Linear Evolution Equations

(1) Preliminaries:

① Def:

i) $S: [0,T] \to X$ is called simple Func. if $S(t) = \sum_1^m \chi_{E_i}(t) u_i$, $u_i \in X$.

ii) $f: [0,T] \to X$. f is strongly measurable if $\exists (S_k(t))$ seq of simple Func's. s.t. $S_k \to f$. a.e. f is weakly measurable if $\forall u^* \in X^*$. $g(t) = \langle u^*, f(t) \rangle$ is m-measurable

iii) $f: [0,T] \to X$ is almostly separable. if $\exists N \subseteq [0,T]$. $m(N) = 0$. $f([0,T]/N)$ is separable.

Thm. f is strongly measurable $\Longleftrightarrow$ f is weakly measurable and almostly separable.

iv) For $S(t) = \sum_1^m \chi_{E_i} u_i$. Define integration: $\int_0^T S(t)\,dt = \sum_1^m m(E_i) u_i$. For strongly measurable func. $f(t)$. If $\int_0^T \| S_k(t) - f(t) \| dt \to 0$. Then define: $\int_0^T f(t)\,dt = \lim_k \int_0^T S_k(t)\,dt$.

Thm. (Bochner)

f is integrable $\Leftrightarrow$ $\|f(t)\|_X$ is integrable.

Besides. $\|\int_0^T f(t)dt\| \leq \int_0^T \|f(t)\|dt$. and

$\langle u^*, \int_0^T f \rangle = \int_0^T \langle u^*, f \rangle$.

② Def:

i) $L^p(0,T;X) = \{u : [0,T] \to X \mid u$ is strongly measurable. $\|u\|_{L^p(0,T;X)} = (\int_0^T \|u\|_X^p dt)^{\frac{1}{p}} < \infty\}$.

ii) $C(0,T;X) = \{u : [0,T] \to X \mid u$ is conti $\|u\|_{C(0,T;X)} = \max_{0\leq t\leq T} \|u(t)\| < \infty\}$.

iii) $u \in L^p(0,T;X)$. We say $v \in L^p(0,T;X)$ is its weak derivation. written in $u' = v$. if:

$\int_0^T \phi'(t) u(t) = -\int_0^T \phi(t) v(t)$. $\forall \phi \in C_c^\infty(0,T)$.

iv) $W^{1,p}(0,T;X) = \{u \in L^p(0,T;X) \mid u'$ exists in weak sense. $\|u\|_{W^{1,p}(0,T;X)} = \begin{cases} (\int_0^T \|u\|^p + \|u'\|^p)^{\frac{1}{p}} < \infty & 1 \leq p < \infty \\ \text{esssup}(\|u\| + \|u'\|) < \infty & p = \infty \end{cases}\}$

③ Properties:

Thm. For $u \in W^{1,p}(0,T;X)$. $1 \leq p \leq \infty$. Then there exists $v \in C(0,T;X)$. st. $u = v$ a.e. Besides.

$v(t) = v(s) + \int_s^t v'(\tau) d\tau$. So. $\max_t \|v(t)\| \leq C\|v\|_{W^{1,p}(0,T;X)}$.

Pf: 1°) Extend u: $u=0$ on $(-\infty,0)$, (T,∞).

2°) $u^\varepsilon = \eta_\varepsilon * u \in C^\infty(\varepsilon, T-\varepsilon)$.

$$\begin{cases} u^\varepsilon \to u & \text{in } L^p(0,T;X) \\ u^{\varepsilon\prime} \to u' & \text{in } L^p_{loc}(0,T;X) \end{cases}$$

Select a.e.-convergent subseq. $\exists V \in L^p(0,T;X)$.

s.t. $V=u$. a.e.

3°) Fix $0<s<t<T$. $u^\varepsilon(t) = u^\varepsilon(s) + \int_s^t u^{\varepsilon\prime}(\tau)d\tau$.

$\therefore V(t) = V(s) + \int_s^t V'(\tau)d\tau$.

<u>Thm.</u> For $u \in L^2(0,T;H_0^1(U))$, $u' \in L^2(0,T;H^{-1}(U))$. Then.

i) $\exists V \in C(0,T;L^2(U))$. $u=V$. a.e. on $[0,T]$.

ii) $\|u(t)\|^2_{L^2(U)} \in AC[0,T]$.

iii) $\frac{d}{dt}\|u(t)\|^2_{L^2(U)} = 2\langle u'(t), u(t)\rangle$ for a.e. $t\in[0,T]$.

with: $\max_{0\le t\le T}\|u(t)\|_{L^2(U)} \le C(T)\left(\|u\|_{L^2(0,T;H_0^1(U))} + \|u'\|_{L^2(0,T;H^{-1}(U))}\right)$

Pf: 1°) $u^\varepsilon = u * \eta_\varepsilon$. $\therefore \frac{d}{dt}\|u^\varepsilon(t)-u^\delta(t)\|^2_{L^2(U)} = 2\langle u^{\varepsilon\prime}-u^{\delta\prime}, u^\varepsilon - u^\delta\rangle$

$\Rightarrow \|u^\varepsilon(t)-u^\delta(t)\|^2_{L^2(U)} = \|u^\varepsilon(s)-u^\delta(s)\|^2_{L^2(U)} + \int_s^t \langle \square, \square\rangle$.

Since $u^\varepsilon \xrightarrow{L^2} u$. Let $\varepsilon,\delta\to 0$. we have:

$\overline{\lim_{\varepsilon,\delta\to 0}} \sup_t \|u^\varepsilon - u^\delta\|^2_{L^2(U)} \to 0$. $\therefore \exists V \in C(0,T;L^2(U))$.

$u^\varepsilon \to V$ in $C(0,T;L^2(U))$. since it's Cauchy

2°) From: $\|u^\varepsilon(t)\|^2 = \|u^\varepsilon(s)\|^2 + 2\int_s^t \langle u^{\varepsilon\prime}, u^\varepsilon\rangle d\tau$.

Let $\varepsilon,\delta\to 0$, replace v by u.

Thm. U is open, bounded. ∂U is smooth

If $u \in L^2(0,T; H^{m+2}(U))$, $u' \in L^2(0,T; H^m(U))$.

Then $\exists\, V \in C(0,T; H^{m+1}(U))$, s.t. $u = V$ a.e.

$$\max_{0 \le t \le T} \|V(t)\|_{H^{m+1}(U)} \le C(U,T,m)\left(\|u\|_{L^2(0,T;H^{m+2})} + \|u'\|_{L^2(\square)}\right)$$

Pf: By induction on m:

1°) $m=0$. Choose V: $U \subset\subset V \subset\subset \mathbb{R}^n$.

extend u to $\bar{u} = Eu$. $\bar{u} \in L^2(0,T; H^2(V))$.

$\therefore \|\bar{u}\|_{L^2(0,T;H^2(V))} \le C\|u\|_{L^2(0,T;H^2(U))}$

replace $\bar{u}, u$ by $\dfrac{\bar{u}(t+\Delta t) - \bar{u}(t)}{\Delta t}$, $\dfrac{u(t+\Delta t) - u(t)}{\Delta t}$.

Let $\Delta t \to 0$. $\therefore \|\bar{u}'\|_{L^2(0,T;L^2)} \le C\|u'\|_{L^2(\square)}$.

2°) Suppose $\bar{u}$ is smooth. (Or approxi by $u * \eta_\varepsilon$)

since $\left|\frac{d}{dt}\int_V |D\bar{u}|^2 dx\right| \le C\left(\|\bar{u}\|^2_{H^2(V)} + \|\bar{u}'\|^2_{L^2(V)}\right)$

By integrating. ($V \in C(0,T; H^1(U))$ is from approxi)

3°) For $m \ge 1$. Let $V = D^\alpha u$. $\forall\, |\alpha| \le m$.

apply $m=0$ case on V. Sum together.

(2) Second-order Parabolic Equations:

① Def:

$$\text{i)} \quad \begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{0\}. \end{cases} \quad (*)$$

$$Lu = \begin{cases} -\sum (a^{ij}(x,t)u_{x_i})_{x_j} + \sum b^i(x,t)u_{x_i} + c(x,t)u. & \text{divergence form.} \\ -\sum a^{ij}(x,t)u_{x_ix_j} + \sum b^i(x,t)u_{x_i} + c(x,t)u & \text{nondivergence form.} \end{cases}$$

We say $\frac{\partial}{\partial t} + L$ is uniformly parabolic if $\exists \theta > 0$.

s.t. $\sum a^{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2$. $\forall \xi \in \mathbb{R}^n$. $\forall (x,t) \in U_T$.

ii) <u>Weak solution:</u>

- Suppose $a^{ij}, b^i, c \in L^\infty(U_T)$. $f \in L^2(U_T)$. $g \in L^2(U)$.

  and $a^{ij} = a^{ji}$

Denote: $B[u,v;t] = \int_U \sum a^{ij}u_{x_i}v_{x_j} + \sum b^i u_{x_i} v + cuv \, dx$.

for $\forall u, v \in H_0^1(U)$. a.e. $0 \leq t \leq T$.

<u>Remark:</u> Note that: $(u', v) + B[u,v;t] = (f, v)$.

$\therefore u' = g^0 + \sum_1^n g^i_{x_i}$. $g_0 = f - \sum b^i u_{x_i} - cu$

$g^i = \sum_j a^{ij} u_{x_j}$. We obtain estimation:

$$\|u_t\|_{H^{-1}(U)} \leq \left(\sum_0^n \|g^i\|^2_{L^2(U)} \right)^{\frac{1}{2}} \leq c\,(\|u\|_{H_0^1(U)} + \|f\|_{L^2(U)})$$

$\Rightarrow u' \in H^{-1}(U)$. rewrite $(u', v) = \langle u', v \rangle$.

<u>Def:</u> For $u \in L^2(0,T; H_0^1(U))$. $u_t \in L^2(0,T; H^{-1}(U))$

is weak solution of I.V.P. (*). if

$$\begin{cases} \langle u', v \rangle + B[u,v;t] = (f,v). \ \forall v \in H_0^1(U). \text{ a.e. } t. \\ u(0) = g \end{cases}$$

① **Existence and Uniqueness:**

i) **Galerkin Approximation:**

1°) Find $(w_k(x))_{k\in\mathbb{N}}$ is orthogonal basis of $H_0^1(U)$.

and orthonormal basis of $L^2(U)$. i.e.

Take (w_k) be the normal eigenfunc's of $L=-\Delta$.

2°) Fix $m\in\mathbb{N}$.

Find $(d_m^k)_{1\le k\le m}$: $u_m(t)=\sum_1^m d_m^k(t)\,w_k : [0,T]\to H_0^1(U)$.

$$\text{s.t.}\ \begin{cases} d_m^k(0)=(g,w_k) & \forall\, 1\le k\le m \quad (\Delta). \\ (u_m',w_k)+B[u_m,w_k;t]=(f,w_k) & \forall\, 0\le t\le T. \end{cases}$$

3°) Send m to infinite.

We desire to find u. $u_m\to u$. solves $(*)$.

Thm. $\forall\, m\in\mathbb{N}$. $\exists$ unique u_m satisfies (Δ).

Pf: $(\Delta)\iff d_m^{k\,\prime}(t)+\sum_l e^{kl}(t)\,d_m^l(t)=f^k(t)$.

where $e^{kl}(t)=B[w_l,w_k;t]$. $f^k(t)=(f,w_k)$

Apply Basic Thm in ODE. solve $(d_m^k)_k$

ii) **Energy Estimation:**

Thm. $\max_{0\le t\le T}\|u_m(t)\|_{L^2(U)}^2+\|u_m\|_{L^2(0,T;H_0^1(U))}+\|u_m'\|_{L^2(0,T;H^{-1}(U))}$

$\le C(U,T,L)\,(\|f\|_{L^2(0,T;L^2(U))}+\|g\|_{L^2(U)})$

Pf: 1°) Multiply $d_k(t)$ for each equation of (Δ).

$(u_m', u_m) + B[u_m, u_m; t] = (f, u_m)$

2°) Note that: $\begin{cases} (u_m', u_m) = \frac{d}{dt}\frac{1}{2}\|u_m\|^2_{L^2(U)} \\ \beta\|u_m\|^2_{H_0^1(U)} \leq B[u_m, u_m; t] + \gamma\|u_m\|^2_{L^2(U)} \end{cases}$

$\therefore \frac{d}{dt}(\|u_m\|^2_{L^2(U)}) + 2\beta\|u_m\|^2_{H_0^1(U)} \leq C_1\|u_m\|^2_{L^2(U)} + C_2\|f\|^2_{L^2(U)}$

3°) Consider $\|u_m\|^2_{L^2(U)}$. $\|u_m\|^2_{H_0^1(U)}$. separately:

From: $\frac{d}{dt}(\|u_m\|^2_{L^2(U)}) \leq C_1\|u_m\|^2_{L^2(U)} + C_2\|f\|^2_{L^2(U)}$

Denote $\eta(t) = \|u_m\|^2_{L^2(U)}$. $\xi(t) = \|f\|^2_{L^2(U)}$

Then $\eta'(t) \leq C_1\eta(t) + C_2\xi(t)$.

$\Rightarrow \eta(t) \leq e^{C_1 t}(\eta(0) + C_2\int_0^t \xi(s)ds)$

$\eta(0) = \|u_m(0)\|^2_{L^2(U)} \leq \|g\|^2_{L^2(U)}$

$\therefore \max_{0\leq t\leq T}\|u_m(t)\|^2_{L^2(U)} \leq C\|g\|^2_{L^2(U)} + C\|f\|^2_{L^2(0,T;L^2(U))}$

Insert into $2\beta\|u\|^2_{H_0^1(U)} \leq C_1\|u_m\|^2_{L^2(U)} + C_2\|f\|^2_{L^2(0,T;L^2(U))}$

By integrate: $\int_0^T\|u_m\|^2_{H_0^1(U)} \leq C(\|g\|^2_{L^2(U)} + \|f\|^2_{L^2(0,T\ldots)})$

4°) Fix $v \in H_0^1(U)$. $\|v\|_{H_0^1(U)} \leq 1$. $v = v^1 + v^2$.

$v^1 \in \mathrm{span}\{w_k\}_1^m$. $(v^2, w_k) = 0$. $\forall 1 \leq k \leq m$.

$\therefore |(u_m', v)| = |(u_m', v^1)| = |(f, v^1) - B[u_m, v^1; t]|$

$\leq C(\|f\|_{L^2(U)} + \|u_m\|_{H_0^1(U)})$

$\therefore \|u_m'\|_{H^{-1}(U)} \leq C(\|f\|_{L^2(U)} + \|u_m\|_{H_0^1(U)})$

By integrating. We have $\|u_m'\|_{L^2(0,T;H^{-1}(U))}$

iii) Existence and Uniqueness:

Thm. Weak solution of (*) exists.

Pf: 1°) By reflexive. boundness:

$$\exists \begin{cases} u_{m_l} \rightharpoonup u & \text{in } L^2(0,T;H_0^1(U)) \\ u_{m_l}' \rightharpoonup v & \text{in } L^2(0,T;H^{-1}(U)) \end{cases}$$

Check: $\langle \int_0^T u\phi', w \rangle = -\langle \int_0^T v\phi, w\rangle$.

for $\forall\ \phi \in C_c^\infty(0,T)$. $w \in H_0^1(U)$.

$\therefore \int_0^T u\phi' = -\int_0^T v\phi$. $u' = v$ in weak sense.

2°) Check $u(0) = g$. Then u is weak solution.

Fix N. Choose $m > N$. $v(t) = \sum_1^N v^k(t) w^k \in C(0,T;H_0^1(U))$

$\int_0^T \langle u_m', v\rangle + B[u_m, v; t]\, dt = \int_0^T (f, v)\, dt$

Let $m \to \infty$. Then it holds for $\forall v \in L^2(0,T;H_0^1(U))$

In particular, $\langle u', v\rangle + B[u,v;t] = (f,v)$. $\forall v \in H_0^1(U)$.

3°) Fix $v \in C(0,T;H_0^1(U))$. $v(T) = 0$. Integrate by part:

$$\begin{cases} -\int_0^T \langle u_{m_l}, v'\rangle + B[u_{m_l}, v; t] = \int_0^T (f,v) + \langle u_{m_l}(0), v(0)\rangle \\ -\int_0^T \langle u, v'\rangle + B[u,v;t] = \int_0^T (f,v) + \langle u(0), v(0)\rangle. \end{cases}$$

Let $m_l \to \infty$. since $u_{m_l}(0) \xrightarrow{L^2} g$.

Thm. The weak solution of (*) is unique.

Pf: check $u \equiv 0$ is the only solution when $f = g \equiv 0$

set $v = u$. Since $B[u,u;t] \geq -\gamma \|u\|_{L^2(U)}^2$

By Gronwall's inequality on $(u', u) + B[u,u;t] = (f,u)$

③ Regularity:

i) Motivation:

For $\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,T] \\ u = g & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$ Assume: $u \in C^\infty$, $u \to 0$ ($|x| \to \infty$)

By integration by part:

$$\int_{\mathbb{R}^n} f^2 dx = \int_{\mathbb{R}^n} (u_t - \Delta u)^2 = \int_{\mathbb{R}^n} u_t^2 + 2Du \cdot Du_t + (\Delta u)^2$$

Note that: $2Du \cdot Du_t = \frac{d}{dt} |Du|^2$, $\int_{\mathbb{R}^n} (\Delta u)^2 = \int_{\mathbb{R}^n} |D^2 u|^2$

Integrate on $\int_0^t$, $\int_0^T$ and sum over:

$$\sup_{0 \le t \le T} \int_{\mathbb{R}^n} |Du|^2 + \int_0^T \int_{\mathbb{R}^n} u_t^2 + |D^2 u|^2 \le C \left(\int_0^T \int_{\mathbb{R}^n} f^2 + \int_{\mathbb{R}^n} |Dg|^2 \right)$$

Set $\tilde{u} = u_t$: $\begin{cases} \tilde{u}_t - \Delta \tilde{u} = \tilde{f} & \text{in } \mathbb{R}^n \times (0,T] \\ \tilde{u} = \tilde{g} & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$

where $\tilde{f} = f_t$. $\tilde{g}(x) = u_t(x,0) = f(x,0) + \Delta g$.

Multiply $\tilde{u}$. Integrate on $(0,t)$, $(0,T)$. Sum over.

$$\Rightarrow \max_{0 \le t \le T} \int_{\mathbb{R}^n} |u_t|^2 + \int_0^T \int_{\mathbb{R}^n} |Du_t|^2 \le C \left(\int_0^T \int_{\mathbb{R}^n} f_t^2 + \int_{\mathbb{R}^n} |D^2 g|^2 + f^2(x,0)\, dx \right)$$

With: $\begin{cases} \max_{0 \le t \le T} \|f\|_{L^2(\mathbb{R}^n)} \le C \left(\|f\|_{L^2(0,T;H_0^1(\mathbb{R}^n))} + \|f_t\|_{L^2(0,T;H^{-1})} \right) \\ -\Delta u = f - u_t \Rightarrow \int_{\mathbb{R}^n} |D^2 u|^2 \le \int_{\mathbb{R}^n} f^2 + u_t^2 \end{cases}$

We obtain estimation concerning u':

$$\sup_{0\le t\le T} \int_{\mathbb{R}^n} |u_t|^2 + |D^2 u|^2 dx + \int_0^T \int_{\mathbb{R}^n} |Du_t|^2 \le C (\int_0^T \int_{\mathbb{R}^n} f^2 + f_t^2 \, dx dt + \int_{\mathbb{R}^n} |D^2 g|^2)$$

ii) Improved Regularity:

. suppose $\{w_k\}$ is eigenfunc's of $-\Delta$. on $H_0^1(U)$. U is open. bounded. ∂U is smooth. a^{ij}. b^i. $c \in C^\infty(\bar{U})$. don't depend on variable t.

Thm. If $g \in H_0^1(U)$. $f \in L^2(0,T; L^2(U))$. u is weak solution of:

$$\begin{cases} u_t + Lu = f & \text{in } U_T \\ u = g & \text{on } U \times \{0\}. \\ u = 0 & \text{on } \partial U \times [0,T]. \end{cases}$$

Then $u \in L^2(0,T; H^2(U)) \cap L^\infty(0,T; H_0^1(U))$, $u' \in L^2(0,T; L^2(U))$

$$\operatorname*{ess\,sup}_{0\le t\le T} \|u\|_{H_0^1(U)} + \|u\|_{L^2(0,T;H^2(U))} + \|u'\|_{L^2(0,T;L^2(U))}$$

$$\le C (\|f\|_{L^2(0,T;L^2(U))} + \|g\|_{H_0^1(U)})$$

With addition: $g \in H^2(U)$. $f' \in L^2(0,T; L^2(U))$

Then $u \in L^\infty(0,T; H^2(U))$. $u' \in L^\infty(0,T; L^2(U)) \cap L^2(0,T; H_0^1(U))$

$u'' \in L^2(0,T; H^{-1}(U))$, with estimation:

$$\operatorname*{ess\,sup}_{0\le t\le T} (\|u\|_{H^2(U)} + \|u'\|_{L^2(U)}) + \|u'\|_{L^2(0,T;H_0^1(U))} + \|u''\|_{L^2(0,T;H^{-1}(U))}$$

$$\le C (\|f\|_{H^1(0,T;L^2(U))} + \|g\|_{H^2(U)})$$

<u>Pf</u>: Only prove the first part:

1°) $(u_m'', u_m') + B[u_m, u_m'] = (f, u_m')$

Separate second-order part: $B[u_m, u_m'] = A + B$

$A = \frac{d}{dt} \frac{1}{2} A[u_m, u_m]$. $A[u,v] = \int_{\mathbb{R}^n} \sum a^{ij} u_{x_i} v_{x_j} \, dx$.

Since $|B| \le \frac{c}{\varepsilon} \|u_m\|_{H_0^1(U)}^2 + \varepsilon \|u_m'\|_{L^2(U)}^2$.

$\Rightarrow \|u_m'\|_{L^2(U)}^2 + \frac{d}{dt}\left(\frac{1}{2} A[u_m, u_m]\right) \le C(\|u_m\|_{H_0^1(U)}^2 + \|f\|^2) + 2\varepsilon \|u_m'\|_{L^2}^2$

with $\begin{cases} \|u_m(0)\|_{H_0^1(U)} \le \|g\|_{H_0^1(U)} \\ A[u,u] \ge \theta \int |Du|^2 \end{cases}$ integrate on t

$\therefore \sup_{0 \le t \le T} \|u_m\|_{H_0^1(U)}^2 \le C(\|g\|_{H_0^1(U)}^2 + \|f\|_{L^2(0,T;L^2(U))}^2)$

<u>Lemma</u>. H is Hilbert space. $u_k \rightharpoonup u$. in $L^2(0,T;H)$

If $\operatorname{essup}_{0 \le t \le T} \|u_k\|_H \le C$. Then $\operatorname{essup} \|u\| \le C$.

<u>Pf</u>: $F_{a,b}(v) = \int_a^b (v, u)\, dt$. $\therefore \lim_k F(u_k) = F(u)$.

$\because |F_{a,b}(u_k)| \le C \|u\| (b-a)$. Let $k \to \infty$.

$\therefore \int_a^b \|u\|_H^2 \le C \|u\|_H (b-a)$.

Let $b \to a$. Apply Lebesgue Diff. Thm.

$\therefore \sup_{0 \le t \le T} \|u\|_{H_0^1(U)} \le C(\|g\|_{H_0^1(U)}^2 + \|f\|_{L^2(0,T;H_0^1(U))}^2)$. a.e.

Return to "$\Rightarrow$". $\therefore \operatorname{essup}_t \|u'\|_{L^2(U)} \le C(\|g\|_{H_0^1(U)}^2 + \|f\|_{L^2(0,T;L^2(U))}^2)$

2°) From $(u', v) + B[u,v] = (f, v)$. a.e.

$\therefore B[u,v] = (f-u', v) \triangleq (h, v)$ By Elliptic Regularity:

$u \in H^2(U)$. $\|u\|_{H^2(U)} \le C(\|f\|_{L^2(U)} + \|u'\|_{L^2(U)} + \|u\|_{L^2(U)})$

<u>Thm.</u> (High order)

If $g \in H^{2m+1}(U)$. $\frac{d^k f}{dt^k} \in L^2(0,T; H^{2m-2k}(U))$. With:

$$\begin{cases} g_0 = g \in H_0^1(U). \quad g_1 = f(0) - Lg_0 \in H_0^1(U). \quad (\text{compatibility conditions}) \text{ holds.} \\ \cdots \quad g_m = \frac{d^{m-1} f(0)}{dt^{m-1}} - L g_{m-1} \in H_0^1(U) \end{cases}$$

Then $\frac{d^k u}{dt^k} \in L^2(0,T; H^{2m+2-2k}(U))$. with estimation:

$$\sum_{k=0}^{m+1} \left\| \frac{d^k u}{dt^k} \right\|_{L^2(0,T;H^{2m+2-2k}(U))} \le C\left(\sum_{k=0}^{m} \left\| \frac{d^k f}{dt^k} \right\|_{L^2(0,T;H^{2m-2k}(U))} + \|g\|_{H^{2m+1}(U)}\right)$$

<u>Pf:</u> By induction on m:

Set $\tilde{u} = u'$. differentite the equation at t:

$$\begin{cases} \tilde{u}_t + L\tilde{u} = \tilde{f} & \text{in } U_T \\ \tilde{u} = 0 & \text{on } \partial U \times [0,T] \\ \tilde{u} = \tilde{g} & \text{on } U \times \{0\}. \end{cases} \qquad \begin{aligned} \tilde{f} &= f_t \\ \tilde{g} &= f(0) - Lg \end{aligned}$$

For $k=0$. similarly. $B[u,v] = (f - u', v)$.

Apply Elliptic Regularity.

<u>Cor.</u> If $g \in C^\infty(\bar{U})$. $f \in C^\infty(\bar{U}_T)$. compatibility condition holds for $m \in \mathbb{Z}^+$. Then $u \in C^\infty(\bar{U}_T)$.

④ Maximum Principles:

i) Weak Maximum Principles:

· Assume L has nondivergence form. a^{ij}, b^i, c are conti. $a^{ij} = a^{ji}$.

Thm. If $u \in C^{1,2}(U_T) \cap C(\bar{U}_T)$. $c \equiv 0$ on U_T.

Then $u_t + Lu \le 0$ in $U_T \Rightarrow \max_{\bar{U}_T} u = \max_{\Gamma_T} u$.

$u_t + Lu \ge 0$ in $U_T \Rightarrow \min_{\bar{U}_T} u = \min_{\Gamma_T} u$.

Pf: 1°) Consider $u_t + Lu < 0$.

Otherwise set $u^\varepsilon = u - \varepsilon t$. Then $\varepsilon \to 0$.

2°) If $\exists (x_0, t_0) \in U_T$. st. $u(x_0, t_0) = \max_{\bar{U}_T} u$.

(a) $0 < t_0 < T$.

Then $u_t(x_0, t_0) = 0$, $Lu \ge 0$ at (x_0, t_0) by elliptic case. contradict!

(b) $t_0 = T$.

Then $u_t(x_0, t_0) \ge 0$. likewise.

$u_t + Lu \ge 0$ at (x_0, t_0). contradict!

Thm. If $u \in C^{1,2}(U_T) \cap C(\bar{U}_T)$. $c \ge 0$ in U_T.

Then
$u_t + Lu \le 0$ in $U_T \Rightarrow \max_{\bar{U}_T} u \le \max_{\Gamma_T} u^+$

$u_t + Lu \ge 0$ in $U_T \Rightarrow \max_{\bar{U}_T} (-u) \le \max_{\Gamma_T} u^-$

Pf: 1°) Consider $u_t + Lu < 0$. (u^2 works as well)

2°) If u attain positive max at $(x_0, t_0) \in U_T$.

Then $Lu \geq 0$, $u_t \geq 0$ at (x_0, t_0). contradict!

Remark: There're various versions of maximal principles for parabolic PDEs. Even if $c(x) \leq 0$.

ii) Harnack's Inequality:

· For $u \in C^{1,2}(U_T)$ solves $u_t + Lu = 0$. If $u \geq 0$ in U_T. $\forall V \subset\subset U$. connected. Then $\forall\, 0 < t_1 < t_2 \leq T$.

$\exists\, C = \text{const}(V, t_1, t_2, L)$. $\sup\limits_V u(x, t_1) \leq C \inf\limits_V u(x, t_2)$

Remark: It holds even when the coefficients are measurable, bounded.

iii) Strong Maximal Principles:

Thm. If $u \in C^{1,2}(U_T) \cap C(\bar{U}_T)$, $c \equiv 0$ in U_T. U is connected.

Then: $u_t + Lu \leq 0 \Rightarrow$ if $\exists\, (x_0, t_0) \in U_T$. $\max\limits_{\bar{U}_T} u = u(x_0, t_0)$

then $u \equiv C$ in U_{t_0}

$u_t + Lu \geq 0 \Rightarrow$ if $\exists\, (x_0, t_0) \in U_T$. $\max\limits_{\bar{U}_T} u = u(x_0, t_0)$

then $u \equiv C$ in U_{t_0}

Pf: 1°) For $W \subset\subset U$. $x_0 \in W$. consider v solves:

$$\begin{cases} v_t + Lv = 0 & \text{in } W_T \\ v = u & \text{on } \Delta_T \end{cases} \quad \Delta_T \text{ is parabolic boundary of } W_T.$$

2°) Note for $w = v - u$ attain min on Δ_T.

$\therefore v \geq u$. Besides: $v \leq \max_{\Delta_T} u \leq u(x_0, t_0) \triangleq M$.

3°) Set $\tilde{v} = M - v$. by 2°). $\tilde{v}(x_0, t_0) = 0$. $\tilde{v} \geq 0$.

solves $\tilde{v}_t + L\tilde{v} = 0$ in U_T.

$\forall V \subset\subset W$. Apply Harnack Inequility:

$\max_V \tilde{v}(x, t) \leq C \inf_V \tilde{v}(x, t_0) \leq C\tilde{v}(x_0, t_0) = 0$.

for $\forall\ 0 < t < t_0$.

$\therefore \tilde{v} \equiv 0$ in $V \times (0, t_0)$. So in W_{t_0}.

$\therefore u \equiv M$ on $\partial W \times [0, t_0]$.

4°) By arbitrary of W. $\therefore u \equiv M$ in U_{t_0}.

(connected $x_1, x_2 \in U$ by ∂W. for some W)

Thm. If $u \in C^{1,2}(U_T) \cap C(\bar{U}_T)$, $c \geq 0$. U is connected.

Then $u_t + Lu \leq 0 \Rightarrow$ If $\exists (x_0, t_0) \in U_T$. $\max_{\bar{U}_T} u =$

$u(x_0, t_0) \geq 0$. Then $u \equiv c$ in U_{t_0}.

$u_t + Lu \geq 0 \Rightarrow$ If $\exists (x_0, t_0) \in U_T$. $\min_{\bar{U}_T} u =$

$u(x_0, t_0) \leq 0$ Then $u \equiv c$ in U_{t_0}.

Pf: 1°) $M = \max_{\bar{U}_T} u = 0$.

The same argument in above Thm.

2°) $M = \max_{\bar{U}_T} u > 0$.

For $x_0 \in W \subset\subset U$. consider V solves

$$\begin{cases} V_t + KV = 0 & \text{in } W_T \\ V = u^+ & \text{on } \Delta_T. \end{cases} \quad KV = LV - cV.$$

$\therefore 0 \le V \le M$. Since $u_t + Ku \le -cu \le 0$ on $\{u \ge 0\}$.

$\therefore M \ge V \ge u$. as well. $\therefore V(x_0, t_0) = M$.

3°) set $\hat{V} = M - V$. $\hat{V}_t + K\hat{V} = 0$ in U_T.

$\Rightarrow \hat{V} \equiv 0$ in $\bar{W}_{t_0}$. $\therefore u^+ \equiv M$ on $\partial W \times [0, t_0]$

Since $u^+ = \max\{u, 0\} = M > 0$. $\therefore u \equiv M$ on $\partial W \times [0, t_0]$.

4°) $u \equiv M$. by arbitrary of W.

(3) Second-Order Hyperbolic Equations:

① Definitions:

i) $$\begin{cases} u_{tt} + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0, T] \\ u = g.\ u_t = h & \text{on } U \times \{0\}. \end{cases}$$

$\frac{\partial^2}{\partial t^2} + L$ is hyperbolic if $\exists \theta > 0$. st.

$\sum_{i,j} a^{ij}(x,t)\xi_i\xi_j \ge \theta |\xi|^2$. $\forall \xi \in \mathbb{R}^n$. $(x,t) \in U_T$

ii) Weak Solutions:

- Suppose $a^{ij}, b^i, c \in C^1(\bar{U}_T)$, $f \in L^2(U_T)$, $g \in H_0^1(U)$, $h \in L^2(U)$, $a^{ij} = a^{ji}$.

See $u, f : [0,T] \to H_0^1(U), L^2(U)$.

i.e. in Time space.

Consider $(u'', v) + B[u, v; t] = (f, v)$, $\forall v \in H_0^1(U)$.

Remark: Analogously, $u'' \in H^{-1}(U)$. We can reinterpret (u'', v) as $\langle u'', v \rangle$.

Def: $u \in L^2(0,T; H_0^1(U))$, $u' \in L^2(0,T; L^2(U))$.

$u'' \in L^2(0,T; H^{-1}(U))$ is weak solution if:

$$\begin{cases} \langle u'', v \rangle + B[u, v; t] = (f, v), \ \forall v \in H_0^1(U) \\ u(0) = g, \ u'(0) = h. \end{cases}$$

② Existence and Uniqueness:

i) Galekin's Method:

Find $d_m^k(t)$:

$$u_m(t) = \sum_1^m d_m^k(t) w_k . \quad \begin{cases} d_m^k(0) = (g, w_k) \\ d_m^{k\prime}(0) = (h, w_k) \end{cases}$$

$(u_m'', w_k) + B[u_m, w_k; t] = (f, w_k)$, $\forall 1 \le k \le m$.

Thm. $\forall m \in \mathbb{Z}^+$. There exists unique $u_m(t)$ satisfies the condition (or say $(d_m^k)_1^m$).

Pf: Similar as parabolic case.

ii) Energy Estimation:

Thm. There exists $C = \text{const.}(U, T, L)$. st.

$$\max_{0 \le t \le T} (\|u_m\|_{H_0^1(U)} + \|u_m'\|_{L^2(U)}) + \|u_m''\|_{L^2(0,T;H^{-1}(U))}$$

$$\le C(\|f\|_{L^2(0,T;L^2(U))} + \|g\|_{H_0^1(U)} + \|h\|_{L^2(U)})$$

Pf: 1°) Multiply $d_m^{k\prime}(t)$ for equations of u_m

$\therefore (u_m'', u_m') + B[u_m, u_m'; t] = (f, u_m')$.

Note: $(u_m'', u_m') = \frac{1}{2}\frac{d}{dt}\|u_m'\|_{L^2(U)}^2$.

2°) For $B[u_m, u_m'; t] = B_1 + B_2$. (separate second-order)

$$B_1 = \frac{d}{dt}\frac{1}{2}A[u_m, u_m; t] - \frac{1}{2}\int_U \sum a_t^{ij} u_{m,x_i} u_{m,x_j}$$

$$\begin{cases} B_1 \ge \frac{1}{2}\frac{d}{dt}A[u_m, u_m; t] - C\|u_m\|_{H_0^1(U)}^2 \\ |B_2| \le C(\|u_m\|_{H_0^1(U)}^2 + \|u_m'\|_{L^2(U)}^2) \end{cases}$$

3°) We obtain: $\frac{d}{dt}(\|u_m'\|_{L^2(U)}^2 + A[u_m, u_m; t])$

$$\le C(\|u_m'\|_{L^2(U)}^2 + \|u_m\|_{H_0^1(U)}^2 + \|f\|_{L^2(U)}^2)$$

$$\le C(\|u_m'\|_{L^2(U)}^2 + A[u_m, u_m; t] + \|f\|_{L^2}^2)$$

Apply Gronwall Inequality:

$$\|u_m'\|_{L^2(U)}^2 + A[u_m, u_m; t] \le C(\|g\|_{H_0^1(U)}^2 + \|h\|_{L^2(U)}^2 + \|f\|_{L^2(0,T;L^2(U))}^2)$$

$\therefore \max_{0\le t\le T} (\|u_m\|^2_{H_0^1(U)} + \|u_m'\|^2_{L^2(U)}) \le C(\|g\|^2_{H_0^1(U)} + \|h\|^2_{L^2(U)} + \|f\|^2_{L^2(0,T;L^2(U))})$

3°) Consider $\|v\|_{H_0^1(U)} \le 1$. $v = v_1 + v_2$.

Similar argue: $|\langle u_m'', v\rangle| \le C(\|f\|_{L^2(U)} + \|u_m\|_{H_0^1(U)})$

iii) Existence and Uniqueness:

Thm. There exists weak solution.

Pf: 1°) By Boundedness:

$\exists u \in L^2(0,T;H_0^1(U))$. $(u_{m_l}) \subset (u_m)$. s.t. $\begin{cases} u_{m_l} \rightharpoonup u & \text{in } L^2(0,T;H_0^1(U)) \\ u_{m_l}' \rightharpoonup u' & \text{in } L^2(0,T;L^2(U)) \\ u_{m_l}'' \rightharpoonup u'' & \text{in } L^2(0,T;H^{-1}(U)) \end{cases}$

2°) To prove: $u(0) = g$. $u'(0) = h$.

similar argument: (As parabolic)

$$\begin{cases} \int_0^T (v'', u) + B[u, v; t]\,dt = \int_0^T (f, v)\,dt - (u(0), v'(0)) + (u'(0), v(0)) \\ \int_0^T (v'', u_m) + B[u_m, v; t]\,dt = \int_0^T (f, v)\,dt - (u_m(0), v'(0)) + (u_m'(0), v(0)) \end{cases}$$

Choose $v(t) \in C^2(0,T;H_0^1(U))$. $v(T) = v'(T) = 0$.

Let $m \to \infty$. Comparing:

$(g - u(0), v'(0)) = (u'(0) - h, v(0))$.

$\Rightarrow$ set $v(t) = (u(0) - g)t + (u'(0) - h)$. ✓.

Thm. The weak solution is unique.

Pf: It suffice to prove:

$u \equiv 0$ when $f = g = h \equiv 0$. in U_T.

1°) Fix $0 \le s \le T$

For balancing the order of differentiation.

set $v(t) = \begin{cases} \int_t^s u(\tau) d\tau. & 0 \le t \le s \\ 0 & . \ s \le t \le T. \end{cases}$ $v \in H_0^1(U). \forall t.$

Consider $\int_0^s \langle u'', v \rangle + B[u, v; t] = 0$.

Since $u'(0) = v(s) = 0$. $v' = -u$. integrate by part:

$\int_0^s \langle u'. u \rangle - B[v'. v; t] \, dt = 0$. Exact the principle:

$\int_0^s \frac{d}{dt} (\frac{1}{2} \|u\|_{L^2(U)}^2 - \frac{1}{2} B[v. v; t]) dt = - \int_0^s C + D \, dt$.

$\begin{cases} C = - \int_U \Sigma b^i u v_{x_i} + \frac{1}{2} b^i_{x_i} u v \, dx \\ D = \frac{1}{2} \int_U \Sigma a^{ij}_t u_{x_i} v_{x_j} + \Sigma b^i_t u_{x_i} v + c_t u v \, dx \end{cases}$

2°) Since $|C| + |D| \le \|v\|_{H_0^1(U)}^2 + \|u\|_{L^2(U)}^2$. $(u = -v')$

$\therefore \|u\|_{L^2(U)}^2 + \|v(0)\|_{H_0^1(U)}^2 \le C (\int_0^s \|v\|_{H_0^1(U)}^2 + \|u\|_{L^2(U)}^2 + \|v(0)\|_{L^2(U)}^2$

set $w(t) = \int_0^t u(\tau) d\tau$. $(0 \le t \le T)$

since $\|v(0)\|_{L^2(U)}^2 = \|w(s)\|_{L^2(U)}^2 \le \int_0^s \|u\|_{L^2(U)}^2 dt$

$\|v(t)\|_{H_0^1(U)}^2 = \|w(s) - w(t)\|_{H_0^1(U)}^2 \le 2 (\|w(s)\|_{H_0^1(U)}^2 + \|w(t)\|_{H_0^1(U)}^2)$

$\Rightarrow \|u(s)\|_{L^2(U)}^2 + (1 - 2sC_1) \|w(s)\|_{H_0^1(U)}^2 \le C (\int_0^s \|w\|_{H_0^1(U)}^2 + \|u\|_{L^2(U)}^2)$

Choose T_1: $1 - 2T_1 C_1 \ge \frac{1}{2}$.

Apply Gronwall Inequality. $\therefore u \equiv 0$. a.e. in $[0. T_1]$.

3°) Consider in $[T_1. 2T_1]$. $[2T_1. 3T_1]$ ...

③ Regularity:

Motivation:

$$\frac{d}{dt}\left(\int_{\mathbb{R}^n} |Du|^2 + u_t^2 \, dx\right) = 2\int_{\mathbb{R}^n} Du\cdot Du_t + u_t u_{tt}$$

$$= 2\int_{\mathbb{R}^n} u_t(u_{tt} - \Delta u) \le 2\int u_t^2 + f^2 \, dx.$$

integrate $\int_0^t$:

$$\therefore \sup_{0\le t\le T}\left(\int_{\mathbb{R}^n} |Du|^2 + u_t^2 \, dx\right) \le C\left(\int_0^T\int_{\mathbb{R}^n} f^2 + \int |Dg|^2 + h^2 \, dx\right)$$

For u_t, u_{tt} part:

Let $\tilde{u} = u_t$
$$\begin{cases} \tilde{u}_{tt} - \Delta \tilde{u} = \tilde{f} & \text{in } \mathbb{R}^n \times (0,T] \\ \tilde{u} = \tilde{g},\ \tilde{u}_t = \tilde{h} & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

$\tilde{f} = f_t$, $\tilde{g} = h$, $\tilde{h} = u_{tt}(x,0) = f(x,0) + \Delta g$.

$$\therefore \sup_{0\le t\le T}\left(\int |Du_t|^2 + u_{tt}^2\right) \le C\left(\|f_t\|^2_{L^2([0,T]\times\mathbb{R}^n)} + \int_{\mathbb{R}^n} |Dg|^2 + |Dh|^2 + f^2(x,0)\right)$$

with
$$\begin{cases} \max_t \|f\|^2_{L^2(U)} \le C\left(\|f\|^2_{L^2([0,T]\times\mathbb{R}^n)} + \|f_t\|^2_{L^2([0,T]\times\mathbb{R}^n)}\right) \\ -\Delta u = f - u_{tt} \Rightarrow \int_{\mathbb{R}^n} |D^2 u| \le C\int_{\mathbb{R}^n} f^2 + u_{tt}^2 \, dx \end{cases}$$

$$\therefore \sup_{0\le t\le T}\left(\int_{\mathbb{R}^n} |Du_t|^2 + |D^2u|^2 + u_{tt}^2\right) \le C\left(\int_0^T\int_{\mathbb{R}^n} f_t^2 + f^2 + \int_{\mathbb{R}^n} |D^2 g|^2 + |Dh|^2\right)$$

$C = \text{const}(T)$.

Thm. $g \in H_0^1(U)$, $h \in L^2(U)$, $f \in L^2(0,T;L^2(U))$.

u solves the hyperbolic equation weakly.

$$\begin{cases} u_{tt} + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g,\ u_t = h & \text{on } U \times \{0\} \end{cases}$$ Then

$u \in L^\infty(0,T;H_0^1(U))$, $u' \in L^\infty(0,T;L^2(U))$

$$\operatorname*{ess\,sup}_t (\|u\|_{H_0^1(U)} + \|u'\|_{L^2(U)}) \le C(\|f\|_{L^2(0,T;L^2(U))} + \|g\|_{H_0^1(U)} + \|h\|_{L^2(U)})$$

With addition: $g \in H^2(U)$, $h \in H_0^1(U)$, $f' \in L^2(0,T;L^2(U))$

Then: $u \in L^\infty(0,T;H^2(U))$, $u_t \in L^\infty(0,T;H_0^1(U))$, $u_{tt} \in L^\infty(0,T;L^2(U))$

$u_{ttt} \in L^2(0,T;H^{-1}(U))$. With estimation:

$$\operatorname*{ess\,sup}_t (\|u\|_{H^2(U)} + \|u'\|_{H_0^1(U)} + \|u''\|_{L^2(U)}) + \|u'''\|_{L^2(0,T;H^{-1}(U))}$$

$$\le C(\|f\|_{H^1(0,T;L^2(U))} + \|g\|_{H^2(U)} + \|h\|_{H^1(U)})$$

<u>Pf.</u> The first part is from: (Apply Lemma before)

$$\sup_t (\|u_m\|_{H^1(U)} + \|u_m'\|_{L^2(U)}) \le C(\|f\|_{L^2(\square)} + \|g\|_{H_0^1(U)} + \|h\|_{L^2(U)})$$

Thm. (High order)

If $g \in H^{m+1}(U)$, $h \in H^m(U)$, $\frac{d^k f}{dt^k} \in L^2(0,T;H^{m-k}(U))$

satisfies m^{th}-order compatibility conditions:

$$\begin{cases} g_0 = g,\ h_1 = h. \\ g_{2l} = \dfrac{d^{2l-2} f}{dt^{2l-2}}(x,0) - L g_{2l-2} \in H_0^1(U), & \text{if } m = 2l \\ h_{2l+1} = \dfrac{d^{2l-1} f}{dt^{2l-1}}(x,0) - L h_{2l-1} \in H_0^1(U), & \text{if } m = 2l+1. \end{cases}$$

Then $\frac{d^k u}{dt^k} \in L^\infty(0,T;H^{m+1-k}(U))$, $\forall\, 0 \le k \le m+1$.

with: $\operatorname*{ess\,sup}_{0\le t\le T} \sum_{k=0}^{m+1} \|\frac{d^k u}{dt^k}\|_{H^{m+1-k}(U)} \le C\left(\sum_{k=0}^{m} \|\frac{d^k f}{dt^k}\|_{L^2(0,T;H^{m-k}(U))} + \|g\|_{H^{m+1}(U)} + \|h\|_{H^m(U)}\right)$

Pf: By induction on m:

Similar argument: consider $\tilde{u} = u_t$ with the

t-differentiated equation. $(1 \le k \le m+1)$

For $k=0$: $B[u,v] = (f-u'', v)$

Apply elliptic regularity Thm.

Thm. If $g, h \in C^\infty(\bar{U})$, $f \in C^\infty(\bar{U}_T)$, satisfies m^{th}-

compatibility conditions, $\forall m \in \mathbb{Z}^+$. Then $u \in C^\infty(\bar{U}_T)$ a.e.

④ Propagation of disturbance:

. Note that maximum principle $\Rightarrow$ Infinite Propagation

However, 2^{nd}-hyperbolic PDEs have opposite phenomenon:

finite propagation of initial disturbance. So the max

principles don't exist for it.

Def: $K = \{(x,t) \mid q(x) < t_0 - t\}$, $q \in C^\infty$ solves: $\begin{cases} \sum_{i,j} a^{ij} q_{x_i} q_{x_j} = 1, \ q>0 \\ q(x_0) = 0. \end{cases}$

$K_t = \{x \mid q(x) < t_0 - t\}$.

$Lu = -\sum a^{ij}(x) u_{x_i x_j}$, $a^{ij} \in C^\infty$

Thm. If $u \in C^\infty$, solves $u_{tt} + Lu = 0$, $u = u_t \equiv 0$ on K_0

Then $u \equiv 0$ in K