Sobolev Space

(1) Function Space:

① Hölder space:

Def: i) $u: U \to \mathbb{R}$. bounded. conti. $\|u\|_{C(\bar{U})} = \sup_{x \in U} |u(x)|$

ii) γ^{th}- Hölder seminorm: $u: U \to \mathbb{R}^1$ is:

$$[u]_{C^{0,\gamma}(\bar{U})} = \sup_{\substack{x \neq y \\ x,y \in U}} \left\{ \frac{u(x)-u(y)}{|x-y|^\gamma} \right\}. \quad 0 < \gamma \leq 1.$$

γ^{th}- Hölder norm is $\|u\|_{C^{0,\gamma}} = \|u\|_{C(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})}$.

iii) For $u \in C^k(\bar{U})$. $\|u\|_{C^{k,\gamma}(\bar{U})} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})}$

Define Hölder space $C^{k,\gamma}(\bar{U})$ is the set:

$\{u \in C^k(\bar{U}) \mid \|u\|_{C^{k,\gamma}(\bar{U})} < \infty\}$. with norm $\|\cdot\|_{C^{k,\gamma}(\bar{U})}$.

Thm. $C^{k,\gamma}(\bar{U})$ is a Banach Space.

Pf: 1°) It's linear space.

2°) $\|u_n - u_m\|_{C^{k,\gamma}(\bar{U})} \leq \varepsilon \Rightarrow \sup_U |u_n - u_m| < \varepsilon$

$\therefore \exists u.\ u_n \xrightarrow{u} u$. Since $u_n \in C^k$. $\therefore u \in C^k$.

And $D^\alpha u_n \xrightarrow{u} D^\alpha u$. $|\alpha| \leq k$.

check $\|u_n - u\|_{C^{k,\gamma}(\bar{U})} \to 0 \ (n \to \infty)$

(Uniform converge can exchang limit)

② Sobolev Space:

i) Weak Derivates:

· We will weaken the smoothness of functions to expand the function space, but guarantee we can apply integration by part.

Def: $u, v \in L^1_{loc}(U)$. We say $v(x)$ is α^{th}-weak partial derivate of u. if for $\forall \phi \in C_c^\infty(U)$

$$\int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v \phi \, dx. \text{ holds.}$$

Remark: v is unique. m-a.e. If $u \in C^k(\bar{U})$

Then $D^\alpha u = v$.

Weak derivates permit the existence of set of discontinuites of measure 0. Since they make sense under integration.

ii) Sobolev Space:

Def: $1 \leq p \leq \infty$. $W^{k,p}(U) = \{u \in L^1_{loc}(U) \mid D^\alpha u$ exists in weak sense. $D^\alpha u \in L^p(U), \forall |\alpha| \leq k\}$

Remark: Denote $H^k(U) = W^{k,2}(U)$. $H^0(U) = L^2(U)$.

$H^k(U) \subseteq L^2(U)$. CLS. $\therefore H^k$ is Hilbert Space.

Def: Norm $\|\cdot\|_{W^{k,p}(U)}$ in $W^{k,p}(U)$ is:

$$\|u\|_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha|\le k} \int_U |D^\alpha u|^p\right)^{\frac{1}{p}}, & 1\le p<\infty \\ \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^\infty(U)}, & p=\infty \end{cases}$$

$u_m \to u$ in $W^{k,p}_{loc}(U)$ means $u_m \to u$ in $W^{k,p}(V)$

for each $V \subset\subset U$.

Denote: $W_0^{k,p}(U) \triangleq \overline{C_c^\infty(U)} \cap W^{k,p}(U)$

$= \{u \in W^{k,p}(U) \mid D^\alpha u = 0 \text{ on } \partial U,\ |\alpha| \le k-1\}$.

$H_0^k(U) \triangleq W_0^{k,2}(U)$.

Remark: When $U \subseteq \mathbb{R}^1$ open. Then:

$u \in W^{1,p}(U) \Leftrightarrow u = g$ a.e. $\exists g \in AC(U)$.

g' exists a.e. $g' \in L^p(U)$

Pf: **Lemma** $f \in L^1_{loc}(a,b)$. If $\int_a^b f(x)\phi'(x)dx = 0$

for any $\phi \in C_c^\infty(a,b)$. Then $f \equiv$ const. a.e.

Pf: Choose $g, h \in C_c^\infty(a,b)$. $\int_a^b h\,dx = 1$.

Define $\phi(x) = \int_a^x g(t)dt - \int_a^x h(t)dt \int_a^b g(t)$

$\therefore \phi \in C_c^\infty(a,b)$ $\phi(b)=0$. $\phi(a)=0$

$\Rightarrow \int_a^b f(x)\left(g(x) - h(x)\int_a^b g(t)dt\right)dx$

$= \int_a^b g(x)\left(f(x) - \int_a^b h(t)f(t)dt\right) = 0$

for $\forall g \in C_c^\infty(a,b)$ $\therefore f(x) \equiv \int_a^b hf$. a.e.

Remark: Weak derivates of f may not exist.

WLOG. suppose $U = (0,1)$. prove: $u(x) = \int_0^x u' \, dt + \text{const. a.e.}$

u' is weak derivates. $\Leftrightarrow$ Check Lemma.

e.g. $(r_k)_{k \in \mathbb{Z}^+} \subseteq B(0,1)$. dense. If $\alpha < \frac{n-p}{p}$. Then

$u(x) = \sum_k |x - r_k|^{-\alpha} / 2^k \in W^{1,p}(U)$. $(U \triangleq B(0,1))$.

However, $u = \infty$ on every open set of U.

Pf: $\|u\|_{L^p} \leq \sum \frac{1}{2^k} \| |x - r_k|^{-\alpha} \|_p \leq \sum \frac{C}{2^k}$.

Check derivates (weak sense) ✓.

$C = \int_U \frac{1}{|x|^{\alpha p}} dx$.

iii) Properties:

Thm. For $u, v \in W^{k,p}(U)$. $|\alpha| \leq k$. Then:

i) $D^\alpha u \in W^{k-|\alpha|, p}(U)$ and $D^\beta (D^\alpha u) = D^\alpha (D^\beta u) = D^{\alpha+\beta} u$.

for $|\alpha| + |\beta| \leq k$.

ii) $\forall \lambda, \mu \in \mathbb{R}$. $\lambda u + \mu v \in W^{k,p}(U)$. $D^\alpha(\lambda u + \mu v)$

$= \lambda D^\alpha u + \mu D^\alpha v$.

iii) For $\zeta \in C_c^\infty(U)$. $\zeta u \in W^{k,p}(U)$. Besides.

$D^\alpha(\zeta u) = \sum_{\alpha \geq \beta} \binom{\alpha}{\beta} D^\beta \zeta \, D^{\alpha-\beta} u$. $\binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha-\beta)!\beta!}$

$(\alpha \geq \beta \Leftrightarrow \beta_k \leq \alpha_k. \ \forall 1 \leq k \leq n.)$

iv) (Chain Rule)

$F \in C^1(\mathbb{R}, \mathbb{R})$. F' is bounded. $u \in W^{1,p}(U)$.

for some $1 \leq p \leq \infty$. U is bounded Then:

$F(u) \in W^{1,p}(U)$: $(F(u))_{x_i} = F'(u) u_{x_i}$. weak sense.

Pf: i). ii) can be check directly.

iii) By induction on $|\alpha|$: $|\alpha|=1$.

$$\int_U u\zeta D^\alpha\phi = \int_U u\,(D^\alpha(\phi\zeta) - \phi D^\alpha\zeta) = -\int_U (uD^\alpha\zeta + \zeta D^\alpha u)\phi.$$

For $|\alpha| = n+1$. $\alpha=\beta+\gamma$. $|\beta|=n$. $|\gamma|=1$.

$$\int_U u\zeta D^\alpha\phi = \int_U u\zeta D^\beta(D^\gamma\phi). \quad D^\gamma\phi\in C_c^\infty(U).$$

iv) $|F(u)-F(0)| \le \|F'\|_\infty |u|$. $\therefore F\in L^p(U)$.

For $u_\varepsilon \to u$. $u_\varepsilon\in C_c^\infty(U)$. in $W^{1,p}(U)$.

$$\left|\int (F(u)-F(u_\varepsilon))\frac{\partial\phi}{\partial x_i}\right| \le \|F'\|_\infty \|u-u_\varepsilon\|_p \left\|\frac{\partial\phi}{\partial x_i}\right\|_{p'}.$$

$\to 0$ $(\varepsilon\to 0)$.

$$\left|\int (F'(u)u_{x_i} - F'(u^\varepsilon)u^\varepsilon_{x_i})\phi\right| \le 2\|F\|_\infty \|u_{x_i}-u^\varepsilon_{x_i}\|_p \|\phi\|_{p'}$$

$\to 0$ $(\varepsilon\to 0)$

Consider $(F(u_\varepsilon))_{\varepsilon>0}$. Let $\varepsilon\to 0$.

Remark: In iv). For U is unbounded. If $F(0)=0$.

Then it will hold. (For $|F(u)|\le\|F\|_\infty|u|$,

then $F(u)$. $F(u)u_{x_i}\in L^p(U)$).

Thm: For $k\in\mathbb{Z}^+$. $1\le p\le\infty$. $W^{k,p}(U)$ is Banach Space.

Pf: 1°) $\|\cdot\|_{W^{k,p}(U)}$ is a norm. $W^{k,p}(U)$ is linear space.

2°) (u_n) Cauchy $\Rightarrow (D^\alpha u_n)$ Cauchy.

Check: $D^\alpha u_n \to u_\alpha$. $u_\alpha = D^\alpha u$.

(2) Approximations:

Fix $k \in \mathbb{Z}^+$. $1 \le p < \infty$. $U_\varepsilon = \{x \in U \mid d(x, \partial U) > \varepsilon\}$.

① Interior Approxi:

Thm. $u \in W^{k,p}(U)$. $u^\varepsilon = u * \eta_\varepsilon \in C^\infty(U_\varepsilon)$. Then

$u^\varepsilon \longrightarrow u$ in $W^{k,p}_{loc}(U)$ $(\varepsilon \to 0)$.

Pf: 1°) Check: $D^\alpha u^\varepsilon = \eta_\varepsilon * D^\alpha u$. $(\eta_\varepsilon \in C_c^\infty(U))$

2°) $D^\alpha u^\varepsilon \to D^\alpha u$. in $L^p(V)$. $\forall V \subset\subset U$. $\forall |\alpha| \le k$.

② Global Approxi:

Thm. (By $C^\infty(U)$ Func.)

U is bounded. $u \in W^{k,p}(U)$. Then there exists.

$u_m \in C^\infty(U) \cap W^{k,p}(U)$. St. $u_m \xrightarrow{W^{k,p}(U)} u$ $(m \to \infty)$

Pf: 1°) $U = \bigcup_{i=1}^\infty U_i$. $U_i = \{x \in U \mid d(x, \partial U) > \frac{1}{i}\}$.

Set $V_i = U_{i+3} - \overline{U_{i+1}}$. Choose $V_0 \subset\subset U$.

$\therefore U = \bigcup_0^\infty V_i$ $\exists$ POU: (ζ_i) Subordinates it.

2°) Set $u^i = \eta_{\varepsilon_i} * (\zeta_i u)$. $\operatorname{supp} u^i \subset W_i \triangleq U_{i+4} - \overline{U_i}$

and $\| u^i - \zeta_i u \|_{W^{k,p}(U)} \le \frac{\delta}{2^i}$.

3°) $v = \sum u^i \in C^\infty(U)$. Since $u = \sum \zeta_i u$.

$\| v - u \|_{W^{k,p}(V)} \le \sum \| u^i - \zeta_i u \| \le \sum \frac{\delta}{2^i}$

for $\forall V \subset\subset U$. Take sup on V.

$\therefore v \longrightarrow u$ in $W^{k,p}(U)$.

(*) For $\sum u^i$ has only finite terms $\neq 0$ when in $V \subset\subset U$. Then exchange limit. integrate.

Remark: It holds when U is unbounded. Choose Exhaustion: $\{|x|<n\} \cap U_n$

Thm (By $C^\infty(\bar{U})$ Function)

U is bounded. ∂U is C^1. $u \in W^{k,p}(U)$. Then exists $u_m \in C^\infty(\bar{U})$ s.t. $u_m \to u$ in $W^{k,p}(U)$.

Remark: $u^\varepsilon \in C^\infty(U_\varepsilon) \not\Rightarrow u^\varepsilon \in C^\infty(\bar{U})$. since u may not belong to $L^p(U + B(0,\varepsilon))$.

$u \in C^\infty(U) \not\Rightarrow u \in C^\infty(\bar{U})$. or $W^{k,p}(U)$

Since there may exist poles on ∂U.

Pf. 1°) Fix $x^0 \in \partial U$. From ∂U is C^1.

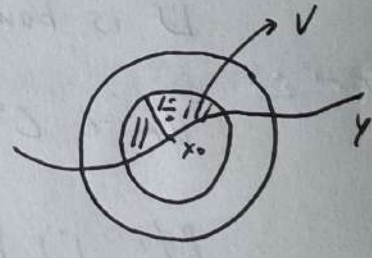
$\exists\, \gamma: \mathbb{R}^{n-1} \to \mathbb{R}^1$ WLOG. suppose for some $r>0$.

$(x_1 \cdots x_{n-1}) \mapsto x_n = \gamma(x_1 \cdots x_{n-1})$ $\quad U \cap B(x^0, r) = \{x \in B(x^0, r) \mid x_n > \gamma(x_1 \cdots x_{n-1})\}$.

Set $V = U \cap B(x^0, \frac{r}{2})$.

2°) $x^\varepsilon = x + \lambda \varepsilon \vec{e}_n$. $\forall x \in V$. $\varepsilon > 0$

Choose λ large enough.



Then $B(x^\varepsilon, \varepsilon) \subseteq U \cap B(x^0, r)$.

for $\forall x \in \bar{V}$. $\forall \varepsilon > 0$. sufficient small. Denote $u_\varepsilon = u(x_\varepsilon)$.

$\therefore V^\varepsilon = \eta_\varepsilon * u_\varepsilon \in C^\infty(\bar{V})$. Check $V^\varepsilon \to u$ in $W^{k,p}(V)$

3°) ∂U is cpt. Find finite balls center on ∂U. which covers ∂U. Use its POU.

Remark: When U is unbounded. it holds. Since there exits countable balls cover ∂U. Submit a POU. Choose $V_i^{\varepsilon_i} \underset{\varepsilon/2^i}{\to} u$ in V_i.

(3) Extension:

Thm. If U is bounded, ∂U is C^1. $1 \le p \le \infty$. for a open bounded set V. s.t. $U \subset\subset V$. Then exists

BLO: $E: W^{1,p}(U) \longrightarrow W^{1,p}(\mathbb{R}^n)$. s.t.

i) $Eu = u$. a.e. in U. ii) $\mathrm{Supp}(Eu) \subset V$.

iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C(p, U, V) \|u\|_{W^{1,p}(U)}$.

Pf: 1°) flatten ∂U to $\{x_n = 0\}$. nearby $x^0 \in \partial U$. $U \subseteq \{x_n > 0\}$.

Denote $B^+ = B(x_0, r) \cap \{x_n > 0\}$. $B^- = B(x_0, r) \cap \{x_n \le 0\}$

2°) Suppose $u \in C^\infty(U)$ firstly.

Let $\bar{u} = \begin{cases} u(x). & x \in B^+ \\ -3u(x', -x_n) + 4u(x', -\frac{x_n}{2}). & x \in B^- \end{cases}$

Check $\bar{u} \in C^1(B)$. (check $D^\alpha(\bar{u}|_{B^+}) = D^\alpha(\bar{u}|_{B^-})$ on $\{x_n = 0\}$.)

3°) $\|\bar{u}\|_{W^{1,p}(B)} \le C \|u\|_{W^{1,p}(B^+)}$.

4°) $\underset{x}{\partial U} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \underset{y}{I}$ suppose ϕ straighten out ∂U. (ϕ, Ψ are C^1-homeomorphisms)

We have prove: $\|\bar{u}(\Psi(y))\| \le C \|u(\Psi(y))\|$.

Convert back to x. since $\dfrac{\max_{1\le i \le n} \sup_I |\Psi_{y_i}(\phi(x))|}{\min_{1 \le i \le n} \inf_I |\Psi_{y_i}(\phi(x))|} < \infty$.

$\therefore \|\bar{u}(x)\| \le C \|u(x)\|$.

5°) ∂U is cpt. cover it by $\bigcup_i^{\sim} W_i \subset\subset V$.

(W_i) submits a POU. suppose u_i is extension on W_i. Let $\bar{u} = \sum_i^{\sim} \zeta_i u_i$.

6°) Generally. Approxi: $u \in W^{1,p}(U)$ by (u_n).

$u_n \in C^\infty(\bar{U})$. Define: $\lim E u_n = \lim \bar{u}_n = Eu$

(This is only fit for $1 \le p < \infty$).

7°) Alternative way for $1 \le p \le \infty$:

$Q = \{(x', x_n) \in \mathbb{R}^n \mid |x'| < 1, |x_n| < 1\}$. $Q^+ = Q \cap \mathbb{R}^n_+$.

<u>Lemma</u>. For $u \in W^{1,p}(Q^+)$. $1 \le p \le \infty$. Extend u

to Q: $u^*(x', x_n) = \begin{cases} u(x', x_n), & x_n \ge 0 \\ u(x', -x_n) & x_n \le 0 \end{cases}$

Then $u^* \in W^{1,p}(Q)$. $\|u^*\|_{L^p(Q)} \le 2\|u\|_{L^p(Q^+)}$

$\|u^*\|_{W^{1,p}(Q)} \le 2\|u\|_{W^{1,p}(Q^+)}$

<u>Pf</u>: prove: $\frac{\partial u^*}{\partial x_k}$ exists $\begin{cases} \frac{\partial u^*}{\partial x_i} = (\frac{\partial u}{\partial x_i})^*. & 1 \le i \le n-1 \\ \frac{\partial u^*}{\partial x_n} = (\frac{\partial u}{\partial x_n})^\wedge \end{cases}$

$\wedge$ is def: $\hat{f}(x', x_n) = \begin{cases} f(x', x_n), & x_n > 0 \\ -f(x', -x_n), & x_n < 0. \end{cases}$

set $\eta(t) \in C^\infty(\mathbb{R})$.

$\eta(t) = \begin{cases} 1. & t > 1 \\ 0. & t < 1/2 \end{cases}$ $\eta_k = \eta(kt)$.

i) $1 \le i \le n-1$:

$$\int_Q u^* \frac{\partial \phi}{\partial x_i} = \int_{Q^+} u \frac{\partial \psi}{\partial x_i}. \quad \psi = \phi(x', x_n) + \phi(x', -x_n)$$

since ψ may $\notin C_c^\infty(Q^+)$. Truncate ψ by $\eta_k(x_n)$:

Consider $\eta_k(x_n)\psi(x', x_n) \in C_c^\infty(Q^+)$. $\frac{\partial(\eta_k \psi)}{\partial x_i} = \eta_k \frac{\partial \psi}{\partial x_i}$.

$$\therefore \int_{Q^+} u \eta_k \frac{\partial \psi}{\partial x_i} = -\int_{Q^+} \frac{\partial u}{\partial x_i} \eta_k \psi. \text{ let } k \to \infty.$$

$$\therefore \int_Q u^* \frac{\partial \phi}{\partial x_i} = -\int_Q (\frac{\partial u}{\partial x_i})^* \phi.$$

ii) For n: $\int_{\mathbb{R}} u^* \frac{\partial \phi}{\partial x_n} = \int_{\mathbb{R}^+} u \frac{\partial \psi}{\partial x_n}$

$\psi = \phi(x', x_n) - \phi(x', -x_n)$. $\psi(x', 0) = 0$. $\therefore |\psi(x', x_n)| \le m|x_n|$

Consider $\eta_k(x_n)\psi(x', x_n) \in C_c^\infty(\mathbb{R}^+)$.

$$\frac{\partial(\psi \cdot \eta_k(x_n))}{\partial x_n} = k\frac{\partial \eta_k}{\partial x_n} \cdot \psi + \eta_k(x_n)\frac{\partial \psi}{\partial x_n}.$$

check: $\int_{\mathbb{R}^+} u k \eta_k'(x_n)\psi \longrightarrow 0 \ (k \to \infty)$. $(\|\eta_k'(t)\|_\infty \to 0, k \to \infty)$

$$\therefore \int_{\mathbb{R}} u^* \frac{\partial \phi}{\partial x} = \lim_k \int_{\mathbb{R}^+} u k \eta_k'(x_n)\psi + u\eta_k \frac{\partial \psi}{\partial x_n}$$

$$= \lim_k -\int_{\mathbb{R}^+} \frac{\partial u}{\partial x_n}\eta_k(x_n)\psi = -\int_{\mathbb{R}^+}\frac{\partial u}{\partial x_n}\psi = \int_{\mathbb{R}} \left(\frac{\partial u}{\partial x_n}\right)^* \phi.$$

$\Rightarrow$ The same operation by before. $\partial U \subseteq \bigcup_i Q_k$. PoU.

<u>Remark:</u> For $k=2$. $W^{2,p}(U)$ can be extended by similar method. if ∂U is C^2.

<u>Cor.</u> For U open bounded. ∂U is C^1. $\forall u \in W^{1,p}(U)$. $1 \le p < \infty$. $\exists (u_n) \subseteq C_c^\infty(\mathbb{R}^n)$. st. $u_n|_U \to u$ in $W^{1,p}(U)$. i.e. $C_c^\infty(\mathbb{R}^n)|_U$ is a dense linear subspace of $W^{1,p}(U)$.

<u>pf:</u> Choose $u_n = T_n(x)(\eta_\varepsilon * Eu) \longrightarrow Eu$.

$$T_n(x) = \begin{cases} x, & |x| \le n \\ 0, & |x| > n \end{cases}$$

<u>Remark:</u> For ∂U isn't bounded. Find n_0 st.

$\|T_{n_0}u - u\|_{W^{1,p}(U)} < \varepsilon$. Approxi. $T_{n_0}u$.

Since it defined on $\{x < n_0\} \cap \partial U$.

(4) Trace:

· Fix $1 \le p < \infty$:

① Thm: (Trace Thm)

U is bounded. ∂U is C^1. Then there exists

BLO: $T: W^{1,p}(U) \longrightarrow L^p(\partial U)$. s.t.

i) $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\bar{U})$.

ii) $\|Tu\|_{L^p(\partial U)} \le C \|u\|_{W^{1,p}(U)}$. $C = Const.(p, U)$

Remark: For $u \in L^p(U)$. such T doesn't exist.

Pf: By contradiction: $\exists T: L^p(U) \to L^p(\partial U)$. BLO.

Let $u_m = \max\{0, 1 - m\,dist(x, \partial U)\}$.

$\therefore Tu_n \equiv 1$. $\|Tu_n\|_{L^p(\partial U)} = |\partial U| > 0$

$\|u_n\| \to 0$. when $n \to \infty$. $\therefore \|T\| \ge \frac{\|Tu_n\|}{\|u_n\|} \to \infty$

Pf: 1°) Suppose $u \in C^\infty(\bar{U})$. ∂U is flat near $x^0 \in \partial U$. locally. lying in $\{x_n = 0\}$. $U \subseteq \{x_n > 0\}$.

$B^+ = B(x^0, 1) \cap \{x_n > 0\}$. $\hat{B} = B(x^0, \frac{1}{2})$. $I = \hat{B} \cap \partial U$.

2°) Find truncate function $\zeta \in C_c^\infty(B(x^0, 1))$

$\zeta = 1$ in $\hat{B}$. $\zeta \ge 0$.

3°) $\int_I |u|^p dx' \le \int_{\{x_n = 0\}} \zeta |u|^p dx' \overset{\text{Green}}{\underset{\text{Formula}}{=}} - \int_{B^+} (\zeta |u|^p)_{x_n} dx$

$\le C \int_{B^+} |u|^p + |Du|^p$. (By Young Inequality)

4°) Convert back: $y = \Phi(x)$.

Since ∂U is cpt. cover by $\cup B_i$.

5°) Approxi. by $u_m \in C^\infty(\bar{U})$. $Tu_m = u_m|_{\partial U}$. $Tu = \lim Tu_m$.

④ **Thm.** (Trace-zero Functions)

If U is bounded. ∂U is C^1. $u \in W^{1,p}(U)$. Then

$u \in W_0^{1,p}(U) \Leftrightarrow Tu = 0$ on ∂U.

Pf. ($\Rightarrow$). $Tu_n = 0$. when $u_n \in C_c^\infty(U) \to u$. in $W^{k,p}(U)$.

$\therefore Tu = \lim Tu_n = 0$. on ∂U.

Remark: For $u \in W_0^{k,p}(U)$. $\Rightarrow D^\alpha u \in W_0^{k-|\alpha|,p}(U) \subseteq W_0^{1,p}(U)$.

for $|\alpha| \leq k-1$. by the thm. we have characterization:

$W_0^{k,p}(U) = \{u \in W^{k,p}(U) \mid D^\alpha u = 0 \text{ in } \partial U. |\alpha| \leq k-1\}$.

(5) **Sobolev Inequalities:**

① **In $W^{1,p}(U)$:**

i) **$1 \leq p < n$:**

First, we want to establish estimate of the form:

$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}$ for $u \in C_c^1(\mathbb{R}^n)$. where C, q

doesn't depend on u. Suppose it holds:

Let $u = u(\lambda x)$. We obtain:

$\|u\|_{L^q(\mathbb{R}^n)} \leq C\lambda^{1-\frac{n}{p}+\frac{n}{q}}\|Du\|_{L^p(\mathbb{R}^n)}$ $\therefore \frac{1}{q} = \frac{1}{p} - \frac{1}{n}$.

otherwise. let $\lambda \to \infty$. or 0. Contradiction!

Def: Sobolev conjugate: $p^* = 1/p^* = 1/p - 1/n$. $1 \leq p < n$.

Thm. (GNS - Inequality)

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C(p,n) \|Du\|_{L^p(\mathbb{R}^n)}. \text{ for } \forall u \in C_c^1(\mathbb{R}^n).$$

Pf: 1°) $p=1$:

$$u(x) = \int_{-\infty}^{x_i} u(x_1 \cdots x_{i-1}, \eta_i, x_{i+1} \cdots x_n) d\eta_i$$

$$\therefore |u(x)| \le \int_{\mathbb{R}^1} |Du(x_1 \cdots \eta_i \cdots x_n)| d\eta_i$$

$$\therefore |u|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left(\int_{\mathbb{R}^1} |Du(\cdots \eta_i \cdots)| d\eta_i\right)^{\frac{1}{n-1}}$$

Integrate on x_1. Apply Hölder Inequality:

$$\int_{\mathbb{R}^1} |u|^{\frac{n}{n-1}} dx_1 = \left(\int_{\mathbb{R}^1} |Du| d\eta_1\right)^{\frac{1}{n-1}} \int_{\mathbb{R}^1} \left(\prod_2^n \left(\int |Du| d\eta_i\right)^{\frac{1}{n-1}}\right) dx_1$$

$$\le \left(\int_{\mathbb{R}^1} |Du| d\eta_1\right)^{\frac{1}{n-1}} \left(\prod_2^n \int_{\mathbb{R}^1}\int_{\mathbb{R}^1} |Du| d\eta_i dx_1\right)^{\frac{1}{n-1}}$$

Repeat the process on $x_2, x_3 \cdots$:

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \le C\left(\int |Du|\right)^{\frac{n}{n-1}}.$$

2°) Let $u = |u|^{\gamma}$. $\gamma > 1$. Choose γ

Thm. (Estimate for $W^{1,p}(U)$)

U is open. bounded. ∂U is C^1. Then.

$W^{1,p}(U) \subseteq L^{p^*}(U)$. i.e. $\|u\|_{L^{p^*}} \le C\|u\|_{W^{1,p}}$

Pf: Extend u to $\bar{u} = Eu$. $\exists u_m \in C_c^\infty(\mathbb{R}^n)$.

$u_m \to \bar{u}$. in $W^{1,p}(\mathbb{R}^n)$. By $\|u_m - u_m\|_{L^{p^*}} \le \|Du_m - Du_m\|_{L^p}$

$\therefore u_m \to \bar{u}$ in $L^{p^*}(\mathbb{R}^n)$. Let $m \to \infty$ on $\|u_m\|_{L^{p^*}} \le C\|Du_m\|_{L^p}$

$$\therefore \|u\|_{L^{p^*}(U)} \le \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \le C\|D\bar{u}\|_{L^p} \le C\|\bar{u}\|_{W^{1,p}} \le C\|u\|_{W^{1,p}(U)}.$$

Thm. (Estimate for $W_0^{1,p}(U)$)

If $u \in W_0^{1,p}(U)$. U is open, bounded. Then:

$\|u\|_{L^{p^*}(U)} \le C \|Du\|_{L^p(U)}$. $C = C(p, n, U)$

Pf: $\exists\, u_m \in C_c^\infty(U) \longrightarrow u$ in $W_0^{1,p}(U)$. extend $\tilde{u}_m = \begin{cases} u_m, & x \in \operatorname{supp} u_m \\ 0, & \text{otherwise.} \end{cases}$

Remark: $L^{p^*}(U) \subseteq L^p(U)$. We can obtain:

Poincaré Inequility: $\|u\|_{L^p(U)} \le C \|Du\|_{L^p(U)}$.

Cor. (Poincaré Inequility for $W^{1,p}(U)$. $1 \le p < n$)

If $u \in W^{1,p}(U)$. Then $C\|u\|_{L^p(U)} \le \|Du\|_{L^p(U)} + \|Tu\|_{L^p(\partial U)}$

for some $C > 0$.

Pf: By contradiction. if $\exists\, (u_m)$. $(n_m) \to \infty$. st.

$\|u_m\|_{L^p(U)} / n_m \ge \|Du_m\|_{L^p(U)} + \|Tu_m\|_{L^p(\partial U)}$

Set $\|u_m\|_{L^p(U)} = 1$. $\forall m$. let $m \to \infty$.

$\therefore$ $Du_m \longrightarrow 0$. $Tu_m \longrightarrow 0$. a.e.

By reflexive. $\exists\, u_{m_k} \longrightarrow u$. $Du_{m_k} \rightharpoonup Du$ in $L^p(U)$.

$\therefore$ $\|Du\|_{L^p(U)} \le \underline{\lim} \|Du_{m_k}\|_p = 0$. $Du = 0$. a.e on U.

$\because$ $u \in W^{1,p}(U) \subset\subset L^p(U)$. $\Rightarrow$ $u_{m_k} \longrightarrow u$ in L^p $\therefore \|u\|_p = 1$.

$\therefore$ $\|Tu\|_{L^p(\partial U)} \le \|T(u_{m_k} - u)\|_{L^p(\partial U)} + \|Tu_{m_k}\|_{L^p(\partial U)} \longrightarrow 0$

$\therefore$ $Tu \equiv 0$. a.e. on ∂U. $\Rightarrow$ $u \in W_0^{1,p}(U)$.

Apply Poincaré Inequility: $\|u\|_{L^p(U)} \le \|Du\|_{L^p(U)} = 0$

$\therefore$ $\|u\|_{L^p} = 0$. contradict with $\|u\|_p = 1$. $\square$

Cor. $C\|u\|_{W^{1,p}(U)} \le \|Du\|_{L^p(U)} + \|Tu\|_{L^p(\partial U)}$

ii) $p=n$:

Note that for U is open. bounded. $\forall p<n$.

$\|u\|_{W^{1,n}(U)} \geq \|u\|_{W^{1,p}(U)} \geq C\|u\|_{L^{p^*}(U)}$.

$p^* = \frac{np}{n-p} \to \infty \; (p\to n)$. $\therefore W^{1,n}(U) \subseteq L^r(U)$.

for $\forall\, 1\leq r<\infty$.

Remark: $W^{1,n}(U) \nsubseteq L^\infty(U)$. e.g. $\ln\ln(1+\frac{1}{|x|})$

$\notin L^\infty(U)$ but belongs to $W^{1,n}(U)$

iii) $n<p<\infty$:

Thm. (Morrey's Inequality)

For $n<p\leq\infty$. $\forall u\in C^1(\mathbb{R}^n)$. we have:

$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$. $\gamma = 1-n/p$.

Pf: 1) prove: $⨍_{B(x,r)} |u(x)-u(y)|\,dy \leq C\int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\,dy$

$\forall B(x,r)\subseteq \mathbb{R}^n$. $C=C(n)$.

Note that: $|u(x+sw)-u(x)| = \left|\int_0^s \frac{d}{dt}u(x+tw)\,dt\right|$

$= \left|\int_0^s Du(x+tw)\cdot w\,dt\right| \leq \int_0^s |Du(x+tw)|\,dt$.

From $\int_{B(x,r)} |u(x)-u(y)| = \int_0^r \int_{\partial B(x,s)} |u(x)-u(z)|\,dS(z)\,ds$

$= \int_0^r \int_{\partial B(0,1)} s^{n-1}|u(x)-u(x+sw)|\,dS(w)\,ds$

$\leq \int_0^r \int_0^s \int_{\partial B(0,1)} s^{n-1}|Du(x+tw)|\,dt\,dS(w)\,ds$

2) Prove: $\sup|u| \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$

Note $|u(x)| \le \fint_{B(x,1)} |u(x)-u(y)|\, dy + \fint_{B(x,1)} |u(y)|\, dy$

$$\le C \int_{B(x,1)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy + C\|u\|_{L^p(B(x,1))}$$

$$\le C\|Du\|_p \left(\int_{B(x,1)} |x-y|^{(1-n)\frac{p}{p-1}}\, dy \right)^{\frac{p-1}{p}} + C\|u\|_p .$$

$$\le C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

3°) $\sup\limits_{x\neq y} \left\{ \frac{|u(x)-u(y)|}{|x-y|^{1-n/p}} \right\} \le C\|Du\|_p$.

$\forall x, y \in \mathbb{R}^n$. $|x-y| = r$. $W = B(x,r) \cap B(y,r)$.

$|u(x)-u(y)| \le \fint_W |u(x)-u(z)|\, dz + \fint_W |u(y)-u(z)|\, dz$

Apply lemma in 1°).

(Refine it: $|u(y)-u(x)| \le C r^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,2r))}$)

Thm. (Estimate for $W^{1,p}(U)$)

U is bounded open. ∂U is C^1. $u \in W^{1,p}(U)$. Then

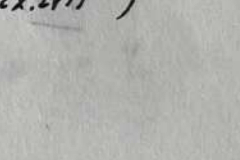
$\exists u^* = u$ a.e. $u^* \in C^{0,\gamma}(\bar{U})$. $\|u^*\|_{C^{0,\gamma}(\bar{U})} \le C\|u\|_{W^{1,p}(U)}$

Pf: Extend $\bar{u} = Eu$. $\exists u_m \in C_c^\infty(\mathbb{R}^n) \longrightarrow \bar{u}$ in $W^{1,p}$.

Similiar argue: (u_m) Cauchy in $C^{0,\gamma}(\bar{U})$ $u_m \to u^* = \bar{u}$ a.e.

Remark: ∂U is C^1 is necessary:

Consider $p=\infty$. $U = B^\circ(0,1) - \{x \ge 0, y = 0\}$ slit.



Choose $u(x_1,x_2) = x_1^2 \operatorname{sgn} x_2$.

$u_{x_1} = 2x_1 \operatorname{sgn} x_2$. $u_{x_2} = 0$. $\therefore u \in W^{1,\infty}(U)$.

But $|u(\frac{1}{2},\varepsilon) - u(\frac{1}{2},-\varepsilon)| / 2\varepsilon = \frac{1}{4\varepsilon} \to \infty$. $(\varepsilon \to 0)$

② General: in $W^{k,p}(U)$

Thm. $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$. bounded. ∂U is C^1. If $u \in W^{k,p}(U)$.

Then. i) If $k < \frac{n}{p}$. then $u \in L^q(U)$. where

$1/q = 1/p - \frac{k}{n}$. i.e. $\|u\|_{L^q} \le C\|u\|_{W^{k,p}(U)}$.

ii) If $k = \frac{n}{p}$. then $u \in L^r(U)$. $\forall r \ge 1$.

but $r \ne \infty$.

iii) If $k > \frac{n}{p}$. then $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\bar{U})$. a.e.

$$\gamma = \begin{cases} [\frac{n}{p}]+1-\frac{n}{p}, & n/p \notin \mathbb{Z}^+ \\ \forall r \in (0,1), & n/p \in \mathbb{Z}^+ \end{cases}$$

pf: i) By GNS Inequality: $\|D^\beta u\|_{p^*} \le C\|u\|_{W^{k,p}}$.

$\forall |\beta| \le k-1$. $\therefore u \in W^{k-1,p^*}(U)$.

Apply again. $u \in W^{k-2,p^{**}}$. $p^{**} = \frac{1}{p} - \frac{2}{n} \cdots$

Then $u \in W^{0,q}(U)$. by repeating k times.

ii) Consider $\forall p$. $1/p > \frac{k}{n}$. let $p \to k/n$.

iii) 1) $n/p \notin \mathbb{Z}^+$.

by i) $u \in W^{k-l,r}(U)$. $1/r = \frac{1}{p} - \frac{l}{n}$.

Let $l = [\frac{n}{p}]$. then $r > n$.

Since $D^\alpha u \in W^{1,r}(U)$. $\forall |\alpha| \le k-l-1$.

$\therefore D^\alpha u \in C^{0,1-n/r}(U)$. By Morrey Inequality.

2) $n/p \in \mathbb{Z}^+$.

Analogously. set $l = n/p - 1$. $\therefore r = n$.

since $D^\alpha u \in W^{1,n}(U)$. $\forall\, |\alpha| \le k-l-1$.

$\therefore D^\alpha u \in L^r(U)$. $\forall\, r \ge 1,\ r \ne \infty$. $\therefore D^\alpha u \in C^{0,1-\frac{n}{q}}(\bar{U})$. $\forall\, |\alpha| \le k-l-2$.

for $\forall\, n < q < \infty$. let $q \to \infty$.

Remark: The ideal is reducing to the case $W^{1,p}(U)$.

(6) Compactness:

Def: X, Y are Banach space. $X \subset Y$. We say X can be compactly embedded into Y. ($X \subset\subset Y$) if

i) $\|u\|_Y \le C\|u\|_X$.

ii) $\forall\, (u_n) \subseteq X$. $\sup_n \|u_n\| < \infty$. Then $\exists\, (u_{n_k}) \subseteq (u_n)$.

$u_{n_k} \to u$. in Y.

Remark: If $u_n \rightharpoonup u$. in X. Then $u_n \to u$ in Y.

Thm. Suppose U is bounded open. ∂U is C^1. If $1 \le p < n$.

Then $W^{1,p}(U) \subset\subset L^q(U)$. $\forall\, 1 \le q < p^*$.

Pf: $W^{1,p}(U) \subset L^q(U)$. $\forall\, 1 \le q < p^*$. we have proved.

Extend $u \in W^{1,p}(U)$ to $W^{1,p}(\mathbb{R}^n)$. where

$\forall u \in W^{1,p}(\mathbb{R}^n)$. $\operatorname{supp} u \subset V$. $U \subset\subset V$.

For $(u_n) \subseteq W^{1,p}(\mathbb{R}^n)$. $\sup_n \|u_n\|_{W^{1,p}(U)} < \infty$. $\operatorname{supp} u_n \subset V$.

1') Suppose (u_n) are smooth.

Since $\exists\, (u_n^m)$. $u_n^m \to u_n$. in $W^{1,p}(\mathbb{R}^n)$. $u_n^m \in C_c^\infty(\mathbb{R}^n)$.

prove: For $u_m^\varepsilon = \eta_\varepsilon * u_m$.

$u_m^\varepsilon \to u_m$ in $L^q(V)$

$\Leftarrow$ Check $u_m^\varepsilon \to u_m$ in L^1. and L^{p^*}.

By interpolation. we have $u_m^\varepsilon \to u_m$ in L^2.

2°) Check for $\forall$ fix $\varepsilon > 0$. (u_m^ε) is satisfing condition of Ascoli Thm.

Check: $|Du_m^\varepsilon|$. $|u_m^\varepsilon|$ are bounded. $\forall m$ (uniform)

3°) $\exists\, (u_{m_k}^\varepsilon) \subseteq (u_m^\varepsilon)$. uniformly converges on every cpt set since it has cpt support. Apply DCT:

$\|u_{m_k}^\varepsilon - u_{m_j}^\varepsilon\|_{L^q(U)} \longrightarrow 0$, $u_{m_k}^\varepsilon \overset{\delta}{\sim} u_{m_k}$. (for some fix ε).

$\therefore$ $\overline{\lim}\, \|u_{m_k} - u_{m_j}\|_{L^q(U)} \leq 2\delta$. Let $\delta = \frac{1}{2}, \frac{1}{3} \cdots \frac{1}{n}$.

by diagnoal Argument.

Remark: i) For $p = n$. $W^{1,n}(U) \subset\subset L^r(U)$. $\forall r \in \mathbb{Z}^+$.

ii) $W_0^{1,p}(U) \subset\subset L^r(U)$. $\forall r \in \mathbb{Z}^+$. for $p \geq n$.

it's from extension on i). $W^{1,p}(U) \subset W^{1,n}(U)$.

And $W_0^{1,p} \subset W^{1,p}(U)$. And it's no need ∂U is C^1. (Since it can be approxi. by C_c^∞.)

(7) Addtional Topics:

① Poincaré's Inequility:

Denote: $(u)_U = ⨍_U u\, dy$.

Thm.

U bounded. open. connected. ∂U is C^1. If $1 \le p \le \infty$.

$u \in W^{1,p}(U)$. Then $\| u - (u)_U \|_{L^p(U)} \le C(n,p,U) \| Du \|_{L^p(U)}$

Pf: Lemma. $v \in W^{1,p}(U)$. $Dv = 0$. a.e. on U. connected

open. Then $v \equiv$ const. a.e.

Pf: $v^\varepsilon = \eta_\varepsilon * u$. $\therefore Dv^\varepsilon = \eta_\varepsilon * Du \equiv 0$. a.e.

$v^\varepsilon \in C^\infty(U_\varepsilon)$ $\therefore v^\varepsilon \equiv C_\varepsilon$ on U_ε

$v^\varepsilon \longrightarrow u$ in $L^p(V)$. $\forall V \subset\subset U$. By DCT:

$\therefore \| u - C \|_{L^p} = 0$. Choose $(C_\varepsilon') \subseteq (C_\varepsilon) \longrightarrow C$.

$\therefore u \equiv C$. a.e. Approx. U by V.

1') By contradiction: $\exists (u_k)$. $\| u_k - (u_k)_U \|_{L^p(U)} > k \| Du_k \|_{L^p(U)}$.

Let $v_k = u_k - (u_k)_U / \| u_k - (u_k)_U \|_p$. $\therefore \| Dv_k \| \le \frac{1}{k}$. $\| v_k \| = 1$. $(v_k)_U = 0$.

2') Since $W^{1,p}(U) \subset\subset L^p(U)$. $\therefore \exists (v_{n_k}) \xrightarrow{L^p} v$.

$\therefore Dv = 0$. a.e. $(v)_U = 0 \Rightarrow v = 0$. a.e. but $\| v \|_p = 1$. Contradict!

Cor. (For a ball)

Under the condition above: $\| u - (u)_{B(x,r)} \|_{L^p(B(x,r))} \le C(n,p) r \| Du \|$

for $u \in W^{1,p}(B(x,r))$.

Pf: Let $U = B^\circ(0,1)$ first. on $u(x)$.

Then let $v(y) = u(x + ry)$. extract "r" variable.

② Difference Quotients:

i) For $W^{1,p}(U)$. $1\le p<\infty$:

Denote: $D_i^h u = u(x+he_i) - u(x) / h$. $0<|h|<\frac{1}{2}d(x,\partial U)$.

Next we suppose $u \in L^1_{loc}(U)$. $D^h u = (D_1^h u, \cdots D_n^h u)$.

Thm. i) If $u \in W^{1,p}(U)$. $\forall, V \subset\subset U$. Then:

$\|D^h u\|_{L^p(V)} \le C \|Du\|_{L^p(U)}$.

ii) If $1<p<\infty$. $u \in L^p(V)$ st. $\|D^h u\|_{L^p(V)} \le C$

for $\forall\ 0<h<d(V,\partial U)$. Then, we obtain:

$u \in W^{1,p}(V)$. $\|Du\|_{L^p(V)} \le C$.

Remark: ii) is false when $p=1$. e.g.

$U = (-\frac{1}{10^3}, 1+\frac{1}{10^3})^n$. $V = (0,1) \subset\subset U$.

$u(x) = \begin{cases} 0, & \text{otherwise} \\ 1, & 0<x_1<\frac{1}{2} \end{cases}$ $u \notin W^{1,p}(U)$. (Consider $D_{x_i}\phi$. $i\ne 1$)

$\|D^h u\| \le \int_{\frac{1}{2}-h}^{\frac{1}{2}} \int_{(0,1)^{n-1}} \frac{1}{|h|} dx = 1$.

Pf: i) Approxi by $C^\infty(\bar{U})$. suppose u is smooth.

From: $|u(x+he_i) - u(x)| \le |h| \int_0^1 |Du(x+the_i)| dt$.

ii) Check: $\int_V u(D_i^h \phi) = -\int (D_i^{-h} u)\phi$. for $\phi \in C_c^\infty(V)$.

only for $h>0$. sufficient small. ($h < d(\text{supp}\phi, V)$)

By reflexive of L^p. $\exists\ (D_i^{-h_k} u) \xrightarrow{L^p} v_k$.

since $\sup \|D^{-h_k} u\|_{L^p(V)} \le C$

We have: $\int u\phi_{x_k} = -\int v_k\phi$. by $h_k \to 0$

ii) For $W^{1,\infty}(U)$:

Thm. (Characterization of $W^{1,\infty}$).

U is open. bounded. ∂U is C^1. Then $\exists u^*$.

Lipschitz conti. $u^* = u$. a.e. $\Leftrightarrow u \in W^{1,\infty}(U)$.

Pf: ($\Leftarrow$) 1°) Extend to $\bar{u} = Du$. has cpt support.

2°) $u^\varepsilon = \eta_\varepsilon * \bar{u}$. (η_ε) is mollifiers.

Claim $\{u^\varepsilon\}$ is uniformly convergent :

$$|u^\varepsilon - u^\delta| \le \int |\eta(y)| \, |\bar{u}(x-\delta y) - \bar{u}(x-\varepsilon y)| \, dy$$

$$= \int |\eta(y)| \, |\bar{u}^*(x-\delta y) - \bar{u}^*(x-\varepsilon y)| \, dy \le C\varepsilon.$$

where $\bar{u}^* = \bar{u}$. a.e. $\bar{u}^* \in C^{0,1}(\mathbb{R}^n)$. by Morrey.

3°) $u^\varepsilon \xrightarrow{u} u^*$. since $u^\varepsilon \xrightarrow{L^1} \bar{u}$.

$\therefore u^* = \bar{u}$. a.e.

4°) $|u^\varepsilon(x) - u^\varepsilon(y)| \le \|Du^\varepsilon\|_\infty |x-y| \le \|D\bar{u}\|_\infty \|\eta_\varepsilon\|_1 |x-y|$

$= \|D\bar{u}\|_\infty |x-y|$. Let $\varepsilon \to 0$

we obtain u^* is Lipschitz Func.

($\Rightarrow$) Similary. $\|D_i^{-h} u\|_\infty \le \mathrm{Lip}(u)$. $\exists v_i \in L^\infty$. $D_i^{+} u \xrightarrow{L^2_{loc}} v_i$.

Remark: There're no correspond : $W^{1,p}(U) \longleftrightarrow C^{0,1-\frac{n}{p}}$. $p > n$.

③ Differentiable. a.e. :

Thm. If $u \in W^{1,p}_{loc}(U)$. $n < p \le \infty$. Then u is differentiable.

a.e. in U. its gradient = weak derivate a.e.

Pf: 1°) $n < p < \infty$:

For $v(y) = u(y) - u(x) - Du(x)\cdot(y-x)$. $v(x) = 0$. $v \in W^{1,p}_{loc}(U)$

By Morrey estimation: $|v(x) - v(y)| \le C r^{1-n/p} \left(\int_{B(x,2r)} |Du(x) - Du(z)|^p dz \right)^{\frac{1}{p}}$

where $r = |x-y|$.

By Lebesgue Differentiate Thm. $\fint_{B(x,r)} |Du(x) - Du(z)|^p dz \to 0 \ (r \to 0)$

$\therefore |u(y) - u(x) - Du(x)\cdot(y-x)| \le o(r). = o(|y-x|)$. a.e. x.

2°) $p = \infty$. By $W^{1,\infty}_{loc}(U) \subseteq W^{1,p}_{loc}(U)$

Cor. (Rademacher's Thm)

$u \in C^{0,1}_{loc}(U) \Rightarrow u$ is differentiable a.e. x.

(Converse is false)

Pf: $C^{0,1}_{loc}(U) \Leftrightarrow W^{1,\infty}_{loc}(U)$.

④ Hardy's Inequality:

Thm. (Local on ball)

If $n \ge 3$, $r > 0$. $u \in H^1(B(0,r))$, Then we have:

$u/|x| \in H^1(B(0,r))$. Besides, we have estimate:

$$\int_{B(0,r)} \frac{u^2}{|x|^2} dx \le C \int_{B(0,r)} |Du|^2 + \frac{u^2}{r^2} dx.$$

Pf: Consider: Integration by part.

Approxi by $C^\infty(B(0,r))$. Suppose $u \in C^\infty(B(0,r))$:

$$\int_{B(0,r)} u^2/|x|^2 = -\int_{B(0,r)} u^2 \cdot D\left(\tfrac{1}{|x|}\right)\cdot \frac{x}{|x|} = -\int_{B(0,r)} \sum \frac{\partial\left(\frac{1}{|x|}\right)}{\partial x_i} u^2 x_i/|x|$$

$$= -\int_{\partial B(0,r)} \sum u^2 x_i/|x| \cdot \nu^i + \int_{B(0,r)} \sum \frac{1}{|x|} \frac{\partial(u^2 x_i/|x|)}{\partial x_i}, \ \nu = \frac{x}{|x|}.$$

prove: $\frac{1}{r}\int_{\partial B(0,r)} u^2 dS \le C \int_{B(0,r)} |Du|^2 + u^2/r^2 dx$.

Thm. (For Global)

If $n \geq 3$. $u \in H^1(\mathbb{R}^n)$. Then $\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq C \int_{\mathbb{R}^n} |Du|^2 dx$

Pf: $|Du + \lambda \frac{x}{|x|^2} u|^2 \geq 0$ i.e. $\lambda^2 \frac{u^2}{|x|^2} + |Du|^2 + 2\lambda \frac{\sum_1^n x_i u_{x_i}}{|x|^2} u \geq 0$.

Integrate on $\mathbb{R}^n$: $\int \lambda^2 \frac{u^2}{|x|^2} + |Du|^2 + \lambda \sum (u^2)_{x_i} \frac{x_i}{|x|^2} \geq 0$.

Integration by part: $\int \lambda^2 \frac{u^2}{|x|^2} + |Du|^2 \geq -\lambda \int \frac{u^2}{|x|^2}$. choose $\lambda = -\frac{1}{2}$

⑤ Fourier Transf Method:

Denote: $\langle \xi \rangle^s = 1 + |\xi|^s$. $\xi \in \mathbb{R}^n$. $s \in \mathbb{Z}^+$.

Lemma. $\langle \xi \rangle^s \leq C(s) (\langle \xi - \eta \rangle^s + \langle \eta \rangle^s)$.

Pf: $1 + |\xi|^s = 1 + (|\xi|^2)^{\frac{s}{2}} \leq 1 + (|\xi - \eta|^2 + |\eta|^2)^{\frac{s}{2}}$

$\leq 1 + 2^{\frac{s}{2}} \max\{|\xi-\eta|^2, |\eta|^2\} \leq 2^{\frac{s}{2}} (\langle \xi - \eta \rangle^s + \langle \eta \rangle^s)$.

prop. $S(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$. $s \in \mathbb{Z}^+$.

Pf: 1°) $S(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. $1 \leq p < \infty$.

$k_\delta = \delta^{-\frac{n}{2}} e^{-\frac{2|x|^2}{\delta}}$. check $k_\delta * f \in S(\mathbb{R}^n)$.

2°) $\forall u \in H^s(\mathbb{R}^n)$. $\langle \xi \rangle^s \hat{u} \in L^2(\mathbb{R}^n)$.

$\exists V_k \in S(\mathbb{R}^n) \xrightarrow{L^2} \langle \xi \rangle^s \hat{u}$.

Choose $u_k = (V_k \langle \xi \rangle^{-s})^\vee$. $u_k \xrightarrow{H^s} u$. Since:

$\|u_k - u\|_{H^s(\mathbb{R}^n)} \leq C \| \langle \xi \rangle^s (\widehat{u_k - u}) \|_{L^2}$

$= C \| V_k - \langle \xi \rangle^s \hat{u} \|_{L^2} \longrightarrow 0$.

Thm. (Characterization)

If. $k \in \mathbb{Z}^+$. $u \in L^2(\mathbb{R}^n)$. Then:

$u \in H^k(\mathbb{R}^n) \iff \langle\eta\rangle^k \hat{u} \in L^2(\mathbb{R}^n)$.

Pf. ($\Rightarrow$) Approxi by $C_c^\infty(\mathbb{R}^n)$. suppose u is smooth.

$\therefore \widehat{D^\alpha u} = (i\eta)^\alpha \hat{u}$. $\therefore \|D^\alpha u\|_{L^2}^2 = \|\eta^\alpha \hat{u}\|_{L^2}^2 < \infty$. $\forall |\alpha| \le k$.

($\Leftarrow$). Denote $u_\alpha = ((i\eta)^\alpha \hat{u})^\vee$. check $u_\alpha = D^\alpha u$. week sense.

Since $\|u_\alpha\|_2^2 = \|\hat{u}(i\eta)^\alpha\|_2^2 \le \int |\eta|^{2|\alpha|} |\hat{u}|^2 \le \|\langle\eta\rangle^k \hat{u}\|_2^2$

prop. (For $H^s(\mathbb{R}^n)$. $s > \frac{n}{2}$).

For $H^s(\mathbb{R}^n)$. $s > \frac{n}{2}$. Then we have conclusions:

i) $H^s(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n)$.

ii) $H^s(\mathbb{R}^n)$ is an algebra. i.e. for $u, v \in H^s(\mathbb{R}^n)$.

$uv \in H^s(\mathbb{R}^n)$. Besides. $\|uv\|_{H^s} \le C\|u\|_{H^s}\|v\|_{H^s}$.

Pf: i) Approxi by $S(\mathbb{R}^n)$ functions for using the inversion formula:

$|u| = |\int \hat{u} e^{ix\cdot\eta} d\eta / (2\pi)^{\frac{n}{2}}| \le C\int |\hat{u}| d\eta$

$= C\int |\hat{u}\langle\eta\rangle^s| / \langle\eta\rangle^s \le C\|\hat{u}\langle\eta\rangle^s\|_2 \|\langle\eta\rangle^{-s}\|_2^2$.

ii) Approxi by Schwarz Func.

$\|uv\|_{H^s} \le C\|\langle\xi\rangle^s \widehat{uv}\|_2 = C\|\langle\xi\rangle^s \hat{u} * \hat{v}\|_2$

$\le C\|\langle\xi-\eta\rangle^s \hat{u} * \hat{v}\|_2 + C\|\langle\eta\rangle^s \hat{u} * \hat{v}\|_2$

$\le C\|\langle\xi\rangle^s \hat{u}\|_2 \|\hat{v}\|_1 + C\|\hat{u}\|_1 \|\langle\eta\rangle^s \hat{v}\|_2$ (Young)

$\le C\|u\|_{H^s} \|\hat{v}\langle\eta\rangle^s\|_2 \|\langle\eta\rangle^{-s}\|_2 + C\|v\|_{H^s} \|\hat{u}\langle\eta\rangle^s\|_2 \|\langle\eta\rangle^{-s}\|_2$

$\le C\|u\|_{H^s}\|v\|_{H^s}$.

⑥ Dual Space H^{-1}:

Denote: $H^{-1}(U)$ is the dual space of $H_0^1(U)$.

Remark: $H_0^1(U) \subsetneqq L^2(U) \subsetneqq H^{-1}(U)$.

we won't identify $H_0^1(U)$ with $H^{-1}(U)$

.... Def: $\|\cdot\|_{H^{-1}(U)}$ is : $\|f\|_{H^{-1}(U)} = \sup\{\langle f, u\rangle \mid \|u\|_{H_0^1(U)} \le 1\}$.

Thm. (Characterization)

i) $\forall f \in H^{-1}(U)$, $\exists (f^i)_0^n \subseteq L^2(U)$. s.t.

$\langle f, v\rangle = \int_U f^0 v + \sum_1^n f^i v_{x_i} \, dx$ (may not unique)

ii) For $f \in H^{-1}(U)$, we have norm representation:

$\|f\|_{H^{-1}(U)} = \inf\{ (\int_U \sum_0^n |f^i|^2)^{\frac{1}{2}} \mid (f^i)_0^n \subseteq L^2(U)$.

satisfies i) }.

Pf: i) Define inner product in $H_0^1(U)$:

$(u, v) = \int_U Du \cdot Dv + uv \, dx$. Apply Riesz Representation.

ii) For $f \in H^{-1}(U)$. $\langle f, v\rangle = \int_U g^0 v + \sum g^i v_{x_i}$. $(g^i) \subseteq L^2(U)$.

1°) $\int \sum_0^n |g^i|^2 \ge \int \sum_0^n |f^i|^2$.

By $\langle f, v\rangle = (u, v)$. Let $v = u$: $(u, u) = \int \sum |f^i|^2$.

$\therefore \int_U \sum |f^i|^2 \le \|u\| (\int \sum |g^i|^2)^{\frac{1}{2}}$

2°) $\int \sum_0^n |f^i|^2 = \|f\|_{H^{-1}(U)}$.

$|\langle f, v\rangle| \le \|f\| = \int \sum_0^n |f^i|^2$. $|v| \le 1$.

Conversely. Let $v = u/\|u\|$.