Ising Model

Background: Ferromagnet exhibits a phase transition by losing its magnetization when heated above a critical temperature.

Ising Model is a model for ferromagnet, to understand the critical temperature.

Suppose $G = (V, E)$. finite graph. $\sigma \in \{\oplus, \ominus\}^V$ is a configuration.

Def: Hamiltonian: $H(\sigma) = -\sum_{x \sim y} \sigma_x \sigma_y$.

Rmk: i) "-1" is for convenience to find the ground state σ^*. st. $H(\sigma^*) = \min_{\sigma} H(\sigma)$

ii) $H(\sigma) = -(\#\text{agree} - \#\text{disagree})$

$= 2\#\text{disagree}(\sigma) - \#E$.

Fix some b.c. $b \in \{\oplus, \ominus\}^{\partial G}$. The Ising model on G with b.c b is a prob. measure:

$\mu^b_{\beta, G}(\sigma) \propto \exp(-\beta H(\sigma))$. $\forall \sigma \in \{\oplus, \ominus\}^G$ and $\sigma = b$ on ∂G. where $\beta > 0$ is inverse temperature.

Rmk: Denote $R = \{\tilde{\sigma} \mid H(\tilde{\sigma}) = \min_{\sigma} H(\sigma)\}$. i.e. set of ground states. Then:

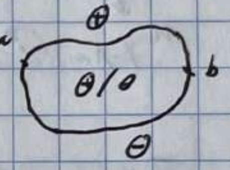
$$\mu^{b}_{\beta,\Lambda_n}(\sigma) \xrightarrow{n\to\infty} \begin{cases} 0, & \sigma \notin R \\ 1/|R|, & \sigma \in R. \end{cases}$$

e.g. Some boundary condition:

i) free b.c. $\mu^{f}_{\beta,G}$: No require on ∂G.

ii) $\mu^{\oplus}_{\beta,G}$ or $\mu^{\ominus}_{\beta,G}$: All $\oplus$ or $\ominus$ on ∂G.

iii) Dobrushin b.c. $\mu^{Dobr}_{\beta,G}$:



iv) b.c. induced by outside

Thm. (Domain Markov Property)

If $G' \subset G$, for b.c. $b \in \{\oplus,\ominus\}^{\partial G}$, $\psi \in \{\oplus,\ominus\}^{G/G'}$ s.t. $\psi = b$ on ∂G. Then:

$$\mu^{b}_{\beta,G}(X \mid \sigma = \psi \text{ on } G/G') = \mu^{\psi}_{\beta,G'}(X)$$

Def: In finite graph G, with b.c. b and $\beta > 0$, define config $\sigma \le \sigma'$ if $\sigma_x \le \sigma'_x$, $\forall x \in V$.

Thm. (FKG inequality)

Fix $\beta > 0$, finite graph G and b.c. b.

$$\mu^{b}_{\beta,G}(A \cap B) \ge \mu^{b}_{\beta,G}(A)\, \mu^{b}_{\beta,G}(B), \quad \forall A, B \uparrow.$$

Pf: Apply Holley inequi. again.

Cor. Fix $\beta>0$. finite graph G. For b.c. $b_1 \le b_2$ and $\forall A \uparrow$. Then:

$$\mu^{b_1}_{\beta.G}(A) \le \mu^{b_2}_{\beta.G}(A).$$

prop. Fix $\beta>0$. Then $\exists$ (possible equal) infinite Volume measures $\mu^{\oplus}_{\beta}$ and μ^{0}_{β}. St. for $\forall A$ depend on finite edges. We have:

$\lim_{n\to\infty} \mu^{\oplus}_{\beta.\Lambda_n}(A) = \mu^{\oplus}_{\beta}(A)$. $\lim_{n\to\infty} \mu^{0}_{\beta.\Lambda_n}(A) = \mu^{0}_{\beta}(A)$.

Pf: It's totally identical as in Random cluster model: $(\mu^{\oplus}_{\beta.\Lambda_n}(A))_n \downarrow$

prop. i) $\mu^{\oplus}_{\beta}$ and μ^{0}_{β} are transition-invariant and ergodic.

ii) For $\mu^{\oplus}_{\beta}$ and μ^{θ}_{β}. There's no ∞-cluster a.s. or exists a unique ∞-cluster a.s.

(1) Critical Value:

Thm. There exists $\beta_c \in (0,\infty)$. St.

$$\begin{cases} \mu^{\oplus}_{\beta}(\sigma_0)=0. & \text{if } \beta<\beta_c \\ \mu^{\oplus}_{\beta}(\sigma_0)>0. & \text{if } \beta>\beta_c \end{cases}$$

Moreover. $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$.

In fact $\mu^{\oplus}_{\beta_c}(\sigma_0)=0$.

Rmk: We consider the expectation of σ_0 (spin at 0) rather than $\{0 \leftrightarrow \infty\}$.

The conclusion is intuitive: when $\beta < \beta_c$, i.e. $T > T_c$. It has high energy so ignores b.c. (In chaos, magnet vanishes)

① Pf by Edwards-Sokal coupling:

Fix $p \in (0,1)$ and a finite graph G.

i) $W \sim$ random cluster model with $(p,2)$, free b.c.

ii) Assign indept to each cluster of w spin ⊕ or ⊖ with prob. $\frac{1}{2}$, get spin config σ.

prop. If $\beta = \frac{1}{2}\log(1-p)$. Then the spin config $\sigma \sim$ Ising model in G with free b.c.

Pf: 1) Note that $P[(w,\sigma)] = \frac{1}{Z^f_{p,2,G}} \cdot \left(\frac{p}{1-p}\right)^{o(w)} 2^{k(w)} \left(\frac{1}{2}\right)^{k(w)}$

$$= \left(\frac{p}{1-p}\right)^{o(w)} / Z^f_{p,2,G}.$$

2) Construct another p.m. Q:

First give $\sigma \sim$ Ising model with free b.c. and $\beta > 0$

Second, for $w \in \{0,1\}^E$, $e = \{x,y\} \in E$.

$$\begin{cases} w(e) = 0 & \text{if } \sigma_x \neq \sigma_y \\ w(e) = 1 \text{ with } p,\ w(e) = 0 \text{ with } 1-p & \text{if } \sigma_x = \sigma_y. \end{cases}$$

Note $H(\sigma) = 2\#\text{disagree}(\sigma) - \#E$.

So: $Q[(\omega,\sigma)] = \frac{1}{Z^f_{\beta,h}} e^{-2\beta\#\text{disagree}} p^{o(\omega)} (1-p)^{c(\omega)-\text{disagree}}$

$$\propto \left(\frac{e^{-2\beta}}{1-p}\right)^{\#\text{disagree}} \left(\frac{p}{1-p}\right)^{o(\omega)}$$

Set $\beta = \frac{1}{2}\log(1-p)^{-1}$. So $P = Q$.

3') Conclusion follows from marginal law of σ under P = marginal of σ under Q = Ising

Return to pf:

Note $\mu^f_{\beta,h}(\sigma_x\sigma_y) = P(\sigma_x\sigma_y) = P(P(\sigma_x\sigma_y|\omega))$

$$= P(I_{\{x\overset{\omega}{\leftrightarrow}y\}} + I_{\{x\overset{\omega}{\not\leftrightarrow}y\}}\cdot 0)$$

$$= \phi^0_{p,2,h}(x\overset{\omega}{\leftrightarrow}y)$$

Since by symmetry $\sigma_x\sigma_y = \pm 1$, with $p=\frac{1}{2}$, if $x\overset{\omega}{\not\leftrightarrow}y$.

$\sigma_x\sigma_y \equiv 1$ if $x\overset{\omega}{\leftrightarrow}y$.

$$\mu^{\oplus}_{\beta,h}(\sigma_x) \overset{(*)}{=} P(\sigma_x) = P(P(\sigma_x|\omega))$$

$$= P(I_{\{x\overset{\omega}{\leftrightarrow}\partial h\}} + I_{\{x\overset{\omega}{\not\leftrightarrow}\partial h\}}\cdot 0)$$

$$= \phi^1_{p,2,h}(x\leftrightarrow\partial h)$$. similarly.

$$\Rightarrow \mu^{\oplus}_{\beta,\Lambda_n}(\sigma_0) = \phi^1_{p,2,\Lambda_n}(0\leftrightarrow\partial\Lambda_n) \xrightarrow{n\to\infty} \phi^1_{p,2}(0\leftrightarrow\infty)$$

i.e. $\mu^{\oplus}_\beta(\sigma_0) = \begin{cases} 0, & p<p_c \\ >0 & p>p_c \end{cases}$ $\beta_c = \frac{1}{2}\log\frac{1}{1-p_c} = \frac{\log(1+\sqrt{2})}{2}$

<u>Rmk:</u> $(*)$ is from replace free b.c. in RCM by $\mu^{\oplus}$ also hold in prop

③ Pf by Krammers - Wannier Dual:

i) High Temperature expansion:

Fix $G = (V, E)$

$\mathcal{E}_G$ is set of even subgraphs w of G.

i.e. $w = (V, E(w))$. $E(w) \subset E$. st. $\forall v \in V$.

degree of v is even in w.

Generally. $\mathcal{E}_G(A)$ is set of subgraphs of G.

for $A \subset V$. st.

i) $\forall v \in V/A$. degree of v in w is even

ii) $\forall v \in A$. degree of v in w is odd.

Rmk: i) $\mathcal{E}_G(\emptyset) = \mathcal{E}_G$

ii) If $\#A$ is odd. Then $\sum_A d(v)$

$+ \sum_{V/A} d(v)$ = odd + even is odd

But $\sum_V d(v) = 2\#E(w)$ in w.

So $\#A$ is odd $\Rightarrow \mathcal{E}_G(A) = \emptyset$.

prop. Fix G is finite. $\beta > 0$. We have:

$$Z^f_{\beta, G} = 2^{\#V(G)} \cosh(\beta)^{\#E(G)} \sum_{w \in \mathcal{E}_G} \tanh(\beta)^{o(w)}$$

where $\cosh(\beta) = \frac{e^\beta + e^{-\beta}}{2}$. $\sinh(\beta) = \frac{e^\beta - e^{-\beta}}{2}$.

Pf: $e^{-\beta\sigma_x\sigma_y} = \cosh(\beta)(1 + \sigma_x\sigma_y \tanh(\beta))$

$$Z^{f}_{p.n} = \sum_{\sigma} e^{\beta \sum_{x\sim y} \sigma_x \sigma_y}$$

$$= \sum_{\sigma} \prod_{\{x,y\}\in E} e^{\beta \sigma_x \sigma_y}$$

$$= \cosh(\beta)^{\#E} \sum_{\sigma} \prod_{\{x,y\}\in E} (1+\sigma_x\sigma_y \tanh(\beta))$$

$$= \cosh(\beta)^{\#E} \sum_{\sigma} \sum_{w\subset E} \tanh(\beta)^{o(w)} \prod_{\{x,y\}\in w} \sigma_x \sigma_y .$$

Note that $\sum_{\sigma} \prod_{\{x,y\}\in W} \sigma_x\sigma_y = 0$ if $\exists\, v \in W$ s.t. degree of v is odd. since by symmetry.

$$\sum_{\sigma} \sigma_x^{k} \prod \square = \sum_{\sigma/\{\sigma_x\}} \prod \square - \sum_{\sigma/\{\sigma_x\}} \prod \square = 0.$$

$$\sum_{\sigma} \prod_{\{x,y\}\in W} \sigma_x\sigma_y = \sum_{\sigma} = 2^{\#V}. \text{ if } W \in \Sigma_n .$$

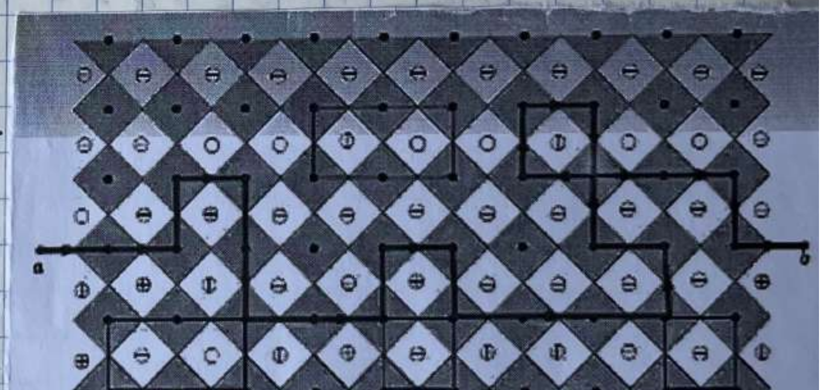
ii) <u>Low Temperature Expansion:</u>

For $\sigma \in \{\oplus, \ominus\}^{V(n)}$.

$w(\sigma) \in \{0,1\}^{E(n^*)}$ def by:

$\forall e = \{x,y\} \in E(n)$.

$$w(\sigma)(e^*) = \begin{cases} 1 & \text{if } \sigma_x \neq \sigma_y \\ 0 & \text{if } \sigma_x = \sigma_y \end{cases}$$



Rmk: i) $W \to \sigma$ is one to one for any b.c.

$W \to \sigma$ is one to two for free b.c. of σ.

since inverse all spins of σ get same w.

ii) $p(\sigma) \propto e^{-\beta H(\sigma)} \Rightarrow p(\sigma) \propto e^{-2\beta\, o(w)}$. by def. directly $(H(\sigma) = 2\,\#\text{disagree} - \#E)$

iii) $W(\sigma) \in \mathcal{E}_{G^*}$ in $\mathcal{M}^{\oplus}_{\beta,G}$ or $\mathcal{M}^{\ominus}_{\beta,G}$.

since it consists of several cycles or curves.

So we have: $Z^{\oplus}_{\beta,G} = e^{\beta \# E} \sum_{w \in \mathcal{E}_{G^*}} e^{-2\beta o(w)}$.

iii) <u>Duality</u>:

<u>Thm.</u> Fix $\beta > 0$. define $\beta^* > 0$ s.t. $\tanh(\beta^*) = e^{-2\beta}$.

Then. $\forall$ graph G. we have:

$$2^{\# V(G^*)} \cosh(\beta^*)^{\# E(G^*)} Z^{\oplus}_{\beta,G} = e^{\beta \# E(G^*)} Z^{f}_{\beta^*, G^*}.$$

<u>Pf</u>: LHS: low Temp expand. RHS: High Temp.

iv) <u>Return to pf</u>:

Set $G = \Lambda_n$. Take "log" in Duality equa. divide $\# E(\Lambda_n^*)$:

$$\frac{\# V(\Lambda_n^*)}{\# E(\Lambda_n^*)} \log 2 + \cosh(\beta^*) + \frac{\log Z^{\oplus}_{\beta,G}}{\# E(\Lambda_n^*)} = \beta + \frac{Z^{f}_{\beta^*, G^*}}{\# E(\Lambda_n^*)}$$

Set $n \to \infty \Rightarrow \log 2 + \cosh(\beta^*) + f(\beta) = \beta + f(\beta^*)$.

Assumption: f has only one point where f is not analytic.

$\Rightarrow$ If $\beta \neq \beta_c$. Then f will have 2 nonanalytic points.

So $\beta_c = \beta_c^*$. i.e. $\tanh(\beta_c) = e^{-2\beta_c}$

$\Rightarrow \beta_c = \frac{1}{2}\log(1+\sqrt{2})$.

<u>Rmk</u>: It's not rigorous since it puts forward a strong assumption.

③ <u>Consequences</u>:

<u>Thm.</u> i) For $\beta \le \beta_c$, we have $\mu_\beta^{\oplus} = \mu_\beta^{\ominus}$, and it's the unique infinite volum measure.

ii) For $\beta > \beta_c$, the set of infinite volum measures is given by: $\{\lambda\mu_\beta^{\ominus} + (1-\lambda)\mu_\beta^{\ominus}\}_{\lambda\in[0,1]}$

<u>Pf</u>: Only prove i):

prove: $\mu_\beta^{\ominus} = \mu_\beta^{\ominus}$ for $\forall A \uparrow$ depend on finite edges (suppose $A \in \sigma(\Lambda_N)$).

Consider $n \gg N$.

1°) Define coupling on $(w, \sigma^{\oplus}, \sigma^{\ominus})$.

$w \sim$ random cluster model with $(p, 2)$. wired b.c. (All $\oplus$).

$\forall v \overset{w}{\leftrightarrow} \partial\Lambda_n$. Set $\sigma_v^{\oplus} = \oplus$, $\sigma_v^{\ominus} = \ominus$.

$\forall v \overset{w}{\not\leftrightarrow} \partial\Lambda_n$. Set $\sigma_v^{\oplus} = \sigma_v^{\ominus} = \oplus/\ominus$.

each with prob. $\frac{1}{2}$.

Note that $(\omega, \sigma^{\oplus}), (\omega, \sigma^{\ominus})$ are Edwards-Sokal coupling, $\sigma^{\oplus} \sim \mu^{\oplus}_{\beta,\Lambda_n}$, $\sigma^{\ominus} \sim \mu^{\ominus}_{\beta,\Lambda_n}$.

2') $\mu^{\oplus}_{\beta,\Lambda_n}(A) - \mu^{\ominus}_{\beta,\Lambda_n}(A) = P(\sigma^{\oplus} \in A, \sigma^{\ominus} \notin A)$

$$\leq \phi^{1}_{p,2,\Lambda_n}(\partial\Lambda_n \leftrightarrow \partial\Lambda_n)$$

$$\to 0 \quad \text{as} \quad n \to \infty.$$

Rmk: For $\beta > \beta_c$, by similar argument as before:

$$\mu^{\ominus}_{\beta}(\sigma_0) = -\phi^{1}_{p,2}(0 \leftrightarrow \infty) = -\mu^{\oplus}_{\beta}(\sigma_0) < 0.$$

(2) Conformal Invariance:

For $(\Lambda^{\diamond}_{\delta}; a_{\delta}, b_{\delta})$ spin-Dobrushin domain. For $z_{\delta} \in \Lambda^{\diamond}_{\delta}$.

Def: i) $\mathcal{E}(a_{\delta}, z_{\delta})$ = collection of contours on Λ_{δ} composed of loops and an interface from a_{δ} to z_{δ}

s.t. $\mathcal{E}(a_{\delta}, z_{\delta}) = \mathcal{E}_{\Lambda^{\diamond}_{\delta}}(\{a_{\delta}, z_{\delta}\})$

ii) Fermionic observable $F_{\delta}(z_{\delta}) =:$

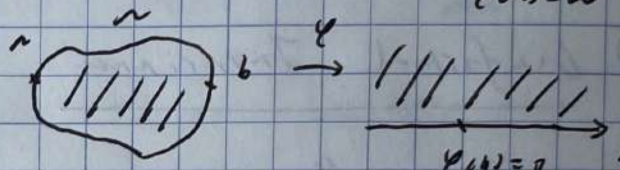
$$\frac{\sum_{\omega \in \mathcal{E}(a_\delta, z_\delta)} e^{-\frac{i}{2} W_{\gamma(\omega)}(a_\delta, z_\delta)} (\sqrt{2}-1)^{o(\omega)}}{\sum_{\omega \in \mathcal{E}(a_\delta, b_\delta)} e^{-\frac{i}{2} W_{\gamma(\omega)}(a_\delta, b_\delta)} (\sqrt{2}-1)^{o(\omega)}}$$

where $\gamma(\omega)$ is the interface from a to b in ω.

Rmk: Recall the setting in low temperature expansion. $p(\sigma) \propto e^{-\beta H(\sigma)} \Rightarrow p(\sigma) \propto e^{-2\beta O(\omega)}$. If $\beta = \beta_c = \frac{1}{2}\log(1+\sqrt{2})$

$\Rightarrow p(\sigma) \propto (\sqrt{2}-1)^{O(\omega)}$. (Motivation)

Thm. $F_\delta(\cdot) \xrightarrow{u.c.c} \sqrt{\varphi'(\cdot)/\varphi'(b)}$ as $\delta \to 0$. where φ is any conformal map: $\Omega \to \mathbb{H}$ send a to ∞ and b to 0.



Pf: 1) Check $F_\delta(\cdot)$ is s-holomorphic. (identical as in FK-Ising model)

2) $H_\delta \xrightarrow{u.c.c} \mathrm{Im}\,\varphi$ $(\delta \to 0)$. (Note: $H_\delta = \frac{1}{2}\mathrm{Im}\int F_\delta^2$)

3) (F_δ) tight. $F_{\delta_n} \to F$. $H_\delta \to \mathrm{Im}\,\varphi$.

$\Rightarrow \mathrm{Im}\,\varphi = \frac{1}{2}\mathrm{Im}\int F^2$.

By holomorphic: $\varphi = \frac{1}{2}\int F^2 + \text{const}$.

Cor. The interface of critical Ising model on $\mathbb{Z}^2$ with Dobrushin b.c. converges to SLE_3.

Cor. (Critical Arm Exponents)

i) $\beta < \beta_c \Rightarrow \mu^{\ominus}_{\beta,\Lambda_n}(\sigma_0) \le e^{-cn}$

ii) $\beta = \beta_c \Rightarrow \mu^{\oplus}_{\beta,\Lambda_n}(\sigma_0) = n^{-\frac{1}{8}+o(1)}$

iii) $\beta > \beta_c \Rightarrow \mu^{\oplus}_{\beta,\Lambda_n}(\sigma_0) \to \mu^{\oplus}_{\beta}(\sigma_0) > 0$.