f|\,dm$

<u>properties</u>: i) f^* is measurable ($f^*>a$ is open, by def)

ii) $m\{f^*>\alpha\} \le \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}$

iii) $f^* < \infty$. a.e. x. (Let $\alpha\to\infty$ in ii))

<u>Vitali's Lemma</u>.

$U = \bigcup_{\alpha\in A} B_\alpha$. $\forall\ c < m(U)$. There exists disjoint $\{B_i\}_1^k$. st. $\sum_1^k m(B_i) \ge \frac{c}{3^d}$.

<u>Pf</u>: By cpt set $K \subseteq U$. $m(K) > c$.

Operate on these finite many balls over K.

For ii): Choose $B_x = \{x \mid \frac{1}{m(B_x)} \int_{B_x} |f|\,dm > \alpha\}$

Since $\{f^*>\alpha\}$ can be cover by $\bigcup B_x$.

Lebesgue differential

Theorem:

. For $f \in L^1_{loc}(\mathbb{R}^d)$. $\overline{\lim\limits_{\substack{m(B)\to 0 \\ x\in B}}} \frac{1}{m(B)} \int_B |f(y)-f(x)|\, dy = 0$. a.e. x.

Pf: Approxi. f by $C_c(\mathbb{R}^d)$ function $g(x)$. firstly.

For conti. Func. the conclusion holds.

$\Rightarrow$ check $E_\alpha = \{ (f-g)^* + |f-g| > 2\alpha \}$

has measure 0 for $\forall \alpha$.

where $(f-g)^* = \overline{\lim\limits_{\substack{m(B)\to 0 \\ x\in B}}} \frac{1}{m(B)} \int_B |g(y)-f(y)|\, dy$

$E_\alpha \subseteq \{ (f-g)^* > \alpha \} \cup \{ |f-g| > \alpha \}$.

By Chebyshev. easy to prove!

Remark: replace $\{B(x,r)\}_{r>0}$ with general sets:

$\{U_\alpha\}_{\alpha\in A}$ is family of sets shrink regularly

at x. if $\exists c>0. \forall \alpha \in A.$ $\exists B$ a ball. s.t. $U_\alpha \subseteq B$.

$m(U_\alpha) \geq c\, m(B)$

$\Rightarrow \overline{\lim\limits_{\substack{m(U_\alpha)\to 0 \\ x\in U_\alpha}}} \frac{1}{m(U_\alpha)} \int_{U_\alpha} |f(y)-f(x)|\, dy = 0$. a.e. x

for $L^1_{loc}(\mathbb{R}^d)$ function f.

ii) $f^*(x) \geq |f(x)|$. $\forall x$.

Pf: $f^*(x) \geq \frac{1}{m(B)} \int_B |f(y)|\, dy \to |f(x)|$. $x \in B$. $m(B) \to 0$.

v) $f \in L^1_{loc}(\mathbb{R}^d)$. $f \not\equiv 0$. Then $f^*(x) \geq \frac{c}{|x|^d}$

for some $c>0$, and $\forall\ |x| \geq 1$

<u>Pf:</u> $f^*(x) \geq \frac{1}{m(B)} \int_B |f(y)|\,dm$. $\forall x \in B$.

Note that $B(0,R) \subseteq B(x, |x|+R) \triangleq \tilde{B}$

(It's for making the integral

indept with x)

$\therefore f^*(x) \geq \frac{1}{m(\tilde{B})} \int_{\tilde{B}} |f|\,dm \geq \frac{1}{m(\tilde{B})} \int_{B(0,R)} |f|\,dm$

$m(\tilde{B}) = (R+|x|)^d\, v_d \leq (R+1)^d |x|^d\, v_d$.

② <u>Application:</u>

<u>Def:</u> Lebesgue density of E:

For E is measurable. x is Lebesgue

density of E. if $\lim\limits_{\substack{m(B)\to 0 \\ x\in B}} \frac{m(B\cap E)}{m(B)} \to 1$

$\Rightarrow$ Actually $\frac{1}{m(B)} m(B\cap E) = \frac{1}{m(B)} \int_B \chi_E\,dm$

$\chi_E \in L^1_{loc}(\mathbb{R}^d)$ $\therefore \forall x \in E$. x is Lebesgue

density. (for a.e. x)

<u>Def:</u> Lebesgue set E for f:

$x \in E$. Then $f(x)$ is finite and

$\lim\limits_{\substack{m(B)\to 0 \\ x\in B}} \frac{1}{m(B)} \int_B |f(y) - f(x)|\,dm = 0$.

$\Rightarrow$ If $f \in L^1_{loc}(\mathbb{R}^d)$. Then almost every $x \in E$.

(2) Differentiation of functions:

. Next, we will find condition for F. st:

$$F(b) - F(a) = \int_a^b F'(x)dx. \text{ (in } \mathbb{R}^1 \text{ only!)}^{(*)}$$

(*) In $\mathbb{R}^1$. the open set $G = \bigcup I_n$. unique : A strong property

① Bounded Variation:

. Recall that for a curve γ. parametrized by $z(t) = (x(t), y(t))$ in $\mathbb{R}^2$. We say it's rectifiable:

if $\exists M>0$. for any : $a = t_0 < t_1 < \cdots < t_N = b$. $N \in \mathbb{Z}_+$.

then $\sum_1^N |z(t_i) - z(t_{i-1})| \leq M$. $\longrightarrow$ Thm. $\gamma \in BV[a,b] \Leftrightarrow \gamma$ is rectifiable!

Def : $\text{length}(\gamma) = \sup_{\{t_i\}_1^N} \sum |z(t_i) - z(t_{i-1})|$

$\Rightarrow$ We say Function $F(x)$ is bounded Variation

if for any partition $\{t_i\}_1^N$ of $[a,b]$.

$\sum |F(t_i) - F(t_{i-1})| \leq M$ (indept of $\{t_i\}_1^N$)

Def: i) Total Variation $T_F(a,x)$

$= \sup_{\{t_i\}} \sum |F(t_i) - F(t_{i-1})|$. $x \leq b$.

$\{t_i\}$ is partition of $[a,x]$.

ii) Negative part: $N_F(a,x) = \sup \sum_{(-)} -(F(t_i) - F(t_{i-1}))$

iii) Positive part: $P_F(a,x) = \sup \sum_{(+)} (F(t_i) - F(t_{i-1}))$

Prop. $T_F(a,x) = N_F(a,x) + P_F(a,x)$. $a \leq x \leq b$

$F(x) = F(a) + P_F(a,x) - N_F(a,x)$. $a \leq x \leq b$

Lemma F is bound variation on $[a,b]$

$\Leftrightarrow$ F is difference of two increasing func.

Pf: Increasing functions on $[a,b]$

are bound variation definitely.

Thm. F is bound variation. Then $F(x)$ is

differentiable a.e. x.

Pf: Rising Sun Lemma:

$G: \mathbb{R} \to \mathbb{R}$. conti. $E = \{x \in \mathbb{R} \mid G(x+h) > G(x)$

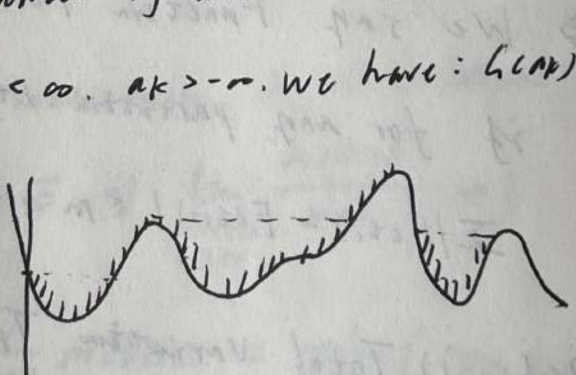
for some $h = h(x) > 0\}$. If $E \neq \emptyset$. Then E

is open, can be written in countable

disjoint union of intervals: $E = \bigcup (a_k, b_k)$

For $b_k < \infty$. $a_k > -\infty$. we have: $G(a_k) = G(b_k)$

Pf:



easy to check in the figure

Sometimes in the end point: $G(a_k) < G(b_k)$

1°) Consider F is conti.

By Lemma. consider F is increasing

$$\Delta_h F(x) = \frac{F(x+h) - F(x)}{h}$$

Pf: Dini Number: $D^+(F)(x) = \overline{\lim_{h\to 0^+}} \Delta_h(F)$. $D_+(F) = \underline{\lim_{h\to 0^+}} \Delta_h(F)$

$D^-(F)(x) = \overline{\lim_{h\to 0^-}} \Delta_h(F)$, $D_-(F) = \underline{\lim_{h\to 0^-}} \Delta_h(F)$

$\Rightarrow$ We will show: i) $D^+(F) < \infty$. a.e. x

ii) $D^+(F) \le D_-(F)$. a.e. x

Apply it on $-F(-x)$. Then: $D^+ \le D_- \le D^- \le D_+ \le D^+ < \infty$. a.e. x

Then $\lim_{h\to 0} \Delta_h(F)(x)$ exists, finite.

For i): Consider $E_\gamma = \{D^+(F) > \gamma\}$

since $E_\gamma = \bigcup_{n\in\mathbb{Z}^+} \{\frac{F(x+\frac{1}{n})-F(x)}{\frac{1}{n}} > \gamma\}$. $\therefore E_\gamma$ is measurable.

Apply Lemma on $F - \gamma x$. $\therefore E_\gamma \subseteq \bigcup (a_k, b_k)$

where $F(b_k) - F(a_k) \ge \gamma(b_k - a_k)$

$\therefore E_\gamma \le \sum m(a_k, b_k) \le \frac{1}{\gamma} \sum F(b_k) - F(a_k) \le \frac{F(b)-F(a)}{\gamma}$

Let $\gamma \to \infty$. $\therefore E_\infty$ is null set.

For ii): Set $E_{R,r} = \{D^+(F) \ge R > r \ge D^-(F)\}$. $R, r \in \mathbb{Q}$.

prove $m(E_{R,r}) = 0$. Then $E = \{D^+(F) > D^-(F)\} = \bigcup_{R,r} E_{R,r}$

E is null set!

a. First Note that $E_{R,r}$ is measurable

b. Suppose $m(E_{R,r}) > 0$.

Approxi. $m(E_{R,r})$ by O open. st. $E \subseteq O$. $m(O) \le \frac{R}{r} m(E)$

$O = \bigcup I_n$. disjoint intervals.

Apply Lemma on: $-F(-x) + rx$ on each $-I_n$

then $E = \bigcup (a_k, b_k) \subseteq I_n$.

Apply Lemma on $F(x) - Rx$, on each (a_n, b_n)

then $O_n = \bigcup_k (a_{k,n}, b_{k,n}) \subseteq (a_n, b_n)$

(i.e. Refine $O = \cup I_n$. $I_n \longrightarrow (a_n, b_n) \longrightarrow (a_{k,n}, b_{k,n})$)

$\Rightarrow$ Find relation of O_n. I_n. Come into contradict!

<u>Cor.</u> F is conti. monotone. Then F' exists. a.e. x.

s.t. $\int_a^b F'(x)\,dx \le F(b) - F(a)$

<u>Pf:</u> Apply Fatou's Lemma on $\dfrac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}}$

<u>Remark:</u> "<" will hold strictly. e.g. F is Cantor-Lebesgue Function of $C_{\frac{1}{3}}$

since $m(C_{\frac{1}{3}}) = 0$. $\therefore F'(x) = 0$. a.e. x

but $\int_0^1 F'(x)\,dm = 0 < F(1) - F(0) = 1$

2) For general $F(x)$ isn't conti.

<u>Def:</u> Jump function $J_F = \sum \alpha_n j_n(x)$

of $F(x)$. where $\{x_n\}$ is discontinuities

of F. $F(x_n^-) = F(x_n) - \theta_n \alpha_n$. $0 < \theta_n \le 1$.

$F(x_n^-) = F(x_n^+) - \alpha_n$.

$$j_n(x) = \begin{cases} 0 & , x < x_n \\ 1 & , x > x_n \\ \theta_n & , x = x_n \end{cases}$$

$\Rightarrow F = F_c + J_F$. We will prove $J_F' = 0$. a.e. x.

② <u>Absolutely conti. Func:</u>

· <u>Def:</u> F is absolutely conti if $\forall \varepsilon > 0$. $\exists \delta > 0$

s.t. $\sum_1^N |F(b_k) - F(a_k)| < \varepsilon$. when $\sum_1^N (b_k - a_k) < \delta$

Remark: i) F is absolutely conti $\Rightarrow$ F is uniformly conti

But converse doesn't hold. Since we can let $N \to \infty$. (e.g. Lebesgue-Cantor Func is conti on cpt, but not absolutely conti)

ii) Absolutely conti $\Rightarrow$ bound variation

moreover. T_F is conti.

iii) $AC[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f$ is absolutely conti$\}$. It forms a linear space.

iv) $F(x) = \int_a^x f(t) dt \in AC[a,b]$.

So absolutely conti is neccessarily for: $F(b) - F(a) = \int_a^b F'(x) dx$

Thm. $F(x) \in AC[a,b]$. Then $F'(x)$ exist. a.e. x.

if $F'(x) = 0$. a.e. x. then $F(x) \equiv C$. $\forall x \in [a,b]$.

for C is a constant.

Cor. $F \in AC[a,b]$. Then $F(b) - F(a) = \int_a^b F'(x) dx$

(Apply Thm on $\int_a^x F' dt - F(x)$)

Pf: Lemma. $m(E) < \infty$. $\forall \delta > 0$. $\exists \{B_k\}_1^N \subseteq \mathcal{B}$ family of Vitali cover, which are disjoint st. $\sum_1^N m(B_k) \geq m(E) - \delta$.

Def: $\mathcal{B}$ is a Vitali cover. if $\forall x \in E$. $\forall \delta > 0$. $\exists B \in \mathcal{B}$. st. $x \in B$. $m(B) < \delta$.

Pf: WLOG. suppose $m(E) > \delta$. $\exists K \underset{cpt}{\subseteq} E$. s.t. $m(K) \geq \delta$.

We will exhaust E in the following:

Apply Vitali covering Lemma:

$\exists \{B_i\}_1^{N_1}$. s.t. disjoint. $\sum_1^{N_1} m(B_i) \geq \frac{\delta}{3^d}$

If $\sum_1^{N_1} m(B_i) > m(E) - \delta$. Then we're done.

Otherwise. Let $E_1 = E - \bigcup_1^{N_1} \bar{B_i}$. $m(E_1) > \delta$.

$\mathcal{B}_1 = \{B \in \mathcal{B} \mid B \cap B_i = \emptyset.\ \forall 1 \leq i \leq N_1\}$ is Vitali cover of E_1 $(*)$

Apply Vitali Lemma on E_1. $\mathcal{B}_1$ again:

obtain $\{B_i\}_{N_1}^{N_2} \subseteq \mathcal{B} / \{B_i\}_1^{N_1}$. $\sum_1^{N_2} m(B_i) > \frac{2\delta}{3^d}$

Since $\frac{k\delta}{3^d} \to \infty$ $(k \to \infty)$. At N^{th} stage. the conclusion holds!

Remark: i) $(*)$ is because $\forall x \in E_1$. $\exists B(x,r) \in \mathcal{B}$.

s.t. $r < \min_{1 \leq i \leq N_1} \{\operatorname{dist}(x, \bar{B_i})\}$. Since $\bigcup_1^{N} \bar{B_i}$ closed!

ii) Cor. $\exists \{B_i\}_1^N \subseteq \mathcal{B}$. s.t. $m(E / \bigcup_1^N B_i) \leq 2\delta$.

Pf: $O \overset{\varepsilon}{\supseteq} E$. $E \subseteq O$. $m(E) + \delta > m(O)$

restrict $\mathcal{B}$ in O:

$\mathcal{B}' = \{B \in \mathcal{B} \mid B \subseteq O\}$. still V-cover of E.

$\Rightarrow$ For $\forall a'.b' \in [a,b]$. We will prove: $F(a') = F(b')$

if we have: $F'(x) = 0$. a.e.x. WLOG. Let $a = a'$. $b = b'$

Since $E = \{x \mid F'(x) = 0.\} = \{x \mid \lim_{h \to 0} \left| \frac{F(x+h) - F(x)}{h} \right| = 0\}$

$m(E) = b - a$. $\forall x \in E$. $\forall \varepsilon > 0$. $\exists \delta > 0$. s.t.

$|F(x+h) - F(x)| < \varepsilon |h|$. when $|h| < \delta$.

Such interval $(x, x+h)$ collection will form a Vitali cover of E. Denote $\{I_k\} = \mathcal{I}$.

Apply the Lemma: we obtain $\{I_j\}_1^N = \{(a_j, b_j)\}_1^N$

s.t. $\sum_1^N m(I_j) > m(E) - \delta$. $\{[\alpha_j, \beta_j]\}_1^n = [a,b] / \{I_j\}$.

$\therefore \sum m(\alpha_j, \beta_j) < \delta$. $[\alpha_j, \beta_j]$ disjoint.

$|F(b) - F(a)| \le \sum |F(b_j) - F(a_j)| + \sum |F(\beta_j) - F(\alpha_j)|$

Part one $\le \sum |b_j - a_j| \varepsilon$. by def of I_k.

Part two $\le \varepsilon$. by absolutely conti of F.

<u>Remark:</u> $|F(b) - F(a)| \le \sum |F(t_i) - F(t_{i-1})|$

Separate the partition into 2 part.

One part F changes almost nothing $\sim \varepsilon$.

Another part the interval $\sim \varepsilon$.

<u>prop.</u> F satisfies Lipschitz condition $\Leftrightarrow$

$F \in AC$ and $|F'(x)| \le M$. for a.e. x

<u>prop.</u> $F \in BV[a,b]$. Then: $\int_a^b |F'(x)| dx \le T_F(a,b)$.

"$=$" holds when $F \in AC[a,b]$.

<u>Pf:</u> Since $F(x) = F(a) + P_F(a,x) - N_F(a,x)$ (*) $\longrightarrow$ P_F, N_F are increasing. So, belong to $BV[a,b]$. P_F', N_F' exists!

$\therefore \int_a^b |F'(x)| dx = \int_a^b |P_F' + (-N_F')|$

$\le \int_a^b |P_F'| + |N_F'| = \int_a^b T_F' = T_F(a,b)$

If $F \in AC[a,b]$. Then $\int_{a_i}^{b_i} F' = F(b_i) - F(a_i)$

Note that $\forall$ partition $\{t_i\}_1^n$

$$\sum |F(t_i)-F(t_{i-1})| = \sum \left| \int_{t_{i-1}}^{t_i} F'(t)\,dt \right|$$

$$\le \sum \int_{t_{i-1}}^{t_i} |F'(t)|\,dt = \int_a^b |F'(t)|\,dt$$

Take suprum of $\{t_i\}_1^n$

$$\therefore T_F(a,b) \le \int_a^b |F'(t)|\,dt.$$

If $\int_a^b |F'(x)|\,dx = T_F(a,b)$

$$\therefore \sum |F(x_i)-F(x_{i-1})| \le \sum T_F(x_i, x_{i+1}) = \int_{\cup(x_i,x_{i+1})} |F'|\,dm$$

$\therefore F \in AC[a,b]$. by property of $\mathcal{L}$-intergrable.

<u>Remark</u>: We obtain: $F \in AC[a,b] \Leftrightarrow \int_a^b |F'| = T_F(a,b)$

<u>Thm.</u> (Variation Formula)

$F \in AC[a,b]$. one-to-one. $f \in L^1[A,B]$

Where $[A,B] = [F(a), F(b)]$. Then:

$$\int_A^B f(x)\,dx = \int_a^b f(F(x))\,dF(x)$$

<u>Pf</u>: Approxi. by simple function.

Check on χ_E, $E \underset{\text{open}}{\subseteq} [A,B]$.

<u>Lemma</u> $m(O) = \int_{F^{-1}(O)} F'(x)\,dx$. for open set O.

<u>Pf</u>: $O = \cup (a_k, b_k)$. Separate $\cup(a_k, b_k)$ into monotone intervals of F.

(There're finite since $[a,b]$ is cpt)

$\therefore \int_a^b \chi_E \, dm = m(E) = \int_{F^{-1}(E)} F'(x)dx$

$= \int_a^b \chi_{F^{-1}(E)} F'(x)dx = \int_a^b \chi_E(F(x))F'(x)dx.$

Suppose E is open.. firstly. Then $E \overset{\varepsilon}{\subset} F$

For general measurable set F:

We have it by monotone converge Thm.

③ Jump Functions :

Thm. $J_F'(x) \equiv 0$. a.e. x.

Pf: Set $E_\varepsilon = \{ x \in [a,b] \mid \overline{\lim_{h\to 0}} \frac{J(x+h)-J(x)}{h} \geq \varepsilon \}$.

1°) $E_\varepsilon = \bigcap_{k \geq n} \{x \mid \sup_{\frac{1}{k}<h<\frac{1}{n}} |J(x+h)-J(x)|/|h| \geq \varepsilon\} \cap [a,b]$.

$\therefore E_\varepsilon$ is measurable.

2°) Suppose $m(E) > \delta > 0$.

$\forall \eta > 0$. $\exists N$. s.t. $\sum_{k>N} \alpha_k < \eta$. Since $J_F = \sum \alpha_k j_k$ converges.

Denote: $J_0(x) = \sum_{k>N} \alpha_k j_k(x)$.

$\therefore E_\varepsilon(J_F) / E_\varepsilon(J_0) = \{ \gamma_k \}_1^N$. i.e. $m(E_\varepsilon(J_0)) > \delta > 0$.

$\therefore K$ cpt $\subset E_\varepsilon(J_0)$. $K \subset \bigcup_1^N I_n = \bigcup_1^N (a_n, b_n)$

$\therefore \eta \geq J_0(b) - J_0(a) \geq \sum J_0(b_n) - J_0(a_n)$

$\geq \sum (b_n - a_n)\varepsilon$. Let $\eta \to 0$, contradict!

Remark: The common way is refining a measurable set. until the property of the ε-difference set is nice!

Prop. (Decomposition of monotone Function)

F is increasing on $[a,b]$. Then $F = F_A + F_C + F_J$

where $F_A \in AC[a,b]$. F_C conti. $F_C' \equiv 0$. a.e. x

F_J is jump function.

Moreover, the decomposition is unique.

up to a const.

Pf: $F = F - J_F + J_F$. $F - J_F$ is conti.

$\therefore F - J_F \in BV[a,b]$. Let $F_J = J_F$

Let $F_A = \int_a^x (F-J_F)' \, dm$. $\in AC[a,b]$

$F_C = F - J_F - F_A$. $F_C' \equiv 0$. a.e. x.

If $F_A^1 + F_C^1 + F_J^1 = F_A^2 + F_C^2 + F_J^2$.

Then $F_J^1 - F_J^2 = F_A^2 + F_C^2 - F_A^1 - F_C^1$

LHS should be conti. $\therefore F_J^1 - F_J^2 \equiv$ Const

Differentiate both sides $\therefore F_A^2 = F_A^1 +$ const.

(3) Application:

① Rectifiable Curves:

. When will " $L = \int_a^b (x'^2 + y'^2)^{\frac{1}{2}} dt$ " hold?

e.g. when $x(t) = y(t) = F(t)$. Lebesgue-

Cantor Function, it doesn't hold!

Thm. For curve $\gamma = (x(t), y(t))$, $t \in [a,b]$.

If $x, y \in AC[a,b]$. Then $L(\gamma) = \int_a^b |\gamma'(t)| dt$.

Pf: This is from: let $F(t) = x(t) + i y(t)$

$\therefore F(t) \in AC[a,b]$. we obtain:

$$\int_a^b |F'(t)| dm = T_F(a,b) = \text{length}(\gamma).$$

It has been proved before!

Remark: Any curve can be realized in different parametrizations:

let $s = s(t) = L(a,t)$, the length of curve $z(t)$ in $[a,t]$.

The arc-length parametrization:

$\tilde{z}(s) = \tilde{z}(s(t)) = z(t)$. $\tilde{z}(s) = \tilde{x}(s) + i\tilde{y}(s)$

Thm. $z(t) = (x(t), y(t))$ a rectifiable of length L.

Consider the arc-length parametrization: $\tilde{z}(s)$

Then $\tilde{x}, \tilde{y} \in AC[a,b]$. $|\tilde{z}'(s)| = 1$. a.e. $x \in [a,b]$

and $L = \int_0^L |\tilde{z}'(s)| ds$.

Pf: Geometrically observe: $|\tilde{z}(s_1) - \tilde{z}(s_2)| \le |s_1 - s_2|$

$\therefore \tilde{z}(s) \in AC[a,b]$. and $|\tilde{z}'(s)| \le 1$.

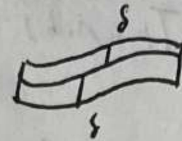
since $\int_0^L |\tilde{z}'(s)| ds = T_{\tilde{z}}(0,L) = \text{length}(z) = L$.

Besides $L \ge \int_0^L |\tilde{z}'(s)| ds = L$. $\therefore |\tilde{z}'(s)| = 1$. a.e.

② Minkovsky Content

of curves:

- Def: $\mu(K) = \lim_{\delta \to 0} \frac{m(K^\delta)}{2\delta}$. $K^\delta = \{x \mid dist(x, K) < \delta\} \subseteq \mathbb{R}^2$.

Geometrically:  : That's why "$\frac{1}{2}$" exists

Def: For $z(t) = (x(t), y(t))$. $a \le t \le b$.

i) $z(t)$ is simple curve if $z(t)$ is injective

ii) $z(t)$ is simple closed curve (Jordan curve) if $z(t)$ is injective on $[a, b)$.

$z(a) = z(b)$

iii) $z(t)$ is quasi-simple curve if:

$z(t)$ is injective on $[a, b] / \{x_i\}_1^n$.

e.g.: 

Thm. For a quasi-simple curve $\gamma: z(t) = (x(t), y(t))$

Its Minkovsky content exists $\Leftrightarrow$

It's rectifiable. when it holds. then:

$\mathcal{M}(\gamma) = length(\gamma)$

Pf: Def: $\mathcal{M}^*(\gamma) = \overline{\lim} \frac{m(\gamma^\delta)}{2\delta}$ prove:

$\mathcal{M}_*(\gamma) = \underline{\lim} \frac{m(\gamma^\delta)}{2\delta}$ $\quad \mathcal{M}^*(\gamma) \le L \le \mathcal{M}_*(\gamma)$

③ Isometric Inequality:

Thm. $\Omega \underset{\text{open}}{\subseteq} \mathbb{R}^2$. bounded. $\gamma = \bar{\Omega}/\Omega$, with length ℓ.

Then: $m(\Omega) \le \frac{\ell^2}{4\pi}$.

Pf: By approxi. of Minkovsky content:

By Minkovsky Inequality(*)

Since $m(\Omega_\delta^+/\Omega_\delta^-) = m(\gamma^\delta)$

estimate: $m(\Omega_\delta^+)$. $m(\Omega)$

Since $\Omega \subseteq \Omega_\delta^- + D(0,\delta)$. $\Omega_\delta^+ \subseteq \Omega + D(0,\delta)$

Remark: (*): Minkovsky Inequality for $\mathcal{L}$-measure:

$$(m(A)^{d} + m(B)^{d})^{\frac{1}{d}} \le m(A+B)$$

ii) Note that "=" holds when Ω is a ball!

iii) For general case, we have:

$$m(\Omega) \le \frac{M^*(\gamma)^2}{4\pi}$$