FK - Ising Model

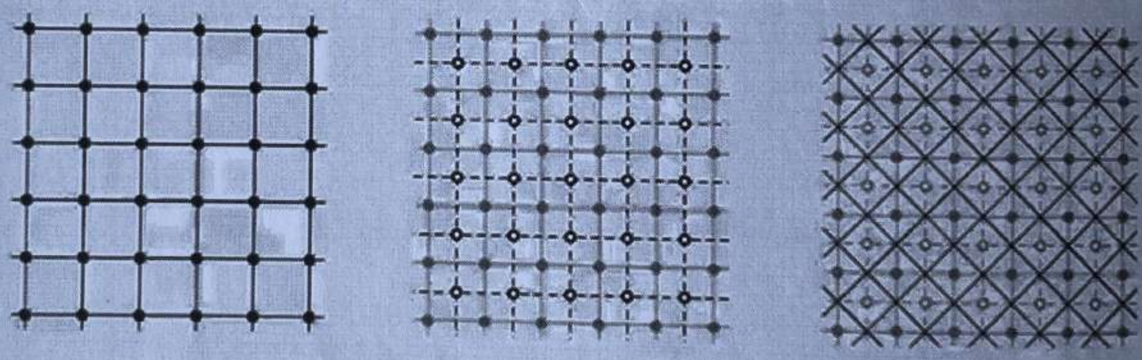
<u>Def:</u> FK-Ising model is random cluster model with $q=2$.

<u>Denote:</u> i) $\mathbb{L} = (\mathbb{Z}^2, E(\mathbb{Z}^2))$. $\mathbb{L}^*$ is its dual lattice.

$\mathbb{L}^\diamond$ is medial lattice where its vertices are centers of edges of $\mathbb{L}$, and edges connect nearest neighbor.

ii) $\mathbb{L}_\delta = \sqrt{2}\delta\mathbb{L}$. $\mathbb{L}^*_\delta$, $\mathbb{L}^\diamond_\delta$ similar defined.

<u>Rmk:</u> $\sqrt{2}$ is for convenience. (mesh of $\mathbb{L}^\diamond_\delta$ is δ)



(a) The square lattice. (b) The dual lattice. (c) The medial lattice.

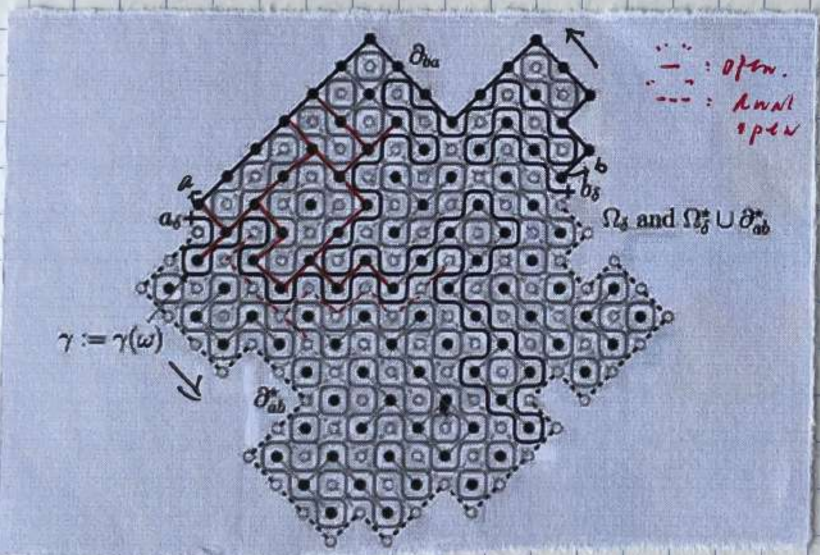
Set: Ω is simply connected domain. $\Omega_\delta = \Omega \cap \mathbb{L}_\delta$.

<u>Def:</u> Dobrushin Domain $(\Omega; a, b)$ where $a, b \in \partial\Omega$ are two distinct boundary points. s.t. edges of (ba) (counterclockwise) are open. And edges of (ab) are closed.

i) Set $(\Omega_\delta^\diamond; a_\delta, b_\delta)$ is approxi. of $(\Omega; a, b)$ which is medial graph of Ω_δ composed of all vertices of $\mathbb{L}_\delta^\diamond$ bordering a black face (means vertices of Ω_δ. White faces mean dual kind) associated to Ω_δ.

ii) b_δ is southeastern corner of a black face.

iii) A self-avoiding loop is a loop will make a $\pm\frac{\pi}{2}$ turn if arriving at a vertice of the medial lattices so as not to cross the open or dual open edges through this vertice.



Def: i) Fermionic Observable of an edge on $\Omega_\delta^\diamond$ is :

$$F_{(\Omega_\delta^\diamond; a_\delta, b_\delta)}(e) = \mathbb{E}_{(\Omega_\delta^\diamond; a_\delta, b_\delta)}\left(\mathbb{I}_{\{e\in\gamma\}}\, e^{\frac{i}{2}W_\gamma(e, b_\delta)}\right)$$

where $\gamma(\omega)$ is curve from a_δ to b_δ (see figure above) $W_\gamma(e, b_\delta)$ is winding (angle) between center of e and b_δ.

ii) FK Fermionic observable of vertices on $\Omega_\delta^\diamond / \partial\Omega_\delta^\diamond$ is :

$$F_{(\Omega_\delta^\diamond ; a_\delta, b_\delta)}(v) = \frac{1}{2} \sum_{e \sim v} F_{(\Omega_\delta^\diamond ; a_\delta, b_\delta)}(e).$$

(sum over 4 medial edges with v as endpoint)

Thm. For Dobrushin domain $(\Omega; a, b)$. $P = P_{S\Lambda(2)}$. $F_\delta(v)$ is vertex Fermionic observable in $(\Omega_\delta^\diamond ; a_\delta, b_\delta)$. Then. $F_\delta / \sqrt{2\delta} \xrightarrow[\text{u.c.c}]{\delta \to 0} \sqrt{\phi'}$. where ϕ is a conformal map from Ω to strip $\mathbb{R} \times (0,1)$ sending a to $-\infty$, b to $+\infty$.

Next, we will prove this main Thm.

(1) Discrete Complex Analysis:

Def: For $x \in \mathbb{L}_\delta$. $h : \mathbb{L}_\delta \to \mathbb{C}$.

$$\Delta_\delta h(x) = \frac{1}{4} \sum_{y : y \sim x} (h(y) - h(x)).$$

i) $h : \Omega_\delta \to \mathbb{C}$ is preharmonic if $\Delta_\delta h = 0$.

ii) $\sim$ is pre-superharmonic if $\Delta_\delta h \le 0$.

iii) $\sim$ is pre-subharmonic if $\Delta_\delta h \ge 0$

Rmk: Set (X_n) is simple random walk on $\mathbb{L}_\delta$ killed at the first time it exits Ω_δ. Then: h is preharmonic $\Leftrightarrow$ $h(X_n)$ is mart.

It's alike the conclusion: (conti. case)

(B_t) is n-dim BM on Ω. Then $h:\Omega\to\mathbb{C}$. harmonic.

$\Leftrightarrow$ $h(B_t)$ is a mart.

<u>Thm.</u> In $(\Omega; a, b)$ Dobrushin domain. $f\in C_B(\partial\Omega/\{a,b\})$ and h is harmonic on Ω. conti. on $\bar{\Omega}/\{a,b\}$. satisfying $h=f$ on $\partial\Omega/\{a,b\}$.

Let $(\Omega_\delta; a_\delta, b_\delta)$ is seq of discrete Dobrshin domains converging to $(\Omega; a, b)$ in Carathéodory sense.

If $f_\delta:\partial\Omega_\delta\to\mathbb{C}$ seq of uniformly bdd func's converges to f uniformly away from a and b.

and h_δ is unique preharmonic func. on Ω_δ. s.t. $h_\delta = f_\delta$ on $\partial\Omega_\delta$.

Then: $h_\delta \xrightarrow{u.c.c} f_\delta$ as $\delta\to 0$.

<u>Def:</u> For $f:\mathbb{L}_\delta\to\mathbb{C}$. $x\in\mathbb{L}_\delta^*$.

i) $\bar{\partial}_\delta f(x) =: \frac{1}{2}(f(E)-f(W)) + \frac{i}{2}(f(N)-f(S))$

ii) $f:\Omega_\delta\to\mathbb{C}$ is preholomorphic if $\bar{\partial}_\delta f = 0$.

<u>prop.</u> i) Sum of preholomorphic func's is preholomorphic.

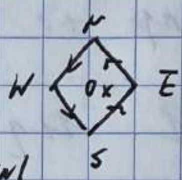
ii) Discrete contour integrals vanish in simply connected domain for pre-holomorphic functions.

iii) Primitive of preholomorphic func. in simply connected domain is well-def.

iv) If family of preholomorphic func. $(f_\delta)_{\delta>0}$ on $\Omega_\delta \xrightarrow{\text{u.c.c.}} f$ on Ω. Then f is holomorphic.

<u>Rmk</u>: Product of preholomorphic func's may not be pre-holomorphic.

<u>Pf</u>: ii) It's enough to prove for one unit face:



(since the integral on communal edges of adjacent faces will counteract each other)

Discrete integral is: $\sum \frac{f(a)+f(b)}{2}(ab) =$

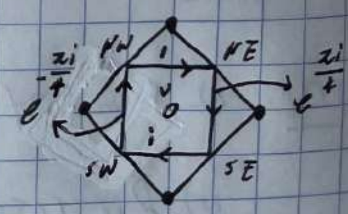
$\frac{1}{2}(f(S)+f(E))\cdot\delta(1+i)+\frac{1}{2}(f(E)+f(N))\cdot$

$\delta(-1+i)+\frac{1}{2}(f(N)+f(W))\,\delta(-1-i)+\cdots$

$=\delta\cdot 2i\cdot\bar\partial f(x)=0$.

iii) follows from ii)

iv) By Morera's Thm.



<u>Def</u>: i) For $e\in\vec{E}(\mathbb{L}^\delta)$, give its an orientation: clockwise around the white face. and associate a direction

$\ell(e)$ as in figure (i.e. $\ell(e) = \sqrt{\bar{e}}$.)

ii) $f : \Omega_\delta^\diamond \to \mathbb{C}$ is spin-holomorphic if $\forall e \in E(\Omega_\delta^\diamond)$. $P_{\ell(e)}(f(x)) = P_{\ell(e)}(f(y))$.

x, y are endpoints of e and $P_{\ell(e)}$ is orthogonal projection in direction $\ell(e)$.

Prop: f is s-holomorphic $\Rightarrow$ f is preholomorphic.

Pf: Dual graph of $\mathbb{L}_\delta^\diamond$ is $\mathbb{L}_\delta \cup \mathbb{L}_\delta^*$.

By s-holo: $\begin{cases} \mathrm{Re}\, f(NW) = \mathrm{Re}\, f(NE) \\ \mathrm{Re}\,(e^{-\frac{\pi i}{4}} f(NE)) = \mathrm{Re}\,(e^{-\frac{\pi i}{4}} f(SE)) \end{cases}$

$$\Rightarrow \begin{cases} f(NW) + \overline{f(NW)} = f(NE) + \overline{f(NE)} \\ f(NE) + i\,\overline{f(NE)} = f(SE) + i\,\overline{f(SE)} \end{cases}$$

Similarly: $\begin{cases} f(SE) - \overline{f(SE)} = f(SW) - \overline{f(SW)} \\ f(SW) - i\,\overline{f(SW)} = f(NW) - i\,\overline{f(NW)} \end{cases}$

$\Rightarrow$ Sum over : $0 = 2(1-i)\,\bar{\partial} f(v)$. $\forall v \in \Omega_\delta^\diamond$.

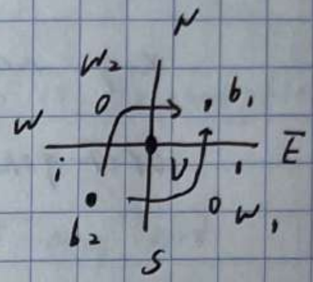
Thm: Ω is simply connected domain. $f : \Omega_\delta^\diamond \to \mathbb{C}$ is s-holomorphic func. $b_0 \in \Omega_\delta$. Then $\exists$ unique func $H : \Omega_\delta \cup \Omega_\delta^* \to \mathbb{C}$. s.t.

$H(b_0) = 1$. $H(b) - H(w) = \delta \cdot |P_{\ell(e)}(f(x))|^2$

for $\forall$ edge $e = \{x, y\}$ on $\Omega_\delta^\diamond$ border by a black $b \in \Omega_\delta$ and a white face $w \in \Omega_\delta^*$.

<u>Rmk</u>: i) Motivation of introducing s-holo func. is to square the func. in the main Thm.

ii) We can obtain:



$H(b_1) - H(b_2) =$

$\frac{\delta}{2} \operatorname{Im}\left(f(v)^2 (b_1 - b_2) \right)$

It follows from: $\begin{cases} H(b_1) - H(w_2) = \delta \, |P_{l(E)} f(v)|^2 \\ H(b_2) - H(w_2) = \delta \, |P_{l(S)} f(v)|^2 \end{cases}$

<u>Pf</u>: Prove: H is well-def

Check $H(b_1) - H(w_1) + H(w_1) - H(b_2) =$

$H(b_1) - H(w_2) + H(w_2) - H(b_2)$

$\Longleftrightarrow |P_{l(S)} f(v)|^2 - |P_{l(E)} f(v)|^2 =$

$|P_{l(W)} f(v)|^2 - |P_{l(N)} f(v)|^2.$

It follows from $|f(v)|^2 = |P_{l(S)} f(v)|^2 + |P_{l(N)} f(v)|^2$

$= |P_{l(W)} f(v)|^2 + |P_{l(E)} f(v)|^2.$

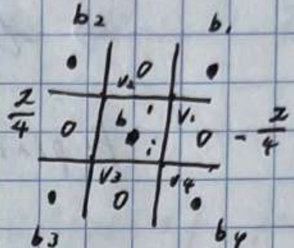
since $l(S) \perp l(N)$, $l(W) \perp l(E)$.

Set $H(b_1) = 1$, which uniquely determines H

<u>Denote</u>: i) $H^{\bullet} = H|_{\Lambda_\delta}$.

ii) $H^{\circ} = H|_{\Lambda_\delta^*}$.

Thm. If f is s-holomorphic. Then $H^\bullet$ is subharmonic and H° is superharmonic.



Pf: Only prove for $H^\bullet$:

Fix a $b \in \Omega_\delta$.

Note: $H(b_j) - H(b) = \frac{\delta}{2} \mathrm{Im}\left(f(v_j)^2 (b_j - b)\right)$

Denote $f(v_j) = a_j + i b_j$. By s-holo:

$$\begin{cases} a_3 = a_4 \triangleq a \\ a_4 - b_4 = a_1 - b_1 \triangleq \sqrt{2}\lambda \\ b_1 = b_2 \triangleq b \\ a_2 + b_2 = a_3 + b_3 \triangleq \sqrt{2} c \end{cases}$$

$$\Rightarrow \frac{2}{\delta} \sum_{j=1}^{4} \left(H(b_j) - H(b)\right) = 4 \sum_{cyc} a^2 + 4\sqrt{2}\left(b\lambda - bc - ac - a\lambda\right)$$

$$\geq 0$$

(2) Conformal Invariance:

For FK-Ising model in $(\Omega_\delta; a_\delta, b_\delta)$ with the Dobrushin b.c.

Lemma. $\phi_{(\Omega_\delta; a_\delta, b_\delta)}(\omega) = \frac{1}{Z} x^{o(\omega)} \sqrt{2}^{\ell(\omega)}$. where $x = p/\sqrt{2}(1-p)$. $\ell(\omega)$ is number of loops

Pf: Note that loops have correspond relation with clusters in $\Omega_\delta \cup \Omega_\delta^*$.

$\Rightarrow \ell(\omega) = k(\omega) + k(\omega^*) - 1$.

with Euler formula:

$\#V - O(w) + f(w) = 1 + k(w)$. $f(w) = k(w^*)$.

$\Rightarrow l(w) = 2k(w) + O(w) + \text{const}$.

Replace in $\phi(w) = \frac{1}{Z}\left(\frac{p}{1-p}\right)^{O(w)} 2^{k(w)}$.

$\Rightarrow \phi(w) = \frac{1}{Z}\left(p/\sqrt{2}(1-p)\right)^{O(w)} \sqrt{2}^{\,l(w)}$.

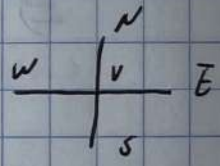
<u>Rmk</u>: For $p = p_{sd}(2) = \sqrt{2}/(1+\sqrt{2})$.

we have $x = 1$. So: $\phi(w) = \frac{1}{Z}\sqrt{2}^{\,l(w)}$.

<u>Lemma</u> For $p = p_{sd}(2)$. medial vertex $v \in \Lambda_\delta^\diamond / \partial\Lambda_\delta^\diamond$

we have: $F_\delta(N) - F_\delta(S) = i(F_\delta(E) - F_\delta(W))$

<u>Pf</u>: $F_\delta(e) = \mathbb{E}\left(I_{\{e\in\gamma\}}\, e^{\frac{i}{2}W_\gamma(e,b)}\right)$

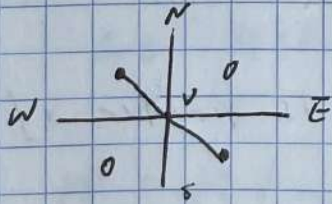


$= \frac{1}{Z}\sum_w \sqrt{2}^{\,l(w)} I_{\{e\in\gamma(w)\}} \exp\left(\frac{i}{2}W_{\gamma(w)}(e,b)\right)$.

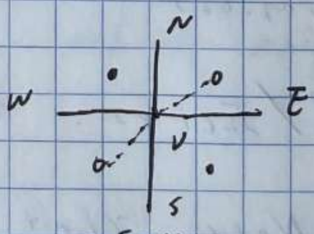
$=: \frac{1}{Z}\sum_w e_w$.

set $F(e) = \sum e_w = \frac{1}{2}\sum(e_w + e_{s(w)})$.

where we correspond w to $s(w)$:



w



$s(w)$

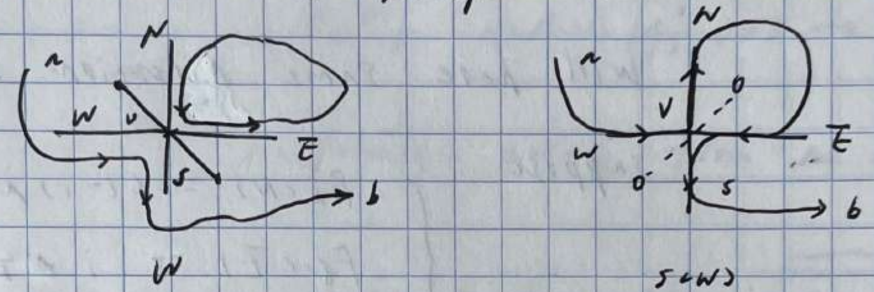
Denote N_w is e_w of $F(N)$. others analog.

Check: $N_W + N_{S(W)} - S_W - S_{S(W)} = i(E_W + E_{S(W)} - W_W - W_{S(W)})$

1') $\gamma(W)$ won't go through any N. S. W. E.

Then it holds trivially.

2') $\gamma(W)$ go through two edges of N. S. W. E (Assume: enter through W and exit through S)



($\gamma(W)$ will cross even edges of v. Since it will enter and exit. And case 2') also contains "go through 4 edges of v")

Note: $W_\gamma(W, b) = W_\gamma(S, b) - \frac{\pi}{2}$ whatever it's in W or $S(W)$

$W_\gamma(W, b) = W_\gamma(E, b) - \pi$
$= W_\gamma(N, b) + \frac{\pi}{2}$

	W	E	N	S
$\Rightarrow$ W	W_W	0	0	$\exp(\frac{1}{4}\pi i) W_W$
S_W	$W_W/\sqrt{2}$	$e^{\frac{\pi i}{2}} W_W/\sqrt{2}$	$e^{-\frac{\pi i}{4}} W_W/\sqrt{2}$	$e^{\frac{\pi i}{4}} W_W/\sqrt{2}$

(Divide $\sqrt{2}$ is because a loop will vanish from W to $S(W)$: $l(W) = l(S(W)) + 1$)

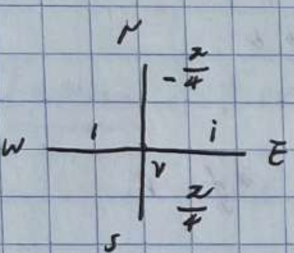
<u>Lemma</u>. Fix $p = p_{sd}(2)$. Then vertex Fermionic observable F_δ is s-holomorphic.

<u>Pf</u>: $F_\delta(v) = \frac{1}{2}(F_\delta(N) + F_\delta(S) + F_\delta(W) + F_\delta(E))$.

WLOG. Assume the direction of γ passing b_δ is $\longrightarrow$ E. (Or multiple $e^{i\theta}$)

Then $F_\delta(e) = \mathbb{E}(I_{e \in \gamma}\, e^{\frac{i}{2} W_\gamma(e, b)})$

will have same direction as $l(e)$.



Suppose:
$$\begin{cases} F_\delta(N) = (1-i)\,a_N, & a_N \in \mathbb{R} \\ F_\delta(E) = i\,a_E, & a_E \in \mathbb{R} \\ F_\delta(W) = a_W, & a_W \in \mathbb{R} \\ F_\delta(S) = (1+i)\,a_S, & a_S \in \mathbb{R}. \end{cases}$$

By Lemma above. we have:

$a_E = a_S - a_N$. $a_W = a_N + a_S$

$\Rightarrow F_\delta(v) = F_\delta(N) + F_\delta(S) = F_\delta(W) + F_\delta(E)$.

with $l(N) \perp l(S)$. $l(W) \perp l(E)$.

So: $F_\delta(N) = P_{l(N)}(F_\delta(v))$ ....

generally $F_\delta(e) = P_{l(e)}(F_\delta(v))$.

for $\forall v \in \Omega_\delta^\diamond$. $v \sim e$.

For $p = p_{sd}(2)$. A is black face bordering Ω_δ

By Lemma above. Def: $H_\delta : \Omega_\delta \cup \Omega_\delta^* \to \mathbb{R}$.

s.t. $H_\delta(A)=1$. $H_\delta(B)-H_\delta(W) = |P_{\ell(e)}(F_\delta(x))|^2$

$= |P_{\ell(e)}(F_\delta(y))|^2$ for medial $e=\{x,y\}$ bordered

by a black face $B \in \Lambda_\delta$. white face $W \in \Lambda_\delta^*$.

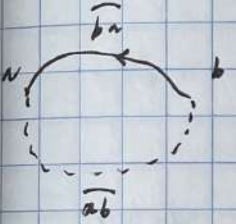
(Apply on $F_\delta(x)/\sqrt{\delta}$. so δ disappears !)

Lemma. i) $H_\delta^\bullet = 1$ on (ba). and $H_\delta^\bullet(B) \to 0$. $(\delta \to 0)$

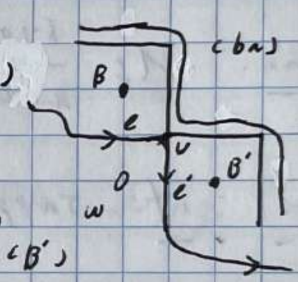
on (ab) uniformly away from a and b.

ii) $H_\delta^0 = 0$ on (a^*b^*) and $H_\delta^\circ(W) \to 1$. $(\delta\to 0)$

on (b^*a^*). uniformly away from a and b.



Pf: Only prove i): For B. B' on (ba)



$H_\delta^\bullet(B) - H_\delta^\bullet(B') =$

$H_\delta^\bullet(B) - H_\delta^\circ(W) + H_\delta^\circ(W) - H_\delta^\bullet(B')$

$= |P_{\ell(e)}(F_\delta(v))|^2 - |P_{\ell(e')}(F_\delta(v))|^2$. $v \in \Lambda_\delta^\diamond$.

$= |F_\delta(e)|^2 - |F_\delta(e')|^2$ (By Argue of Lemma above)

$= 0$.

Since if $\gamma(\omega)$ crosses e. then it will

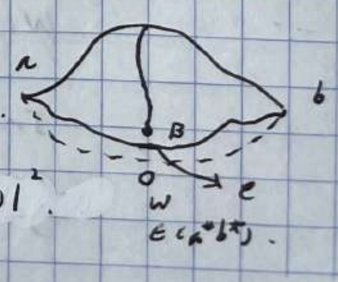
also cross e'. $W_\gamma(e,b) = W_\gamma(e',b) - \frac{\pi}{2}$.

$\Rightarrow H_\delta^\bullet(B) = H_\delta(A) = 1$. $\forall B$ on (ba).

Similarly. $H_\delta^0 \equiv 0$. on (a^*b^*).

As for $B \in (ab)$. and its

adjacent white vertex $W \in (a^*b^*)$.



$H_\delta^\bullet(B) = H_\delta^\bullet(B) - H_\delta^\circ(W) = |F_\delta(e)|^2$.

$= \mathbb{E}_\phi^2 (I_{[e \in \gamma]} \exp (\frac{i}{2} W_\gamma (e, b_\delta)))$

$\le \phi (e \in \gamma)^2 \le \phi^2 (B \leftrightarrow (ba))$

Suppose B is at distance r from (ba).

Set $U_\delta = B + [-r, r]^2$.

$\phi (B \leftrightarrow (ba)) \le \phi^1_{U_\delta} (B \leftrightarrow \partial U_\delta) \xrightarrow[u]{\delta \to 0} 0$

(Since $\delta \to 0$. there will be infinite number of edges in path from B to ∂U_δ. $\phi^1_{p_c} (0 \leftrightarrow \infty) = 1$)

Similar for $H_\delta^0 \to 1$ along (ba) away $a.b$.

<u>Lemma.</u> $H_\delta \xrightarrow[u.c.c]{\delta \to 0} \mathrm{Im}\,\phi$.

<u>Pf</u>: Suppose: $h_\delta^\bullet$ preharmonic with same boundary condition as $H_\delta^\bullet$.

h_δ° preharmonic with same boundary condition as H_δ°.

$\Rightarrow H_\delta^\bullet \le h_\delta^\bullet$ and $H_\delta^\circ \ge h_\delta^\circ$. (By def)

For $b \in \Omega_\delta$. $w \in \Omega_\delta^*$. neighbours.

$h_\delta^\circ (w) \le H_\delta^\circ (w) \le H_\delta^\bullet (b) \le h_\delta^\bullet (b)$

(Note $H_\delta^\bullet (B) - H_\delta^\circ (w) = \delta \, | P_{\ell(e)} (f \cdots) |^2 \ge 0$)

With $h_\delta^\bullet . h_\delta^\circ \xrightarrow{\delta \to 0} \mathrm{Im}\,\phi$ (harmonic. have same boundary cond. as $h_\delta^\circ . h_\delta^\bullet$.)

So: $H_\delta \longrightarrow \mathrm{Im}\,\phi$. as $\delta \to 0$.

Return to Pf of Main Thm:

$(H_\delta)_{\delta>0}$ u.c.c. converge $\Rightarrow$ $(\frac{1}{\sqrt{2\delta}} F_\delta)_{\delta>0}$ is tight:

For $\forall$ convergent subseq. $\frac{1}{\sqrt{2\delta_n}} F_{\delta_n} \xrightarrow{n\to\infty} f$

Since $H_{\delta_n}(\eta) - H_{\delta_n}(x) = \frac{1}{2\delta_n} \operatorname{Im} \int_x^\eta F_{\delta_n}^2(z)\,dz$.

$\xrightarrow{u.c.c} \operatorname{Im}(\phi(\eta) - \phi(x)). = \operatorname{Im} \int_x^\eta f^2(z)\,dz$

Fix x, we have both sides are holomorphic of η.

By uniqueness Thm: $\phi(\eta) - \phi(x) = \int_x^\eta f^2\,dz + \text{const}(x)$

$\Rightarrow \phi'(\eta) = f^2(\eta)$.

So: $\frac{1}{\sqrt{2\delta}} F_\delta \xrightarrow{u.c.c} \sqrt{\phi'}$. as $\delta \to 0$.

<u>Cor</u>. The exploration path of FK-Ising with Dobrushin b.c. converges to $SLE_{\frac{16}{3}}$.