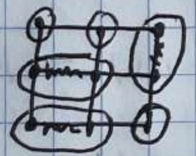
Random Cluster Model.

- RCM. also called FK-percolation. It's generalization of Bernoulli percolation where it kept on edges.

For $G=(V,E)$. finite graph.

configuration $w \in \{0,1\}^E$. $o(w)$. $c(w)$. $k(w)$ are number of open edges. closed edges, clusters respectively. where a cluster is a set containing most vertexs connected by open edges.

e.g. w



"~" means open edge.

Then $o(w)=3$. $c(w)=9$. $k(w)=6$

For edge-parameter $p \in [0,1]$. cluster parameter $q>0$. FK-percolation is a p.m. on G:

$$\phi_{p,q,G}(w) \propto p^{o(w)}(1-p)^{c(w)} q^{k(w)}$$

i.e. $\phi_{p,q,G}(w) = p^{o(w)}(1-p)^{c(w)} q^{k(w)} / Z_{p,q,G}$. where

$Z_{p,q,G} = \sum_w p^{o(w)} \cdots q^{k(w)}$. normalization.

(1) Boundary Conditions:

Def: Boundary condition ξ is a partition of ∂G and identify vertices in same component as a cluster. (Before Cnfg.)

Rmk: Under b.c. ξ, k will depend ξ:

$$\phi^{\xi}_{p,q,G}(w) \propto p^{o(w)}(1-p)^{c(w)} 2^{k(w,\xi)}.$$

e.g. i) Wire. b.c. $\phi^{1}_{p,q,G}$: $\{\partial G\}$.

ii) Free. b.c. $\phi^{0}_{p,q,G}$: $\bigcup_{e\in\partial G}\{e\}$.

iii) Induced by a cnfg. outside G.

(i.e. ∂G has partition from G_0/G. if $G\subset G_0$)

iv) Dobrushin. b.c. : It's kind of sym. cnfg.

Prop. (Domain Markov Property)

For $\tilde{G}\subset G$. $\psi\in\{0,1\}^{E(G)/E(\tilde{G})}$. Then:

$$\phi^{\xi}_{p,q,G}(X \mid w(e)=\psi(e).\ \forall e\in E(G)/E(\tilde{G})) = \phi^{\psi\xi}_{p,q,\tilde{G}}(X).$$

where $\psi\xi$ is b.c. induced by $G/\tilde{G}$. and

$\phi^{\xi}_{p,q,G}(X) =: \mathbb{E}_{\phi}(X)$

Pf: $\forall A\in\sigma(\psi)$. $I_A\, d\phi^{\xi} = d\phi^{\psi\xi}$.

(2) **FKG Inequility**:

Def: i) For b.c. ξ,η. $\xi\leq\eta$ if: vertex x,y belongs to same component under ξ $\Rightarrow$ So they do under η.

ii) p.m. μ_1. μ_2. $\mu_1 \le_{st} \mu_2$ if $\mu_1(A) \le \mu_2(A)$

for $\forall A$ increasing event.

<u>Lemma.</u> (Holley Inequality)

$\forall w$. $\mu_1(w)$. $\mu_2(w) > 0$ on G. finite. If

$\mu_2(w^e)\mu_1(\eta_e) \ge \mu_2(w_e)\mu_1(\eta^e)$. $\forall e \in G$.

and $\forall \eta \le w$. Then $\mu_1 \le_{st} \mu_2$.

<u>Thm.</u> (FKG. Inequality)

Fix $p \in [0,1]$. $q \ge 1$. Finite graph G and b.c.

ξ. Then: $\phi^{\xi}_{p,q,G}(A \cap B) \ge \phi^{\xi}_{p,q,G}(A)\,\phi^{\xi}_{p,q,G}(B)$ for

$\forall$ increasing events A, B.

<u>Pf</u>: $\Leftrightarrow$ $\phi^{\xi}_{p,q,G}(B \mid A) \ge \phi(B)$. fix A.

Set $\mu_2 = \phi(\cdot \mid A)$. $\mu_1 = \phi(\cdot)$

Check cond. of Lemma:

$$\frac{\mu_1(\eta^e)}{\mu_1(\eta_e)} = \frac{p}{1-p}\, q^{k(\eta^e,\xi) - k(\eta_e,\xi)}$$

$$\frac{\mu_2(w^e)}{\mu_2(w_e)} = \frac{p}{1-p}\, q^{k(w^e,\xi) - k(w_e,\xi)}.$$ (Restrict $w_e \in A$ for $\mu_2 > 0$ and $w^e \in A$ as well)

$\Leftrightarrow$ prove: $k(\eta_e,\xi) - k(\eta^e,\xi) \ge k(w_e,\xi) - k(w^e,\xi)$.

It follows from: $w \ge \eta$.

(Both sides take values in $\{0, 1\}$)

So we proved it on A. If $\phi(A) = 0$. Then trivial.

Cor. Fix $p \in [0,1]$. $q \geq 1$. finite graph G.

$\forall$ b.c. $\xi \leq \eta$. We have: $\phi^{\xi}_{p,q,G} \leq_{st} \phi^{\eta}_{p,q,G}$.

Pf: Set $Y(\omega) =: q^{k(\omega,\eta) - k(\omega,\xi)}$. $\uparrow$ on ω.

(follows from argue above).

$$\Rightarrow \quad d\phi^{\eta} / d\phi^{\xi} = Y / \phi^{\xi}(Y). \quad \text{R-N. derivate.}$$

(By uniqueness. check directly).

$$\forall A \uparrow. \quad \phi^{\eta}(A) = \frac{\phi^{\xi}(Y I_A)}{\phi^{\xi}(Y)} \overset{(FKG)}{\geq} \phi^{\xi}(A).$$

Cor. (Monotonicity)

Fix $p \leq p'$. b.c. ξ on finite graph G. $q \geq 1$.

We have: $\phi^{\xi}_{p,q,G} \leq_{st.} \phi^{\xi}_{p',q,G}$.

Pf: Set $Y(\omega) = \left(\frac{p'}{1-p'} \cdot \frac{1-p}{p}\right)^{o(\omega)}$ $\uparrow$ on ω.

$$\Rightarrow \quad d\phi_{p'} / d\phi_p = Y / \phi_p(Y).$$

$$\phi_{p'}(A) = \frac{\phi_p(Y I_A)}{\phi_p(Y)} \overset{(FKG)}{\geq} \phi_p(A). \quad \forall A \uparrow.$$

Cor. (Finite Energy Property)

Fix $p \in [0,1]$. $q \geq 1$. finite graph G. b.c. ξ.

We have: $\frac{p}{p + (1-p)q} \leq \phi^{\xi}_{p,q,G}(w(f) = 1 \mid w(e) = \psi(e)$.

$\forall e \in E(G)/\{f\}) \leq p$. for $\psi \in \{0,1\}^{E(G)/\{f\}}$.

Pf: By Domain Mark: $\phi^{0}(w(f)=1) \leq \sim \leq \phi^{1}(w(f)=1)$

(3) Phase Transition:

① Infinite Volum Measure:

Note that $\phi^{1}_{p,q,G}$ depends on the graph G is finite. We want to extend it to infinite graph.

Def: (ξ_n) is seq of b.c. on (Λ_n). We say: $\phi^{\xi_n}_{p,q,\Lambda_n}$ converges to infinite volum measure $\phi_{p,q}$ if:

$\lim_{n\to\infty} \phi^{\xi_n}_{p,q,\Lambda_n}(A) = \phi_{p,q}(A)$. for $\forall A$.

only depends on finite edges.

Prop. Fix $p\in[0,1]$. $q\geq 1$. There exist two infinite volum measure $\phi^{0}_{p,q}$. $\phi^{1}_{p,q}$. s.t.

$$\begin{cases} \lim_{n} \phi^{1}_{p,q,\Lambda_n}(A) = \phi^{1}_{p,q}(A) \\ \lim_{n} \phi^{0}_{p,q,\Lambda_n}(A) = \phi^{0}_{p,q}(A). \end{cases}$$

for $\forall A$ depends on finite edges.

Pf: Only prove it for ϕ^{1}. (ϕ^{0} similar)

1) $\phi^{1}_{\Lambda_{n+1}}(A) = \phi^{1}_{\Lambda_{n+1}}(\phi^{1}_{\Lambda_n}(A \mid \Lambda_{n+1}/\Lambda_n))$

$\overset{DMP}{=} \phi^{1}_{\Lambda_{n+1}}(\phi^{\xi}_{\Lambda_n}(A))$

$\overset{(KFG)}{\leq} \phi_{\Lambda_m}(A)$. $\therefore (\phi_{\Lambda_n}(A)) \downarrow$

Set $p(A) = \lim_{n} \phi'_{\Lambda_n}(A)$.

2°) Note (1°) is for A is increasing event.

For general event, it can be expressed

in increasing events. (By "$\cup$", "$\cap$", "c".)

3°) Extend $p(\cdot)$ from algebra to σ-algebra.

which becomes a p.m. denoted by $\phi'_{p,q}$.

② Ergodic :

Lemma. For infinite volume measure $\phi'_{p,2}$, $\phi^0_{p,2}$.

They're translation-invariant and ergodic.

Pf: Only prove it for ϕ'. (ϕ^0 similar)

1°) For $A \uparrow$ depend on finite edges. $|x|=1$.

$$\phi'(A) = \lim \phi'_{\Lambda_n}(A)$$

$$= \lim \phi'_{z_x\Lambda_n}(z_xA)$$

Note $\Lambda_{n-1} \subset z_x\Lambda_n \subset \Lambda_{n+1}$.

$$\phi'_{z_x\Lambda_n} = \phi'_{z_x\Lambda_n}(\phi(z_xA \mid z_x\Lambda_n/\Lambda_{n-1}))$$

$$\leq \phi'_{\Lambda_{n-1}}(z_xA).$$

$\phi'_{\Lambda_{n+1}}(z_xA) \leq \phi'_{z_x\Lambda_n}(z_xA)$. Similar.

$$\Rightarrow \phi'(z_xA) = \phi'(A).$$

2') Prove: $B, C \uparrow$ in Λ_n. Then:

$$\phi'(B \cap \tau_x C) \xrightarrow{|x| \to \infty} \phi'(B)\phi'(C).$$

First. $\phi'(B \cap \tau_x C) \overset{(FKG)}{\geq} \phi'(B)\phi'(\tau_x C)$

$$= \phi'(B)\phi'(C).$$

Conversely, for $n \gg |x| \gg N$.

$$\phi'_{\Lambda_n}(B \cap \tau_x C) = \phi'_{\Lambda_n}(\phi'_{\Lambda_n}(B \cap \tau_x C \mid \Lambda_n / \Lambda_{\frac{|x|}{2}}))$$

$$= \phi'_{\Lambda_n}(I_{\tau_x C}\, \phi'_{\Lambda_n}(B \mid \Lambda_n / \Lambda_{\frac{|x|}{2}}))$$

$$\overset{(DPC)}{\leq} \phi'_{\Lambda_n}(\tau_x C)\, \phi'_{\Lambda_{\frac{|x|}{2}}}(B).$$

Set $n \to \infty$. $|x| \to \infty$.

Similar as before: $\phi'(A) = \phi'(A)^2 + O(\varepsilon)$

<u>Lemma</u>: Fix $q \geq 1$. For $\phi'_{p,q}$ or $\phi^0_{p,q}$: either there's no ∞-cluster a.s. or exist unique ∞-cluster a.s.

<u>Pf</u>: 1') $2 \leq n_0 < \infty$:

$\forall \varepsilon > 0$. $\exists N$. s.t. ϕ'(All n_0 ∞-clusters intersect Λ_N) $\geq 1 - \varepsilon$.

Denote E_N = { All edges in Λ_N open }.

$\Rightarrow \phi'(A_1) \geq \phi'(E_N \cap \{\text{All } n_0 \cdots \Lambda_N\})$

$$\geq (1-\varepsilon)\left(\frac{p}{p+(1-p)q}\right)^{\#E(\Lambda_N)} > 0$$

follows from DMP and Finite energy prop.

2') For $n_0 = \infty$:

Use trifurcation and replace "indept" with "Finite energy property". argue as in Bernoulli bond percolation.

③ Critical Point:

Thm. Fix $q \geq 1$. There $\exists\ p_c = p_c(q) \in [0,1]$. s.t.

i) For $p > p_c$. $\forall$ infinite volum measure has ∞-cluster a.s.

ii) For $p < p_c$. $\forall$ infinite volum measure has no ∞-cluster. a.s.

Rmk: i) p_c is indept of choice of infinite volum measures

ii) ∞-cluster may not be unique under some infinite volum measure if $p > p_c$.

Lemma. $\phi^0(w(e)=1) = \phi^1(w(e)=1) \Rightarrow \phi^1 = \phi^0$.

Pf: Prove: $\phi^0(A) = \phi^1(A)$. $\forall A \uparrow$ depends on Λ_N. (Then extend to $\mathcal{F}$).

Note: $\forall n$. $\phi^1_{\Lambda_n} \geq_{st} \phi^0_{\Lambda_n}$.

So. $\exists$ monotone coupling $(w_0, w_1) \in \{0,1\}^E \times \{0,1\}^E$

with p.m. $\mathbb{P}_n((w_0, w_1))$, s.t. its marginal

dist. is $\phi^0_{\Lambda_n}$ and $\phi^1_{\Lambda_n}$. $\mathbb{P}_n(w_0 \le w_1) = 1$

$$0 \le \phi^1_{\Lambda_n}(A) - \phi^0_{\Lambda_n}(A) = \mathbb{P}_n(w_1 \in A) - \mathbb{P}_n(w_0 \in A)$$

$$\overset{(w \le w')}{=} \mathbb{P}_n(w_1 \in A, w_0 \notin A)$$

$$\le \mathbb{P}_n(\exists e \in \Lambda_n, w_1(e) = 1, w_0(e) = 0)$$

$$\le \sum_{e \in \Lambda_n} \mathbb{P}_n(w_1(e) = 1, w_0(e) = 0)$$

$$= \sum_{\Lambda_n} (\phi^1_{\Lambda_n}(w(e)=1) - \phi^0_{\Lambda_n}(w(e)=1)) \xrightarrow[n\to\infty]{\text{cond.}} 0$$

<u>Lemma.</u> Fix $q \ge 1$. $\phi^0_{p,q} = \phi^1_{p,q}$ for all but countably

many $p \in [0,1]$.

<u>Rmk:</u> It's intuitive. Since when $\Lambda_n \xrightarrow{n\to\infty}$

G. infinite graph. b.c. can be ignored

<u>Pf:</u> By Lemma above. We need to prove:

$\phi^0_p(w(e)=1) = \phi^1_p(w(e)=1)$. for $p \sim \ldots$

Define free energy:

$f(p,q) = \lim\limits_{n\to\infty} \log Z^{\xi}_{p,q,\Lambda_n} / \# E(\Lambda_n)$. where

$Z^{\xi}_{p,q,\Lambda_n} = \sum\limits_{w} p^{o(w)} (1-p)^{c(w)} q^{k(\xi, w)}$.

<u>Rmk:</u> f is indept of choice of ξ in

the limit. Since $|\partial\Lambda_n| \sim O(n)$. But

$|E(\Lambda_n)| \sim O(n^2)$.

Note: $Z^{s}_{p,q,\Lambda_n} = \sum_{w} \left(\frac{p}{1-p}\right)^{O(w)} 2^{k(w,s)} (1-p)^{|E(\Lambda_n)|}$

set $\pi = \log(p/1-p)$. $\therefore \frac{p}{1-p} = e^{\pi}$.

Set $\tilde{f}^{s}_{n}(\pi,q) = \ln(1+e^{\pi}) + \ln Z^{s}_{p,q,n} / |E(\Lambda_n)|$

$= \ln \sum_{w} e^{\pi O(w)} 2^{k(w,q)} / |E(\Lambda_n)|$

$\partial_{\pi} \tilde{f}^{s}_{n}(\pi,q) = \phi^{s}_{p,q,\Lambda_n}(O(w)) / |E(\Lambda_n)| \geq 0$

which is increasing on π.

$\Rightarrow \tilde{f}^{s}_{n}(\pi,q)$ is convex.

So: $\tilde{f}^{s}_{n}(\pi,q) \to \ln(1+e^{\pi}) + \tilde{f}(\pi,q)$.

$= \ln(1+e^{\pi}) + f(p,q)$ convex.

$\tilde{f}(\pi,q)$ is differentiable on π except for countable points.

claim: $\phi^{1}_{p,q}(w(e)=1) = \phi^{0}_{p,q}(w(e)=1)$ on differentiable points of $\tilde{f}(\pi,q)$.

Note: $\partial_{\pi} \tilde{f}^{1}_{n}(\pi,q) = \frac{1}{|E(\Lambda_n)|} \sum_{E(\Lambda_n)} \phi^{1}_{p,q,\Lambda_n}(w(e)=1)$

$\to \phi^{1}_{p,q}(w(e)=1)$

$\partial_{\pi} \tilde{f}^{1}_{n}(\pi,q) \to \partial_{\pi} \tilde{f}$. at diff. points.

So as $\partial_{\pi} \tilde{f}^{0}_{n}(\pi,q)$ does. ($\to \partial_{\pi}\tilde{f}$ and $\phi^{0}_{p,q}$)

Return to pf:

Set $p_c = \sup\{p \mid \phi^0_{p,q}(0 \leftrightarrow \infty) = 0\}$.

1') $p > p_c$.

$$\phi_{p,q}(0 \leftrightarrow \infty) \geq \phi^0_{p,q}(0 \leftrightarrow \infty) \overset{\text{(Ergodic)}}{=} 1$$

(Note $\{\exists!\ \infty\text{-cluster}\}$ isn't a increasing event. So we can't conclude it)

2') $p < p_c$.

$$\exists\ \tilde{p} \in (p, p_c).\ \text{s.t.}\ \phi^1_{\tilde{p},q}(0 \leftrightarrow \infty)$$

$$\overset{\text{(Lemma)}}{=} \phi^0_{\tilde{p},q}(0 \leftrightarrow \infty) = 0 \geq \phi_{\tilde{p},q}(0 \leftrightarrow \infty)$$

$$\geq \phi_{p,q}(0 \leftrightarrow \infty) \geq 0.$$

(4) Critical Value: Self-dual Point:

Thm: Consider Random Cluster model on $\mathbb{Z}^2$ with cluster weight $q \geq 1$. Then critical point $p_c(q) = \sqrt{q}/(1+\sqrt{q})$.

Rmk: We call $p_c(q)$ by Self-dual point p_{sd}.

e.g. $q = 1$. $p_c(1) = \frac{1}{2}$. i.e. in $BBP(\mathbb{Z}^2)$

Lemma. The dual configuration of random cluster model on G with (p, q) and b.c. ξ. is random cluster model with (p^*, q) on G^* with b.c. ξ^*. st.

$pp^* / ((1-p)(1-p^*)) = q$. where $G^* : w^*(e^*)$
$= 1 - w(e)$.

Pf: Note that: $o(w) = c(w^*) = \#E^* - o(w^*)$.

with Euler's Formula:

$\#V - o(w) + f(w) = 1 + k(w)$. $\quad f(w) = k(w^*)$

i.e. $k(w) = k(w^*) + o(w^*) + \text{const}$.

Note $\phi^*(w^*) = \phi(w) \propto \left(\frac{p}{1-p}\right)^{o(w)} q^{k(w)}$

$$\propto \left(\frac{p}{1-p}\right)^{-o(w^*)} q^{k(w^*)+o(w^*)}$$

$$= \left(\frac{q(1-p)}{p}\right)^{o(w^*)} q^{k(w^*)}$$

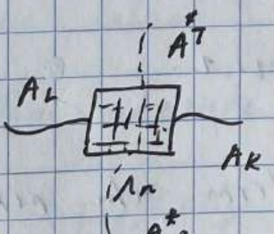
$$= \left(\frac{p^*}{1-p^*}\right)^{o(w^*)} q^{k(w^*)}.$$

Rmk: Set $p = p^* \Rightarrow p^* = \sqrt{q}/(1+\sqrt{q})$.

So that's why we call it self-dual point.

Lemma. Fix $q \geq 1$. we have: $\phi^0_{p_{sd}(q), q}(0 \leftrightarrow \infty) = 0$.

Pf: It's identical as we have proved in BBP:



If $\phi(0 \leftrightarrow \infty) > 0$. Then $\phi(0 \leftrightarrow \infty) = 1$.

$\mathbb{P}(A_L \cap A_R \cap A_T^* \cap A_B^* \cap E_n) \geq$ (By FKG)

$(1-4\varepsilon)\left(p/(p+(1-p)q)\right)^{\#\Lambda_n} > 0$.

(Note ϕ, ϕ^* are identical if $p = p_{sd}(q)$).

which is contradicted with "∞-cluster is unique".

<u>Lemma.</u> (Exponential Decay)

i) $p < p_c$. $\Rightarrow \exists\ c(p) > 0$. s.t. $\forall n \geq 1$.

$$\phi^1_{p,q,\Lambda_n}(0 \leftrightarrow \partial \Lambda_n) \leq e^{-c(p)n}.$$

ii) $p > p_c$. $\Rightarrow \exists\ c$. s.t. $\phi^1_{p,q}(0 \leftrightarrow \infty) \geq c(p-p_c)$

<u>Return to pf of Thm:</u>

We have prove $p_c(q) \geq p_{sd}(q)$. by Lemma.

If $p_c(q) > p_{sd}(q)$. Then contradict with exponential decay and RSW estimate

Next. we consider continuity of ϕ at $p_c(q)$:

<u>Thm.</u> i) For $1 \leq q \leq 4$. We have: $\phi^1_{p_c,q}(0 \leftrightarrow \infty) = 0 = \phi^0_{p_c,q}(\square)$

ii) For $q > 4$. We have: $\phi^1_{p_c,q}(0 \leftrightarrow \infty) > 0$ and.

$\phi^0_{p_c,q}(0 \leftrightarrow \infty) = 0$.

<u>Cor.</u> i) For $1 \leq q \leq 4$. $\phi^1_{p_c(q),q} = \phi^0_{p_c(q),q}$

ii) For $q > 4$. $\phi^1_{p_c(q),q} \neq \phi^0_{p_c(q),q}$.

<u>pf:</u> It's immediately follows from the next Lemma:

<u>Lemma.</u> $p \in [0,1]$. $q \geq 1$. If $\phi^1(0 \leftrightarrow \infty) = 0$. Then

$$\phi^0(w(e) = 1) = \phi^1(w(e) = 1)$$

<u>Rmk:</u> By Lemma we have proved before.

$$\Rightarrow \phi^0 = \phi^1.$$

Pf: Note that $\phi^1(w(e)=1) = \lim_{m\to\infty} \phi^1_{\Lambda_m}(w(e)=1)$

Set $n < m$. Consider $C = \{x \in \Lambda_n \mid x \leftrightarrow \partial\Lambda_n\}$

$\Rightarrow \{0 \nleftrightarrow \partial\Lambda_n\} = \{0 \notin C\}$. (Consider finite first)

$\therefore \phi^1_{\Lambda_m}(w(e)=1,\ 0 \nleftrightarrow \partial\Lambda_n) =$

$\sum_{0\notin A} \phi^1_{\Lambda_m}(w(e)=1,\ C=A) =$

$\sum_{0\notin A} \phi^1_{\Lambda_m}(w(e)=1 \mid C=A)\, \phi^1_{\Lambda_m}(C=A)$

$= \sum_{0\notin A} \phi^0_u(w(e)=1)\, \phi^1_{\Lambda_m}(C=A)$

$\leq \phi^0(w(e)=1)\, \phi^1_{\Lambda_m}(0 \nleftrightarrow \partial\Lambda_n)$

where $u = \Lambda_m / A$. (Note ∂u must be closed)

containing 0. The last inequi follows from

$\phi^1_{\Lambda_m}(A) \uparrow$ on n. if $A \uparrow$.

Set $m \to \infty$. $n \to \infty$. with $\phi^1(0 \nleftrightarrow \infty) = 1$.

So: $\phi^0(w(e)=1) \geq \phi^1(w(e)=0)$.

Thm. Fix $q \geq 1$. The unique edge-weight $p \in [0,1]$ for which there can exist distinct ∞-volume measure is $p_c(q) = p_{sd}(q)$.

Pf: For $p < p_{sd}(q)$. $\exists\, \tilde{p} \in (p, p_{sd}(q))$. st $\phi^0_{\tilde{p}} = \phi^1_{\tilde{p}}$

$\phi^1_p(0 \leftrightarrow \infty) \leq \phi^1_{\tilde{p}}(0 \leftrightarrow \infty) = \phi^0_{\tilde{p}}(0 \leftrightarrow \infty)$

$= 0$

For $p > p_{sd}(q)$. $\Rightarrow p^* < p_{sd}(q)$. So:

$\phi^1_p = \phi^0{}_{p^*} = \phi^1{}_{p^*} = \phi^0_p$.