Bernoulli Bond Percolation.

Def: i) $G = (V, E)$ is a graph. ("V": vertex. "E": edge)

configuration $w = (w(e))_{e \in E} \in \{0,1\}^E$

Rmk: Each w can be viewed as subgraph of G.

ii) Denote $e \in E$ by $\{x,y\}$ for x, y are vertexs in V.

Set $\Lambda_n = [-n,n]^d$ in $\mathbb{Z}^d \subseteq \mathbb{R}^d$. $\partial G = \{x \in V \mid \exists y \in V.$ st. $\{x,y\} \notin E\}$

iii) Probability space: $(\{0,1\}^E, \mathcal{F}, \mathbb{P}_p)$. where $\mathcal{F}$ is σ-algebra generated by events depending on finite edges e in E. And $\mathbb{P}_p(w(e_i)=1,\ 1 \le i \le k) = p^k$. $\forall (e_i)_1^k \subseteq E$. (Each edge "open" with prob p. i.e. $p(w(e)=1)=p$. and "close" with prob $1-p$. i.e. $p(w(e)=0) = 1-p$. $(w(e))_{e \in E}$ is seq of indept. r.v.'s)

Rmk: $p \in [0,1]$ is called edge-weight

(1) Pre:

① Monotoricity:

Def: i) Partial order on $\{0,1\}^E$ "$\le$" is:

$w \le w'$ if $w(e) \le w'(e)$. $\forall e \in E$.

ii) $A \in \mathcal{F}$ is increasing if $w \in A$. $w' \geq w \Rightarrow w' \in A$.

<u>Lemma.</u> $p \leq p'$. Then $\forall A \in \mathcal{F}$. increasing. we have:

$\mathbb{P}_p(A) \leq \mathbb{P}_{p'}(A)$.

<u>Rmk</u>: It's intuitive: $p \uparrow$. the number of open edges $\uparrow$.

<u>Pf</u>: (Use Coupling)

Set $(U_e)_{e \in E}$. $\underset{\mathbb{P}}{\overset{\text{i.i.d.}}{\sim}}$ Uniform $[0,1]$.

$W_p(e) = I_{\{U_e \leq p\}}$ Prove $W_p \sim \mathbb{P}_p$.

Note: $W_p(e) \leq W_{p'}(e)$. $\forall e$.

$\Rightarrow \mathbb{P}_p(A) = \mathbb{P}(W_p \in A) \leq \mathbb{P}(W_{p'} \in A) = \mathbb{P}_{p'}(A)$

<u>Lemma.</u> (FKG inequality)

$p \in [0,1]$. $\forall f. g \uparrow$ on $\{0,1\}^E$. Then:

$\mathbb{E}_p(fg) \geq \mathbb{E}_p(f)\,\mathbb{E}_p(g)$.

<u>Rmk</u>: For $A. B \in \mathcal{F}$ increasing. Set $f = I_A$. $g = I_B$.

$\Rightarrow \mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\,\mathbb{P}_p(B)$.

i.e. $\mathbb{P}_p(A \mid B) \geq \mathbb{P}_p(A)$. It means the increasing events have positive effect on each other.

Pf: Set $E^{(N)} = \{e_1 \cdots e_N\}$. Induct on N:

1°) $N=1$: It equi. : $p(1-p)(f(1)-f(0))(g(1)-g(0)) \geq 0$

2°) $N=n-1$ holds. For $N=n$:

$$LHS = \mathbb{E}_p(f(w(e_1)\cdots w(e_n))\, g(w(e_1)\cdots w(e_n)))$$

$$= \mathbb{E}_p(\mathbb{E}_p(f(\cdots)g(\cdots) \mid w(e_n)))$$

$$= p\,\mathbb{E}_p(f(w(e_1)\cdots w(e_{n}), 1))\, g(w(e_1)\cdots, 1))$$

$$+ (1-p)\,\mathbb{E}_p(f(w(e_1), \cdots 0))\, g(w(e_1)\cdots 0))$$

$$\overset{(hypo)}{\geq} p\,\mathbb{E}_p(f(\cdots 1))\,\mathbb{E}_p(g(\cdots, 1)) + (1-p)$$

$$\mathbb{E}_p(f(\cdots 0))\, g(\cdots 0))$$

$$\overset{(N=1)}{\geq} \mathbb{E}_p(f)\,\mathbb{E}_p(g).$$

3°) For $N=\infty$:

We know: $\mathbb{E}_p(f_n g_n) \geq \mathbb{E}_p(f_n)\,\mathbb{E}_p(g_n)$

$f_n = \mathbb{E}_p(f \mid w(e_1)\cdots w(e_n))$. $g_n = \mathbb{E}_p(g \mid (w(e_k))_1^n)$

Note $f_n \to f$. $g_n \to g$. a.s. By Fatou's. $n \to \infty$.

② <u>Russo's Formula</u>:

Def: i) $w^e(\tilde{e}) = \begin{cases} w(\tilde{e}). & \tilde{e} \neq e. \\ 1. & \tilde{e} = e \end{cases}$. $w_e(\tilde{e}) = \begin{cases} w(\tilde{e}). & \tilde{e} \neq e \\ 0. & \tilde{e} = e \end{cases}$

ii) $A \in \mathcal{F}$ increasing. e is pivot for A if:

$w^e \in A$. $w_e \notin A$.

<u>Rmk</u>: $\{e$ is pivotal for $A\}$ indept of states of e.

Lemma: A increasing, depends on finite edges.

Then: $\frac{d}{dp}\mathbb{P}_p(A) = \sum_{e\in E} \mathbb{P}_p(e \text{ is pivot for } A)$.

Pf: Suppose A depends on $\{e_k\}_1^N$.

Denote: p_i is prob. of e_i opens.

Set $\vec{p} = (p_1 \cdots p_N) \overset{\text{orpd}}{\sim} \mathbb{P}_{\vec{p}}$.

$\lim_{\varepsilon\to 0} \mathbb{P}_{\vec{p}+\varepsilon e_j}(A) - \mathbb{P}_{\vec{p}}(A)/\varepsilon$ is operation

on 2 different prob. measure.

Consider Coupling again:

$W_{\vec{p}}(e_i) = I\{u_i \le p_i\} \sim \mathbb{P}_{\vec{p}}.\ u_i \underset{\mathbb{P}}{\overset{\text{i.i.d}}{\sim}} U[0,1]$.

Note $W \triangleq W_{\vec{p}} \le W_{\vec{p}+\varepsilon e_j} \triangleq W'$.

$(W(f) = W'(f).\ f \ne e_j.\ W(e_j) \le W'(e_j))$

$LHS = \lim_{\varepsilon\to 0} (\mathbb{P}(W'\in A) - \mathbb{P}(W\in A))/\varepsilon$

$= \lim \frac{1}{\varepsilon}\mathbb{P}(U(e_j) \in [p_j, p_j+\varepsilon].\ W^{e_j}\in A.\ W_{e_j}\notin A)$

$= \lim_{\varepsilon\to 0} \frac{\varepsilon}{\varepsilon}\cdot \mathbb{P}_{\vec{p}}(e_j \text{ is pivotal for } A)$

Set $p_i = p$. We obtain the conclusion.

(2) Phase Transition:

Lemma. $\{0 \leftrightarrow \infty\}$ means $\exists$ open-edge path connect

0 and ∞ in $\mathbb{Z}^d$. Then: $\{0 \leftrightarrow \infty\} \in \mathcal{F}$.

Pf: $\{0 \leftrightarrow \infty\} = \bigcap_{n \geq 1} \{0 \leftrightarrow \partial \Lambda_n\} \in \mathcal{F}$.

Next, consider $G = \mathbb{Z}^2$:

Thm. $\exists\, p_c \in (0,1)$. s.t. $\mathbb{P}_p(0 \leftrightarrow \infty) = 0$. for $p < p_c$

$\mathbb{P}_p(0 \leftrightarrow \infty) > 0$. for $p > p_c$

Rmk: i) p_c is trivial or not depends on the Graph G.

ii) $p = p_c$ depends on continuity of $\mathbb{P}$.

Pf: 1) Existence of p_c:

$\theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty)$ $\uparrow$ on p follows from $\{0 \leftrightarrow \infty\}$ is increasing event.

Set $p_c = \sup\{p \mid \theta(p) = 0\}$.

2) prove $p_c > 0$: i.e. $\mathbb{P}_p(0 \leftrightarrow \infty) = 0$ for p small.

Note $\mathbb{P}_p(0 \leftrightarrow \infty) \leq \mathbb{P}_p(\exists$ open path of length n. start at $0)$

$\leq \sum_{l \in \mathcal{L}_n} \mathbb{P}_p($All edges of l are open$)$

where $\mathcal{L}_n$ is set of path of length n and starts at origin.

$\Rightarrow \# \mathcal{L}_n \leq 4^n$ (4 direction. 4 choice)

$\therefore$ RHS $\le 4^n \cdot p^n$. choose $p < \frac{1}{4}$. set $n \to \infty$.

i.e. $p_c \ge \frac{1}{4}$.

3') Def: For $G = (V, E)$ plane graph. We say $G^* = (V^*, E^*)$ is dual graph of G if:

i) It has a vertex for each face of G.

ii) It has an edge if two faces of G are separated from each other by an edge of G.

Rmk: i) Only graph lies in plane has a dual plane

ii) Dual graph of $\mathbb{Z}^2$ is $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$

For $w \in \{0,1\}^E$. define $w^*(e^*) = 1 - w(e)$

$\Rightarrow w \sim \mathbb{P}_p$ on G. then: $w^* \sim \mathbb{P}_{1-p}$ on G^*.

Note that $0 \not\leftrightarrow \infty$. $\Leftrightarrow$ $\exists$ open cluster in w^*. contains 0.

Set $\mathcal{L}_{m,n} = \{$ path of length m. passing through $(n+\frac{1}{2}, \frac{1}{2})\}$. $\# \mathcal{L}_{m,n} \le 4^m$.

So: $\mathbb{P}_p(0 \leftrightarrow \infty) \leq \sum_n \sum_{m \geq 2n} \# \mathcal{L}_{m,n} \cdot (1-p)^m$

$$\leq \sum_n \sum_{m \geq 2n} (4-4p)^m$$

$$= \sum_n (4-4p)^{2n} / 4p-3$$

$$= (4-4p) / (4p-3)^2.$$

$\Rightarrow \exists p < 1$. s.t. $\mathbb{P}_p(0 \leftrightarrow \infty) > 0$.

Def: i) $\tau x : \{0,1\}^{E(\mathbb{Z}^2)} \to \{0,1\}^{E(\mathbb{Z}^2)}$ is shift operator of $x \in \mathbb{Z}^2$. defined by $\tau_x w(\{a,b\}) = w(\{a+x, b+x\})$. $\forall \{a,b\} \in E(\mathbb{Z}^2)$ ($E(\mathbb{Z}^2)$ is set of edges of $\mathbb{Z}^2$).

ii) For $A \in \mathcal{F}$. $\tau x A = \{w \mid \tau_x w \in A\}$. it's invariant if $\tau x A = A$. $\forall x \in \mathbb{Z}^2$.

Lemma Bernoulli bond percolation on $\mathbb{Z}^2$ is ergodic. i.e. $\forall A \in \mathcal{F}$. invariant. satisfies : $\mathbb{P}_p(A) \in \{0,1\}$.

Pf. 1') $\forall \varepsilon > 0$. $\exists B \in \mathcal{F}$. depends on finite edges. s.t. $\mathbb{P}_p(A \Delta B) \leq \varepsilon$. Suppose B depends edges in Λ_N.

2') $\mathbb{P}_p(\tau x B \cap B) + O(\varepsilon) = \mathbb{P}_p(\tau x A \cap A)$

$= \mathbb{P}_p(A) = \mathbb{P}_p(B) + O(\varepsilon)$

Choose X is large enough. s.t.

Then $\tau_x B$ depend on edges of

$\tau_x \Lambda_n \cap \Lambda_n = \emptyset$.

$\Rightarrow \mathbb{P}_p(\tau_x B)\, \mathbb{P}_p(B) = \mathbb{P}_p(\tau_x B \cap B)$

set $\varepsilon \to 0$. $B \to A$.

So: $\mathbb{P}_p(A) = \mathbb{P}_p^2(A)$. $\mathbb{P}_p(A) \in \{0,1\}$

<u>Thm.</u> (Uniqueness of ∞-cluster)

For Bernoulli bond percolation of $\mathbb{Z}^2$. Either

no ∞-cluster (i.e. path with infinite open edges)

or $\exists$ unique ∞-cluster.

<u>Pf</u>: 1) $\theta(p) = 0$. $\Rightarrow$ No ∞-cluster.

2) $\theta(p) > 0$. $\Rightarrow \mathbb{P}_p(\exists\, \infty\text{-cluster}) \geq \theta(p) > 0$

With $\{0 \leftrightarrow \infty\}$ is invariant.

$\Rightarrow \mathbb{P}_p(\exists\, \infty\text{-cluster}) = 1$.

3) Next, we prove: $\exists!$ ∞-cluster

when $\theta(p) > 0$.

Set $A_k = \{\exists\ k\ \infty\text{-cluster}\}$. $1 \leq k \leq \infty$.

Note A_k is invariant. $\mathbb{P}_p(A_k) \in \{0,1\}$

With $\sum_{k\geq 1} \mathbb{P}_p(A_k) = \mathbb{P}_p(\exists\, \infty\text{-cluster}) = 1$.

$\exists!\ k_0 \in \overline{\mathbb{Z}^+}$. s.t. $\mathbb{P}_p(A_{k_0}) = 1$.

If $k_0 \in (1, \infty)$:

Note: $\exists\, n$ large. $\mathbb{P}_p($All k_0 ∞-cluster $\cap \Lambda_n)$

> 0. $\mathbb{P}_p($All edges in Λ_n open$) > 0$.

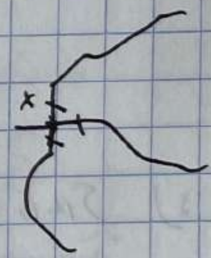
Besides. these two events indept.

$\Rightarrow \mathbb{P}_p(\{\cdots\} \cap \{\cdots\}) = \mathbb{P}_p\{\cdots\}\, \mathbb{P}_p\{\cdots\} > 0$.

So: $\mathbb{P}_p(A_1) > 0$. contradict!

If $k_0 = \infty$:

<u>Def</u>: Trifurcation point is a vertex in W if close its edges will produce three connected components. which are ∞-clusters.



set $\mathcal{T}_n = \{$All trifurcations in $\Lambda_n\}$

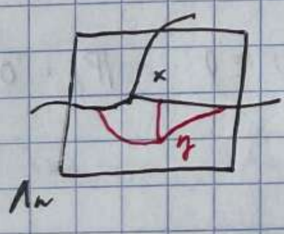
$T_0 = \{0$ is trifurcation$\} \in \mathcal{F}$.

$\Rightarrow \mathbb{E}_p(\#\mathcal{T}_n) = \sum_{\Lambda_n} \mathbb{P}_p(x$ is a trifurcation$)$

$= \#\Lambda_n \cdot \mathbb{P}_p(T_0)$

by translation invariant.

Claim: $\#\mathcal{T}_n \leq \#\partial\Lambda_n$.



If $x \in \mathcal{T}_n$. $y \in \mathcal{T}_n$. But y won't correspond a new point $\in \mathcal{T}_n$. Then it must walks like the way in figure.

Which will contradict with $x, y \in \mathcal{T}_n$.

$\Rightarrow \mathbb{E}_p(\# \mathcal{T}_n) \le \# \partial \Lambda_n$ i.e.

$\mathbb{P}_p(T_0) \cdot \# \mathrm{int}\,\Lambda_n \le \# \partial \Lambda_n$.

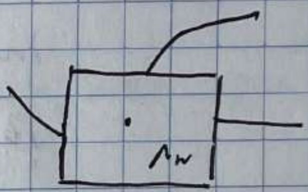
Note $\# \mathrm{int}(\Lambda_n) \sim O(n^2)$. $\# \partial \Lambda_n \sim O(n)$

Set $n \to \infty$. $\therefore \mathbb{P}_p(T_0) = 0$

But when n is large. since $k_0 = \infty$.

$\Rightarrow \mathbb{P}_p(\exists$ at least 3 ∞-cluster $\cap \Lambda_n \neq \emptyset) \ge \frac{1}{2}$.

$\Rightarrow \mathbb{P}_p(T_0) > 0$.



(Operate the edges in Λ_n. for T_0 holds)

(3) <u>Subcritical: exponential decay:</u>

For Bernoulli bond percolation on $\mathbb{Z}^2$.

<u>Thm.</u> i) If $p < p_c$. Then $\exists\, c = c(p)$. s.t. $\forall n \ge 1$.

$$\mathbb{P}_p(0 \leftrightarrow \partial \Lambda_n) \le e^{-cn}$$

ii) If $p > p_c$. Then $\mathbb{P}_p(0 \leftrightarrow \infty) \ge \frac{p - p_c}{p(1-p_c)}$

<u>Rmk:</u> For $p = p_c$. $\theta(p_c) = 0$. $\mathbb{P}_p(0 \leftrightarrow \partial \Lambda_n)$

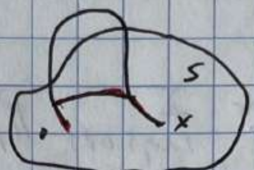
$= O(n^{-5/48})$

Denote: i) $S \subset V(\mathbb{Z}^2)$ is finite set contains 0.

ii) For $p \in [0,1]$. Define:

$$\varphi_p(S) =: p \sum_{\{x,y\} \in \partial S} \mathbb{P}_p(0 \overset{S}{\longleftrightarrow} x) \text{. where}$$

$\{0 \overset{S}{\longleftrightarrow} x\}$ means "$\exists$ path inside S st. connects 0 to x".

e.g.  $w \in \{0 \overset{S}{\longleftrightarrow} x\}$.

Rmk: $\{0 \overset{S}{\longleftrightarrow} x\}$ is a increasing event.

So: $\varphi_p(S)$ is increasing on p.

iii) $\tilde{p}_c =: \sup\{p \in [0,1] \mid \exists S \text{ st. } \varphi_p(S) < 1\}$.

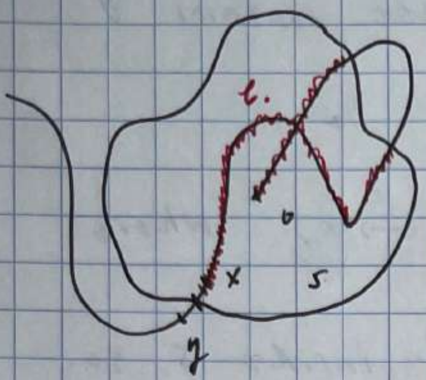
Pf: Goal: For $p < \tilde{p}_c$ or $> \tilde{p}_c$ it satisfies the conditions in Thm. Then: $\tilde{p}_c$ is critical value. (set $n \to \infty$) $\Rightarrow p_c = \tilde{p}_c$.

1°) If $p < \tilde{p}_c$. Then $\exists S \subset \Lambda_{N-1}$ st. $\varphi_p(S) < 1$.

Next, we want to estimate the prob. of events $\{0 \leftrightarrow \partial\Lambda_{jN}\}_{j \in \mathbb{Z}}$, which may produce a contraction.

Def: $\mathcal{C} = \{x \in S \mid 0 \overset{S}{\longleftrightarrow} x\}$ r.v.

Decompose the event of $\{0 \leftrightarrow \Lambda_{jN}\}$:



$\{0 \leftrightarrow \partial\Lambda_{jN}\} = \{\exists\, (x,\eta) \in \partial S,$

$0 \overset{S}{\leftrightarrow} x,\ \{x,\eta\}$ open, $\eta \overset{\mathcal{C}^c}{\leftrightarrow} \partial\Lambda_{jN}\}$

$=: \{\exists\,(x,\eta)\in\partial S,\ ①, ②, ③\}$

(Actually, it's easy to see:

①, ②, ③ are 3 indept events.)

$\Rightarrow \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_{jN}) \le \sum_{\{x,\eta\}\in\partial S} \mathbb{P}_p(0 \overset{S}{\leftrightarrow} x,\ \{x,\eta\} \text{ open},\ \eta \overset{\mathcal{C}^c}{\leftrightarrow} \partial\Lambda_{jN})$

$= \sum_{\{x,\eta\}\in\partial S} \sum_{C\subset S} \mathbb{P}_p(①, ②, ③,\ \mathcal{C} = C)$

$= \sum_{\{x,\eta\}\in\partial S} \sum_{C\subset S} \mathbb{P}_p(0 \overset{S}{\leftrightarrow} x,\ \mathcal{C} = C)\cdot p\cdot \mathbb{P}_p(\eta \overset{C^c}{\leftrightarrow} \partial\Lambda_{jN})$

(Note that we can obtain it since $\mathcal{C}$ isn't a r.v. (or say it's determined) any more)

So: $\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_{jN}) \le \sum_{\{x,\eta\}} \sum_{C\subset S} \mathbb{P}_p(\cdots)\, p\ \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_{(j-1)N})$

$= \varphi_p(S)\ \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_{(j-1)N})$

follows from translation invariant. $\eta \in \Lambda_N$.

$\Rightarrow \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_{jN}) \le \varphi_p(S)^j$.

2°) Lemma: For $p\in[0,1]$, $n\ge 1$, we have:

$$\frac{d}{dp}\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n) \ge \frac{1}{p(1-p)}\left(\inf_{0\in S\subset\Lambda_n} \varphi_p(S)\right)(1 - \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n))$$

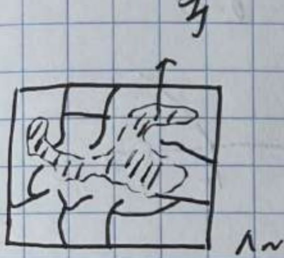
<u>Pf:</u> By Russo's Lemma:

$$LHS = \sum_{e \in E(\Lambda_n)} \mathbb{P}_p(e \text{ is pivotal to } \{0 \leftrightarrow \partial\Lambda_n\})$$

$$= \sum_{e \in E(\Lambda_n)} \mathbb{P}_p(e \text{ is pivotal for } \{\cdots\},\ e \text{ closed}) \cdot 1/(1-p) \qquad (\text{indept}).$$

$$= \sum_{E(\Lambda_n)} \mathbb{P}_p(e \text{ is pivotal for } \{\cdots\},\ 0 \nleftrightarrow \Lambda_n) \cdot 1/(1-p)$$

Set $\mathcal{Y} = \{x \in \Lambda_n \mid x \nleftrightarrow \partial\Lambda_n\}$.



Note $\forall\, 0 \in S \subset \Lambda_n$. on $\{\mathcal{Y} = S\}$:

$\{e$ is pivotal for $\{0 \leftrightarrow \partial\Lambda_n\},\ 0 \nleftrightarrow \partial\Lambda_n\}$

$= \{0 \overset{S}{\leftrightarrow} x,\ e \in \partial S\}$.

So: $LHS = \sum_{0 \in S \subset \Lambda_n} \sum_{\{x,y\} \in \partial S} \mathbb{P}_p(0 \overset{S}{\leftrightarrow} x,\ \mathcal{Y} = S) / 1-p$.

$\{\mathcal{Y} = S\}$ only depends on the edges of $S^c \cap \Lambda_n$. So indept of $\{0 \overset{S}{\leftrightarrow} x\}$.

$$\Rightarrow LHS = \frac{1}{p(1-p)} \sum_{0 \in S \subset \Lambda_n} \varphi_p(S)\, \mathbb{P}_p(\mathcal{Y} = S)$$

$$\geq \frac{1}{p(1-p)} \inf_{0 \in S \subset \Lambda_n} \varphi_p(S) \sum_{0 \in S \subset \Lambda_n} \mathbb{P}_p(\mathcal{Y} = S)$$

$$= \frac{1}{p(1-p)} \inf_{0 \in S \subset \Lambda_n} \varphi_p(S)\, (1 - \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n))$$

Return to pf:

Set $f(p) = \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n)$. Then:

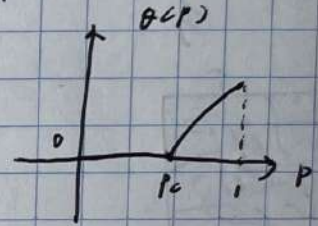
$f'(p) / 1-f(p) \geq 1/p(1-p)$. integrate $\int_p^{\tilde{p}_c}$:

$$\frac{p(1-\tilde{p}_c)}{\tilde{p}_c(1-p)} \leq \frac{1-f(\tilde{p}_c)}{1-f(p)} \leq 1/1-f(p)$$

$$\Rightarrow f(p) = \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n) \geq \frac{p-\tilde{p}_c}{p(1-\tilde{p}_c)}$$. Set $n \to \infty$.

<u>Thm.</u> For Bernoulli bond percolation on $\mathbb{Z}^2$.

we have $p_c = \frac{1}{2}$. $\theta(p_c) = 0$.



<u>pf</u>: <u>Lemma</u>: $\theta(\frac{1}{2}) = 0$

<u>Pf</u>: consider dual configuration:

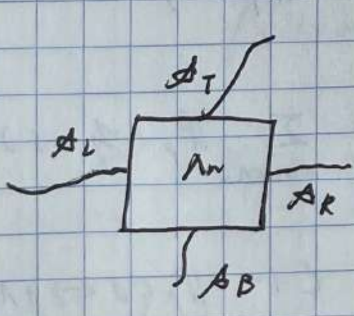
$w^* \sim \mathbb{P}_{\frac{1}{2}}$ on $(\mathbb{Z}^2)^*$ (same p.m.)

By contradiction, $\theta(\frac{1}{2}) > 0$.

Then : $\mathbb{P}_{\frac{1}{2}}(\exists\, \infty\text{-cluster}) = 1$

Note : $\mathbb{P}_{\frac{1}{2}}(\Lambda_n \leftrightarrow \infty) \to 1 \ (n \to \infty)$

$\therefore \exists N,\ n \geq N \quad \mathbb{P}_{\frac{1}{2}}(\Lambda_n \leftrightarrow \infty) \geq 1 - \varepsilon^4$.



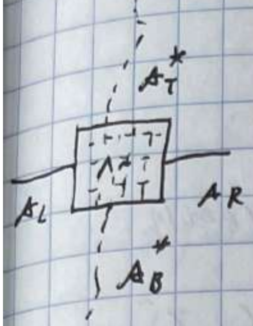
set A_T means event $\{$Top side of $\Lambda_n \leftrightarrow \infty\}$. $A_L \cdots$ similarly def.

$\Rightarrow \mathbb{P}(A_T \cup A_B \cup A_R \cup A_L) \geq 1 - \varepsilon^4$.

By FKG inequi. with invariant of rotate

$\mathbb{P}(A_L^c)^4 = \mathbb{P}(A_L^c) \cdots \mathbb{P}(A_T^c) \leq$

$\mathbb{P}(\cap A_L^c) \leq \varepsilon^4$.



$\Rightarrow \mathbb{P}(A_L^c) \le \varepsilon$.

Set $A_T^*, \cdots$ is same event as $A_T \cdots$

but in $(\mathbb{Z}^2)^*$. $S_n = \{$all edges closed in $\Lambda_n\}$.

$\Rightarrow \mathbb{P}_{\frac{1}{2}}(A_L \cap A_R \cap A_T^* \cap A_B^*) \ge 1 - 4\varepsilon$.

So: $\mathbb{P}_{\frac{1}{2}}(\exists\, 2\ \infty\text{-cluster}) \ge \mathbb{P}(S_n \cap A_L \cap A_R \cap A_T^* \cap A_B^*)$

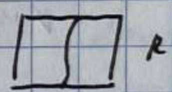
$= \mathbb{P}(S_n)\,\mathbb{P}(A_L \cap \cdots) > 0$. contradict!

Return to pf: We have $p_c \ge \frac{1}{2}$. Next, prove $p_c \le \frac{1}{2}$.

1°) RSW estimate:

Set $R = [0, n]^2$. Consider $p = \frac{1}{2}$.

$C_h(R) =: \{\exists$ horizontal open path in $R\}$



$C_v(R) =: \{\exists$ vertical open path in $R\}$

Note: $(C_v(R))^c = C_h(R^*)$.

And $\mathbb{P}_{\frac{1}{2}}(C_h(R^*))$

$\le \mathbb{P}_{\frac{1}{2}}(C_h(R))$. ($R^*$ is longer than R)

$\Rightarrow 1 = \mathbb{P}(C_h(R^*)) + \mathbb{P}(C_h(R)) \le 2\,\mathbb{P}(C_h(R))$

So: $\mathbb{P}_{\frac{1}{2}}(C_h(R)) \ge \frac{1}{2}$

<u>Rmk</u>: RSW estimate is indept of scaling or position of R.

2°) If $p_c > \frac{1}{2}$. Then: $\mathbb{P}_{\frac{1}{2}}(0 \leftrightarrow \Lambda_n) \le e^{-cn}$.

But $\frac{1}{2} \le \mathbb{P}_{\frac{1}{2}}(C_h(R)) = \sum_{x \in \partial_L \Lambda_n} \mathbb{P}(x \leftrightarrow x + \partial\Lambda_n)$

$\leq n \cdot e^{-cn} \longrightarrow 0$. contradict!

(4) <u>BBP in $\mathbb{Z}^d$, $d \geq 3$:</u>

It's difficult to extend $\mathbb{Z}^2$ to $\mathbb{Z}^d$ directly:

i) Pfs above strongly depend on structure of $\mathbb{Z}^2$.

ii) Dual graph only exists in plane ($\mathbb{Z}^2$).

But some conclusions still hold:

i) Phase transition:

$\exists\, p_c \in (0,1)$. s.t. $\theta(p) = 0$ if $p < p_c$

$\theta(p) > 0$. if $p > p_c$. But the critical

value of p_c is an open problem.

ii) Exponential Decay:

$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n^d) \leq e^{-cn}$. $\exists\, c > 0$. if $p < p_c$

As for continuous of $\theta(p)$ at p_c:

It's conti. when $d \geq 11$.

But it's unknown when $3 \leq d \leq 10$.

<u>Rmk:</u> $p_c = \frac{1}{2}$ in $\mathbb{Z}^2$ is called dual critical

point.. It's intuitively understood.