Bernoulli Site Percolation.

Triangular Lattice

$G = (V(\mathbb{T}), E(\mathbb{T}))$.

For $G = (V, E)$.

Configuration: $w \in \{\bullet, \circ\}^V$.

with p.m. $\mathbb{P}_p$ on G:

$\mathbb{P}_p(w(x) = \bullet) = p$. $\mathbb{P}_p(w(x) = \circ) = 1 - p$.

vertex of G are indept as BBP in $\mathbb{Z}^2$.

Rmk: i) The graph (triangular lattice) is equi.:

It's honeycomb lattice (ignore edges of $G(\mathbb{T})$)

ii) One advantage of G is that it doesn't permit the white path intersects with the black path.

iii) Similarly argue:

Monotonicity of $\mathbb{P}_p$. FKG Inequality, Russo's. Formula all holds in BSP.

iv) Phase transition and exponential decay hold:

$\theta(p) = \mathbb{P}_p(0 \overset{B}{\longleftrightarrow} \infty)$. $\uparrow$ on p. where $\{0 \overset{B}{\longleftrightarrow} \infty\}$ means: $\exists$ Black path connects 0 with ∞.

$\exists p_c \in (0, 1)$. $\theta(p) > 0$ if $p > p_c$; $= 0$ if $p < p_c$

$\mathbb{P}_p(0 \longleftrightarrow \partial H_n) \leq e^{-cn}$. $\exists c > 0$. if $p < p_c$

(1) Critical value:

Thm For BSP in $\mathbb{T}$. We have $p_c = \frac{1}{2}$.

and $\theta(p)$ conti. at p_c.

Pf: It follows from the next Thm.

with identical argue as BBP.

(RSW estimate will also be proved

as follow).

Thm For BSP on $\mathbb{T}$. There $\exists\, c > 0$ s.t.

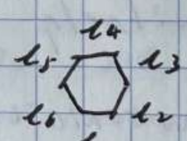
$\forall n \geq 1$. $\mathbb{P}_{\frac{1}{2}}(0 \xleftrightarrow{B} \partial H_n) \leq n^{-c}$.

where H_n is hexagon of side of

length n:  . center at 0.

Pf: 1°) RSW estimate:

consider  (with $\forall$ scaling, position)

Number the sides: 

Claim: $\mathbb{P}_{\frac{1}{2}}(l_1 \xleftrightarrow{B} l_4) \geq 1/9$.

Note $\{l_1 \xleftrightarrow{B} l_3 \cup l_4\}^c = \{l_2 \xleftrightarrow{W} l_5 \cup l_6\}$.

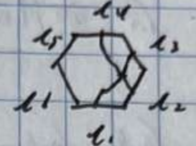
By symmetry, they have same prob.

So: $\mathbb{P}_{\frac{1}{2}}(l_1 \xleftrightarrow{B} l_3 \cup l_4) = \frac{1}{2}$

$\leq \mathbb{P}_{\frac{1}{2}}(l_1 \xleftrightarrow{B} l_3) + \mathbb{P}_{\frac{1}{2}}(l_1 \xleftrightarrow{B} l_4)$

To replace $\mathbb{P}_{\frac{1}{2}}(l_1 \overset{B}{\leftrightarrow} l_3)$:

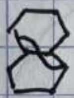
$$\mathbb{P}_{\frac{1}{2}}(l_1 \overset{B}{\leftrightarrow} l_4) \geq \mathbb{P}_{\frac{1}{2}}(\{l_1 \overset{B}{\leftrightarrow} l_3\} \cap \{l_2 \overset{B}{\leftrightarrow} l_4\})$$

$= \mathbb{P}_{\frac{1}{2}}($  $)$

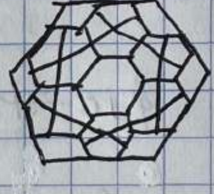
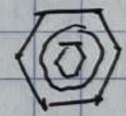
(FKG) $\geq \mathbb{P}_{\frac{1}{2}}(l_1 \overset{B}{\leftrightarrow} l_3)^2$. by sym.

$\Rightarrow \mathbb{P}_{\frac{1}{2}}(l_1 \overset{B}{\leftrightarrow} l_4) \geq 1/9$.

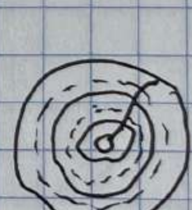
Cor. $\mathbb{P}_{\frac{1}{2}}($  $) \geq \frac{1}{18^2} > 0$.

Pf: $\mathbb{P}_{\frac{1}{2}}($  $) = \frac{1}{2}$ consider: 

Cor. For $A_n = H_{3n}/H_n$. $\mathbb{P}_{\frac{1}{2}}(\exists$ Black circuit in $A_n) = \mathbb{P}_{\frac{1}{2}}($  $) \geq c^6 > 0$. $c = 1/18^2$.

Pf: $\{$  $\} \subseteq \{$  $\}$

2) $\mathbb{P}_{\frac{1}{2}}(\exists$ white circuit in $H_{3^n}/H_{3^{n-1}}) \geq \tilde{c} > 0$.

$\Rightarrow \mathbb{P}_{\frac{1}{2}}(\partial H_3 \overset{B}{\leftrightarrow} \partial H_{3^n}) \leq (1-\tilde{c})^n$. 

So: $\mathbb{P}_{\frac{1}{2}}(0 \leftrightarrow \partial H_n) \leq n^{-\epsilon}$.

Cor. $\theta(\frac{1}{2}) = 0$. in particular.

Rmk: $c = -5/48 + o(1)$. more precisely.

and $\theta(p) = (p - p_c)^{-5/48 + o(1)}$ as $p \downarrow p_c$

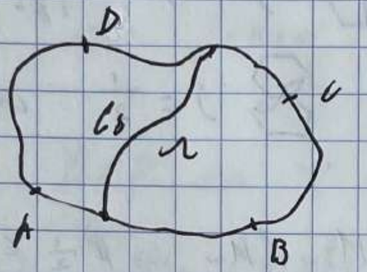
(2) Cardy's Formula:

Denote: i) G : Honeycomb. G^* : triangular

ii) G_δ, G^*_δ : scaled by $\delta > 0$

iii) Ω is a simply connected bdd nonempty domain. (Nonempty)

$(\Omega; A, B, C, D)$ is a quad: (Topological rectangle)



iv) $(\Omega_\delta; A_\delta, B_\delta, C_\delta, D_\delta)$ is approxi. of iii) in G_δ in Caratheory sense:

$\exists\ \Phi_\delta : U \to \Omega_\delta,\ \delta > 0,\ \phi : U \to \Omega.$

all conformal maps. st. $\Phi_\delta \xrightarrow{\delta \to 0} \phi$

locally uniform and satisfies:

$\Phi_\delta^{-1}(P_\delta) \xrightarrow{\delta \to 0} \phi^{-1}(P)$. for $\forall P \in \{A, B, C, D\}$.

v) $C_\delta(\Omega; A, B, C, D) = \{\exists$ Black path in Ω_δ connects $(A_\delta B_\delta)$ to $(C_\delta D_\delta)\}$.

Thm. (Smirnov)

$$\mathbb{P}_{\frac{1}{2}}(C_\delta(\Omega; A, B, C, D)) \xrightarrow{\delta \to 0} f(\Omega; A, B, C, D)$$

s.t. i) f is conformal invariance. i.e. $\forall \phi$

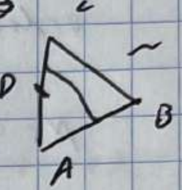
conformal on Ω. $f(\phi(\Omega); \phi(A) \cdots \phi(D))$

$= f(\Omega; A, B, C, D)$.

ii) If Ω is equilateral triangle with

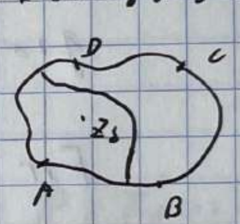
vertices $A. B. C.$ $D \in (AC)$. $\Rightarrow$

$f(\Omega; A, \cdots D) = \dfrac{|CD|}{|CA|}$.



<u>Def</u>: i) For $z \in \Omega$. $E_A^\delta(z) = [\exists$ black path in Ω_δ separate $\{A_\delta, z_\delta\}$ and $\{B_\delta, C_\delta\}\}$.

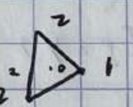
$H_A^\delta(z) = \mathbb{P}(E_A^\delta(z))$



For $B. C. D.$ similarly.

ii) $H^\delta(z) = H_A^\delta(z) + \tau H_B^\delta(z) + \tau^2 H_C^\delta(z)$.

where $\tau^3 = 1$. (



)

$S^\delta(z) = H_A^\delta(z) + H_B^\delta(z) + H_C^\delta(z)$.

<u>Pf</u>: 1°) Tightness:

Note that $\{H_A^\delta, H_B^\delta, H_C^\delta\}$ uniform bdd.

With RSW estimate: $\exists\, C, \alpha > 0$

$|H_p^\delta(x) - H_p^\delta(y)| \leq C|x-y|^\alpha$. $\forall p \in \{A, B, C\}$.

$\Rightarrow$ By Ascoli. Thm. $\exists$ u.c.c subseq.

2°) Denote the subseq limit is H, S. of H^δ, S^δ. Prove: They're holomorphic:

Only prove for H. (S is similar)

To apply Morera Thm. Fix γ simply closed.

i.e. prove: $\oint_\gamma H = 0$.

Suppose γ_δ in G_δ is a approxi. of γ.

order each edge of γ_δ: $(\gamma_\delta(k))_{0\le k\le N_\delta}$.

$N_\delta = O(1/\delta)$. Discreten $\oint_\gamma H$:

$$I_\delta(\gamma) =: \sum_0^{N_\delta-1} \frac{1}{2}\left(H^\delta(\gamma_\delta(k)) + H^\delta(\gamma_\delta(k+1))\right)(\gamma_\delta(k+1)-\gamma_\delta(k)) \to \oint_\gamma H \quad (\text{as } \delta\to 0)$$

goal: prove $I_\delta(\gamma) = o(1)$. $(\delta\to 0)$.

For $e=(x,y)$. Denote: $H^\delta(e) = \frac{1}{2}(H^\delta(x)+H^\delta(y))$

$$\partial_e H^\delta = H^\delta(y) - H^\delta(x).$$

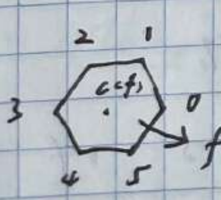
$$\Rightarrow I_\delta(\gamma) = \sum_{e\in\gamma_\delta} e\, H^\delta(e)$$

$$= \sum_{f\in \operatorname{int}\gamma_\delta}\ \sum_{e\in\partial f} e\, H^\delta(e). \quad (\text{Localized})$$

(f is unit hexagon. It holds because the public edges will counterfeit: )

"To integrate by part":

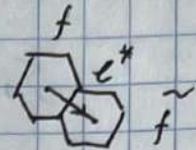
$$\sum_{e\in\partial f} e\, H^\delta(e) = \sum_0^5 \frac{1}{2}\left(H^\delta(x_{k+1})+H^\delta(x_k)\right)(x_{k+1}-x_k)$$



$$= -\sum_0^5 \left(\frac{x_k+x_{k+1}}{2} - c(f)\right)\left(H^\delta(x_{k+1}) - H^\delta(x_k)\right)$$

For $e \subset \operatorname{int}\gamma_\delta$. it will turn up twice:

Denote $e = [x, \eta]$:



$$\left(\frac{x+\eta}{2} - c(f)\right)\left(H^\delta(\eta) - H^\delta(x)\right) +$$

$$\left(\frac{x+\eta}{2} - c(\tilde f)\right)\left(H^\delta(x) - H^\delta(\eta)\right) =$$

$$\left(c(\tilde f) - c(f)\right)\left(H^\delta(\eta) - H^\delta(x)\right) = -e^* \partial_e H^\delta .$$

For e on γ_δ :

$$\sum_{e \in \gamma_\delta} \left(\frac{x_{k+1}+x_k}{2} - c(f)\right)\left(H^\delta(x_{k+1}) - H^\delta(x_k)\right) =$$

$$O(1/\delta) \cdot O(\delta) \cdot o(1) = o(1). \quad (\delta \to 0)$$

$$\Rightarrow I_\delta(\gamma) = \sum_{e \in \text{int}\gamma_\delta} e^* \partial_e H^\delta + o(1).$$

goal: $\sum_{\text{int}\gamma_\delta} e^* \partial_e H^\delta = 0$.

Note $\partial_e H^\delta_A = H^\delta_A(\eta) - H^\delta_A(x)$

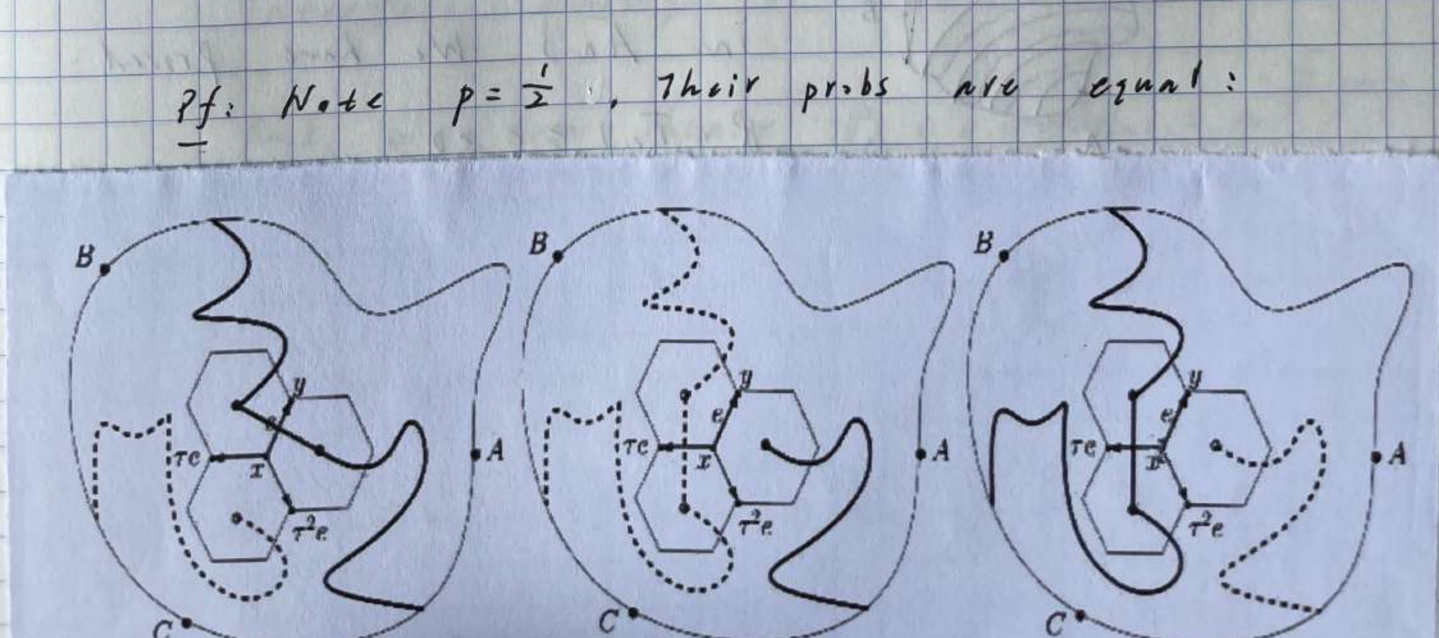
$$= \mathbb{P}(E^\delta_A(\eta) / E^\delta_A(x)) - \mathbb{P}(E^\delta_A(x) / E^\delta_A(\eta))$$

Denote $\mathbb{P}_A(e) =: \mathbb{P}(E^\delta_A(\eta) / E^\delta_A(x))$.

<u>Lemma</u>. (Color Switching)

$$\mathbb{P}_A(e) = \mathbb{P}_B(\tau e) = \mathbb{P}_C(\tau^2 e)$$

<u>Pf</u>: Note $p = \frac{1}{2}$, Their probs are equal:



S_1: $\sum_{e \in \mathrm{int}\, Y_\delta} e^* \partial_e H^\delta =$

$2\sum e^* (P_A(e) + z P_B(e) + z^2 P_C(e)) =$

$2\sum e^* (P_A(e) + z P_A(ze) + z^2 P_A(z^2 e))$

$= 2\sum (e^* + z(ze)^* + z^2 (z^2 e)^*) P_A(e)$

$= 0$. (Sum exchange)

$\Rightarrow I_\delta(\gamma) = O(1)$

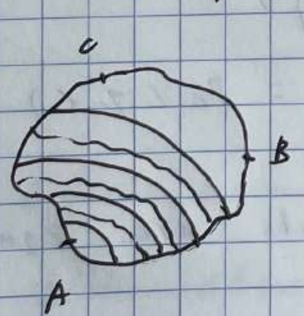
3') Boundary Value (Carathéodory $\Rightarrow$ Extend H.S. on $\bar{\Omega}$):

We have $S \equiv 1$. and $H_A(z) = \dfrac{2\,\mathrm{Re}\, H(z) + 1}{3}$

where H is conformal from Ω to

. Sends $(A.B.C)$ to $(1.z.z^2)$.

First. prove: $H_A^\delta(A) \to 1$ (as $\delta \to 0$)



Set $E_n = \{ \exists$ white circuit in $A_n \}$. We have proved: $\mathbb{P}(E_n) \geq c > 0$.

$\Rightarrow H_A^\delta(A) \geq \mathbb{P}(\bigcup_1^{N_\delta} E_n) \geq 1 - (1-c)^{N_\delta} \to 1$.

So: $H_A(A) = 1$. $H_B(A) = H_C(A) = 0$

Similarly. So: $H(A) = 1$. $H(B) = z$. $H(C) = z^2$.

Note S is bdd. real-value. $\in \Theta(\bar{\Omega}) \Rightarrow S \equiv 1$.

On (BC):

$H_A(z) = 0$. $H_B(z) + H_C(z) = 1$.

$H_B(z)$ moves from 1 to 0 conti. as $z : B \xrightarrow{\text{move}} C$.

$\Rightarrow H$ is one-to-one conti. from (BC) to $(\tau\, \tau^2)$.

Similarly, so H is conti. bijection: $\partial\Omega \to \partial D$.

Finally, by: $\begin{cases} H_A + H_B + H_C = 1 \\ H_A + \tau H_B + \tau^2 H_C = H \end{cases}$.

Take real-part of H: $H_A - \frac{1}{2}H_B - \frac{1}{2}H_C = \mathrm{Re}\,H$.

So: $H_A = \frac{1}{3}(1 + 2\,\mathrm{Re}\,H)$.

4') $H_A(D) = f(\Omega; A, B, C, D) = \frac{1}{3}(1 + 2\,\mathrm{Re}\,H(D))$

$= |CD| / |CA|$.

(From above: $\mathbb{P}_{\frac{1}{2}}(C_\delta(\Omega; A, \cdots D)) = H_A^\delta(D)$)

So we proved i), ii), simultaneously.

Cor. Bernoulli site Percolation on $\mathbb{T}$ has its interface converges to $SLE(6)$.

Rmk: We can obtain $\mathbb{P}_{p_c}(0 \overset{B}{\longleftrightarrow} \partial\Lambda_n) = n^{-\frac{5}{48} + o(1)}$ from this corollary!