Four kinds of PDEs.

(1) Transport Equations

· Consider $u(\vec{x},t): \mathbb{R}^n \times [0,+\infty) \longrightarrow \mathbb{R}^1$. $\vec{b} \in \mathbb{R}^n$.

Solve: $u_t + \vec{b} \cdot D_x u = 0$.

$\Rightarrow$ Define: $z(s) = u(\vec{x} + s\vec{b}, t+s)$. Then

$z'(s) = u_t + \vec{b} \cdot D_x u = 0$. $z(s) \equiv 0$. $\forall s \in \mathbb{R}^1$.

It means $u \equiv c$ on the direction $(\vec{b}, 1)$.

① Initial value Problem:

$$\begin{cases} u_t + \vec{b} \cdot D_x u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g(x) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Then by the discussion before:

$u(\vec{x},t) = u(\vec{x} + s\vec{b}, t+s) = u(\vec{x} - t\vec{b}, 0) = g(\vec{x} - t\vec{b})$

② Nonhomogenous:

$$\begin{cases} u_t + \vec{b} \cdot D_x u = f(\vec{x},t). & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u(\vec{x},0) = g(\vec{x}) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Then $z'(s) = f(\vec{x} + s\vec{b}, t+s)$

$\therefore \int_{-t}^{0} z'(s)\,ds = u(\vec{x},t) - g(\vec{x} - t\vec{b}) = \int_0^t f(\vec{x} + (s-t)\vec{b}, s)\,ds$.

$\therefore u(\vec{x},t) = g(\vec{x} - t\vec{b}) + \int_0^t f(\vec{x} + (s-t)\vec{b}, s)\,ds$.

(2) Laplace's Equation:

For $u: \bar{U} \to \mathbb{R}^1$. $f: U \to \mathbb{R}^1$. $U \subseteq \mathbb{R}^n$.

Laplace's equation: $\Delta u = 0$

Poisson's equation: $-\Delta u = f$

Physical interpretation:

u denotes the density of some quantity in equilibrim. Then $\forall V \subseteq U$, the net flux of u through ∂V is zero. i.e.

$\int_{\partial V} \vec{F} \cdot \vec{\nu} \, ds = 0$. $\vec{F}$ is flux density. which is propositional to $-Du$. $\vec{F} = -a Du$

Then by $\int_V Div(\vec{F}) \, dx = 0$. $Div(\vec{F}) = 0$ ∴ $\Delta u = 0$.

① Fundamental Solution:

Lemma. Laplace's Equation is invariant under rotations. i.e. $\Delta u \equiv 0 \Rightarrow v = u(O\vec{x})$. $\Delta v \equiv 0$ where O is orthonormal matrix.

Pf: $O = (O_1 \cdots O_n)$. then $O_i^T O_j = \delta_{ij}$

$$\frac{\partial^2 u}{\partial x_i^2} = \sum_{k,j} O_{ki} O_{ji} u_{kj}. \quad \therefore \Delta u(O\vec{x}) \equiv 0.$$

⇒ Firstly. search for radical solution:

$u(\vec{x}) = v(r)$. $r = |x| = (\sum_i^n x_i^2)^{\frac{1}{2}}$.

Then $\Delta u \equiv 0 \iff v'' + \frac{n-1}{r} v' = 0$.

$\therefore V(r) = \begin{cases} b\log r + c, & n=2. \\ b/r^{n-2} + c, & n\geq 3 \end{cases}$. $b, c \in$ Const.

Def: $\phi(\vec{x}) = \begin{cases} -\frac{1}{2\pi}\log|x|, & n=2. \\ 1/n(n-2)\alpha(n)|x|^{n-2}, & n\geq 3 \end{cases}$ $x\in\mathbb{R}^n$. $x\neq\vec{0}$.

is the fundamental solution of Laplace equation.

Remark: i) $|D\phi| \leq \frac{C}{|x|^{n-1}}$. $|D^2\phi| \leq \frac{C}{|x|^n}$

ii) The constant is choosen for normalization:

Note that $\Delta\phi(\vec{x}-\vec{\eta}) = \begin{cases} 0, & \vec{x}\neq\vec{\eta} \\ \infty, & \vec{x}=\vec{\eta}. \end{cases}$ Besides.

$-\int_{\mathbb{R}^n} \Delta\phi(x-\eta)\,dx = -\int_{B(0,1)} \Delta\phi(x)\,dx = -\int_{\partial B(0,1)} D\phi\cdot\vec{\nu}\,dx$

$\vec{\nu} = \vec{x}/|\vec{x}|$. $\therefore -\int_{\mathbb{R}^n}\Delta\phi(x-\eta)\,dx = 1$.

$\Rightarrow \Delta\phi(x-\eta) = -\delta_\eta$. Dirac measure at $\vec{\eta}$.

② Poisson Equation:

· Knowing that $\Delta\phi$ is Dirac measure, we can construct solution for Poisson Equation.

For $f\in C_c^2(\mathbb{R}^n)$, $u = \phi * f$ is the solution

Thm: i) $u\in C^2(\mathbb{R}^n)$

ii) $-\Delta u = f$ on $\mathbb{R}^n$.

pf: i) check: $\frac{u(x+he_i)-u(x)}{h}$. by supp f is cpt.

$\frac{f(x-\eta+he_i)-f(x-\eta)}{h} \to f_{x_i}(x-\eta)$. uniform with $x-\eta$.

ii) keep in mind: $\begin{cases} \int_{B(0,1)} \frac{d\vec{x}}{|x|^n}. & n \geq N-1. \text{ diverges.} \\ \int_{B(0,1)^c} \frac{d\vec{x}}{|x|^n}. & n > N-1. \text{ converges.} \end{cases}$

For $\Delta u = \int \phi(y) \Delta_x f(x-y) dy$. separate into 2 parts

$\Delta u = \int_{B(0,\varepsilon)} + \int_{B(0,\varepsilon)^c} \triangleq I_\varepsilon + J_\varepsilon$. (Excavate the pole 0 of ϕ)

1°) $|I_\varepsilon| \leq \|\Delta f\|_\infty \int_{B(0,\varepsilon)} |\phi(y)| dy \leq \begin{cases} C\varepsilon^2 \log\varepsilon. & n=2. \\ C\varepsilon^2. & n \geq 3 \end{cases}$

2°) For J_ε. apply Green formular:

We want to use $\Delta\phi \equiv 0$. So convert the differentiation.

③ Harmonic Functions:

i) Mean Value Formula:

Thm. $u \in C^2(U)$. $\Delta u \equiv 0$. Then for $\forall B(x,r) \subseteq U$

$$u(x) = ⨍_{B(x,r)} u \, d\vec{y} = ⨍_{\partial B(x,r)} u \, ds.$$

Pf. Denote $\phi(r) = ⨍_{\partial B(x,r)} u \, ds$. Prove: $\phi'(r) = 0$.

1°) $\phi(r) = \frac{1}{n r^{n-1} \alpha(n)} \int_{\partial B(x,r)} u(y) \, ds(y)$.

$y = x + rz$. $z \in \partial B(0,1)$.

Since $ds(y) = \sqrt{1+|Dy_n|^2} \, dy_1 \cdots dy_{n-1}$

$= r^{n-1} \sqrt{1+|Dz_n|^2} \, dz_1 \cdots dz_{n-1}$

$= r^{n-1} \, ds(z)$.

$\therefore \phi(r) = ⨍_{\partial B(0,1)} u(x+rz) \, ds(z)$.

Apply Green Formula. assert: $\phi'(r)=0$

2°) $\phi(r) = \lim_{t\to 0} \phi(t) = u(x)$

3°) $\int_{B(x,r)} u(y)dy = \int_0^r (\int_{\partial B(x,s)} u(y)dS(y))ds.$

Thm. (Converse)

For $u\in C^2(U)$. satisfies $⨍_{\partial B(x,r)} u(y)dS(y) = u(x)$.

Then $\Delta u \equiv 0$.

Pf: $\phi'(r) = \frac{r}{n} ⨍_{B(x,r)} \Delta u(y)dy$ from above.

ii) Strong Maximal Principles:

Thm. If $u\in C^2(U)\cap C(\bar{U})$. $\Delta u\equiv 0$ in U.

Then. $\max_{\bar{U}} u = \max_{\partial U} u$. Moreover. if U is connected. exist $x_0\in U$. attain the maximal. then $u\equiv$ const. in U.

Pf: Apply Mean Value prop. $\{u\equiv M\}$ is clopen.

$\therefore$ In every component. (connected). u is const.

(If $\exists x_0\in U$. attain the maximum)

Remark: Let $u=-u$. We obtain the minimum case.

$\therefore$ Harmonic Func attain extremum at boundary.

$u\in C(\bar{U})$ guarantees $u\equiv$ const. in U then.

$u\equiv$ const on $\bar{U}$.

Cr. For $g \in C(\partial U)$. $f \in C(U)$. There exists at most one solution $u \in C^2(U) \cap C(\bar{U})$ of:

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$ boundary-value problem.

iii) Regularity:

Thm. If $u \in C(U)$, satisfies mean-value property. Then $u \in C^\infty(U)$

pf: $(\eta_\varepsilon)_{\varepsilon>0}$ is set of mollifiers.

Consider $u^\varepsilon = \eta_\varepsilon * u$ in $U_\varepsilon = \{ \operatorname{dist}(x, \partial U) > \varepsilon \}$.

Check $u^\varepsilon \equiv u$ in U_ε (since $\operatorname{supp} \eta_\varepsilon = \overline{B(0,\varepsilon)}$). $\forall \varepsilon > 0$.

Remark: $C(U)$ (or $L^1_{loc}(U)$) + mean-value prop $\Rightarrow$ Harmonic $\begin{cases} u \in C^2(U) \\ \Delta u = 0 \text{ in } U \end{cases}$

iv) Local Estimation:

Thm. (Estimate on derivates)

u is harmonic in U. Then $|D^\alpha u(x_0)| \le \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$

for $\forall B(x_0, r) \subset U$. $\forall \alpha$. $|\alpha| = k \ge 0$.

where $C_0 = 1/\alpha(n)$. $C_k = (2^{n+1} n k)^k / \alpha(n)$

pf: $k=0$ is from mean-value.

$k=1$: Consider $u(x)$ by $⨍_{B(x,\frac{r}{2})} u(y)\,dy$:

$|u_{x_i}(x_0)| \le \frac{n}{r} \|u\|_{L^\infty(\partial B(x_0,\frac{r}{2}))}$. $1 \le i \le n$.

For $\forall x \in \partial B(x, \frac{r}{2}) \subseteq B(x_0, r)$.

$|u(x)| \le \frac{1}{\alpha(n)(\frac{r}{2})^n}\|u\|_{L^1(B(x,\frac{r}{2}))} \le \frac{2^n}{\alpha(n)r^n}\|u\|_{L^1(B(x_0,r))}$

$\therefore \|u\|_{L^\infty(\partial B(x_0,r/2))} \le \frac{2^n}{\alpha(n)r^n}\|u\|_{L^1(B(x_0,r))}$.

$\therefore |D^\alpha u| \le \frac{2^{n+1}n}{\alpha(n)}\frac{1}{r^{n+1}}\|u\|_{L^1(B(x_0,r))}$. $|\alpha|=1$.

$k\ge 2$: By induction. Consider $x\in\partial B(x_0,\frac{r}{k})$. $B(x,\frac{k-1}{k}r)\subseteq B(x_0,r)$.

Cor. (Liouville Thm)

$u:\mathbb{R}^n\longrightarrow\mathbb{R}^1$. Harmonic. bounded. Then $u\equiv$ Const.

Pf: $|Du(x_0)| \le \frac{\sqrt{n}C_1}{r^{n+1}}\|u\|_{L^1(B(x_0,r))} \le \frac{\sqrt{n}C_1\alpha(n)}{r}\|u\|_\infty \to 0$

As $r\to\infty$. ($|Du|\le\sqrt{n}\max_{1\le i\le n}|D_i u|$.)

Thm (Representation Formula)

$f\in C_c^2(\mathbb{R}^n)$. $N\ge 3$. Then any bounded solution

of $-\Delta u=f$ in $\mathbb{R}^n$. has form:

$\int_{\mathbb{R}^n}\Phi(x-y)f(y)dy + C = u(x)$. C is const.

Remark: $N=2$. $\Phi=-\frac{1}{2\pi}\log|x|\to\infty\ (|x|\to\infty)$

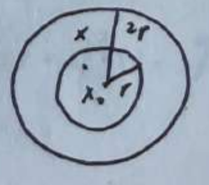
So may be $\Phi * f$. (Not exist).

V) Analyticity:

Thm. u is Harmonic in U. Then u is analytic in U.

Pf. Fix $x_0\in U$. Denote $r=\mathrm{dist}(x_0,\partial U)/4$.

$M=\frac{1}{\alpha(n)r^n}\|u\|_{L^1(B(x_0,2r))}$



$\forall x\in B(x_0,r)$. $B(x,r)\subseteq B(x_0,2r)$

For the convergence of Taylor series:

estimate the reminder term: $R_N(x) = \sum_{|\alpha|=N} \frac{D^\alpha u(x_0+t(x-x_0))(x-x_0)^\alpha}{\alpha!}$

where $0 \le t \le 1$. depend on x.

$\Rightarrow$ Estimate: $\|D^\alpha u\|_{L^\infty(B(x_0,r))} \le m\left(\frac{2^{n+1} n|\alpha|}{r}\right)^{|\alpha|}$

since $\forall x \in B(x_0,r)$. $|D^\alpha u(x)| \le \frac{C_{|\alpha|}}{(2r)^{n+|\alpha|}} \|u\|_{L^1(B(x_0,2r))}$. locally.

$\Rightarrow$ Estimate $|\alpha|^{|\alpha|}$:

we perfer find λ. st. $\lambda^{|\alpha|}\alpha! \ge |\alpha|^{|\alpha|}$. λ fixed.

1°) $e^\lambda \ge \frac{\lambda^k}{k!}$ $\therefore$ $|\alpha|^{|\alpha|} \le e^{|\alpha|}|\alpha|!$

2°) $n^{|\alpha|} = (1+1+1\cdots+1)^{|\alpha|} = \sum_{|\beta|=|\alpha|} \frac{|\alpha|!}{\beta!} \ge \frac{|\alpha|!}{\alpha!}$

$\Rightarrow$ $|\alpha|^{|\alpha|} \le (ne)^{|\alpha|}\alpha!$ Choose : $|x-x_0| \le \frac{r}{2^{n+2}n^3 e}$.

3°) Note that $\sum_{|\alpha|=N} 1 \le \sum_{|\alpha|=N} \frac{|\alpha|!}{\alpha!} = (1+1+1\cdots+1)^{|\alpha|}$

$= n^{|\alpha|} = n^N$. Let $N \to \infty$. $R_N \to 0$.

vi) Harnack Inequility:

Denote : $V \subset \bar{V} \subset U$. by $V \subset\subset U$. ($\bar{V}$ cpt)

Thm. $V \subset\subset U$. connected open. Then $\exists C$ depends on V. st. $\sup_V u \le C \inf_V u$. for every

$u \ge 0$. harmonic. on U. ($C > 0$).

Pf: 1°) By mean value. Consider $B(x,2r)$

$\forall y \in B(x,r)$. $B(y,r) \subseteq B(x,2r)$.

Check: $2^n u(y) \ge u(x) \ge \frac{1}{2^n} u(y)$

2°) Cover $\bar{V}$ by $(B_k)_1^n$. $r(B_k) < r$.

④ Green's Function:

· Suppose $U \subseteq \mathbb{R}^n$. open. bounded. ∂U is C^1.

We want to express solution of $\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$

i) Derivation:

For $u \in C^2(\bar{U})$. Fix $x \in U$. $B(x,\varepsilon) \subset U$. choose $\varepsilon > 0$.

Denote $V_\varepsilon = U - B(x,\varepsilon) \longrightarrow U$ $(\varepsilon \to 0)$. $\partial V_\varepsilon = \partial U + \partial B(x,\varepsilon)$.

$\therefore \int_{V_\varepsilon} u(y) \Delta \phi(y-x) - \Delta u(y) \phi(y-x) \, dy =$

$\int_{\partial V_\varepsilon} u(y) \frac{\partial \phi(y-x)}{\partial \nu} - \frac{\partial u}{\partial \nu} \phi(y-x) \, dS(y)$ $(\Delta \phi(y-x) = 0 \, . \, y \neq x)$.

1') $\int_{\partial B(x,\varepsilon)} u(y) \frac{\partial \phi(y-x)}{\partial \nu} \, dS(y) = ⨍_{\partial B(x,\varepsilon)} u(y) \, dS \to u(x)$

2') $\left| \int_{\partial B(x,\varepsilon)} \frac{\partial u}{\partial \nu} \phi(y-x) \, dS(y) \right| \leq \begin{cases} C\varepsilon \\ C\varepsilon^{n-1} \log \varepsilon \end{cases} \to 0.$

$\therefore u(x) = \int_{\partial U} \phi(y-x) \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial \phi(y-x)}{\partial \nu} \, dS(y) - \int_U \phi(y-x) \Delta u(y) \, dy.$

Remark: We can express $u(x)$. if we know:
the value: $\frac{\partial u}{\partial \nu}$. u on ∂U. Δu in U

$\Rightarrow$ Find a corrector function $\phi^x(y)$. avoid us to use "$\frac{\partial u}{\partial \nu}$ on ∂U". which satisfies:

$\begin{cases} \Delta \phi^x(y) = 0 & \text{in } U \\ \phi^x(y) = \phi(y-x) & \text{on } \partial U \end{cases}$

Def: Green Function $G(x,y) = \phi(y-x) - \phi^x(y)$. $y \neq x$.

Remark: $\begin{cases} \Delta G = -\delta_x & \text{in } U \\ G = 0 & \text{on } \partial U \end{cases}$

Thm. The $C^2(\bar{U})$ solution $u(x) =$

$$-\int_{\partial U} g(\eta) \frac{\partial G}{\partial \nu} \, dS(\eta) + \int_U f(\eta) G \, d\eta.$$

Pf: Check by Green Formula.

prop. (Symmetry)

$\forall x, \eta \in U,\ x \neq \eta,\ G(x,\eta) = G(\eta, x)$

Pf: $v(z) = G(x,z)$, $u(z) = G(\eta, z)$.

$V_\varepsilon = U - B(x,\varepsilon) \cup B(\eta,\varepsilon)$.

Apply Green Formula. (Note: $\Delta v = \Delta u \equiv 0$)

ii) Green Function for half Space:

Consider $\mathbb{R}^n_+ = \{ \vec{x} = (x_1 \cdots x_n) \mid x_n > 0 \}$.

Denote: reflection $\tilde{x} = (x_1 \cdots x_{n-1}, -x_n)$ for $\vec{x} = (x_1 \cdots x_n)$.

Then $\phi^x(\eta) = \phi(\eta - \tilde{x})$. is the corrector. (Ideal is remove singularity x.)

Check $\frac{\partial G}{\partial \nu} = -\frac{2x_n}{n\alpha(n)} \frac{1}{|x-\eta|^n}$. $\vec{\nu} = (0, \cdots, 1)$. Denote by $K(x,\eta)$.

Remark: For U is bounded. we can prove directly:

$$\int_{\partial U} \frac{\partial G}{\partial \nu} \, dS = \int_U \Delta G \, dS = -1.$$

But for $U = \mathbb{R}_n^+$. check: $\int_{\partial \mathbb{R}_n^+} K(x,\eta) \, d\eta = 1$

Thm. Suppose $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$. for $u(x) =$

$$\int_{\partial \mathbb{R}^n_+} K(x,\eta) g(\eta) d\eta \text{. Then } u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+).$$

satisfies i) $\Delta u \equiv 0$ in $\mathbb{R}^n_+$

ii) $\lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^n_+}} u(x) = g(x_0)$. $\forall x_0 \in \partial \mathbb{R}^+_n$.

(*) for $\begin{cases} \Delta u = f \\ u = g \end{cases}$.

u may not converge even if $f \in C_c(\mathbb{R}^n_+)$

i.e. solve $\begin{cases} \Delta u \equiv 0. & \text{in } \mathbb{R}^n_+ \ (*) \\ u = g & \text{on } \partial \mathbb{R}^n_+. \end{cases}$

Pf: 1°) $|u(x)| \le \|g\|_\infty \left|\int K(x,\eta)\right| = \|g\|_\infty$

$\therefore u(x)$ is bounded. (well-def).

2°) $\Delta K \equiv 0$. $\therefore u \in C^\infty(\mathbb{R}^n_+)$.

3°) For $|u(x) - g(x_0)| \le \int_{\partial \mathbb{R}^n_+ \cap B(x_0,\delta)} K |g(\eta) - g(x_0)|$

$$+ \int_{\partial \mathbb{R}^n_+ - B(x_0,\delta)} K(x,\eta) |g(\eta) - g(x_0)| \triangleq I + J.$$

$I \le \varepsilon$. easy to see. (For small δ)

For J: control $|x - x_0| < \frac{\delta}{2}$.

Then $|\eta - x| \ge \frac{1}{2}|\eta - x_0|$. $\therefore J \le 2\|g\|_\infty \int_\square \frac{2^{n+2} x_n \, d\eta}{|\eta - x_0|^n \alpha(n)}$.

iii) Green's Func for a ball:

Def: For $x \in \mathbb{R}^n / \{0\}$. $\tilde{x} = \frac{x}{|x|^2}$. it's dual point of x w.r.t $\partial B(0,1)$.

Remark: Similar with half-plane. We want to remove the singularity by symmetry.

Consider $\phi^x(y) = \phi(|x|(y-\tilde{x}))$. for $x \in B^\circ(0,1)$.

Thm. Suppose $g \in C(\partial(B(0,r)))$, $\tilde{u}(x) = u(rx)$.

where $u(x) = \int_{\partial B(0,1)} K(x,y)\, g(y)\, dy$. solve

$$\begin{cases} \Delta u = 0 & \text{in } B(0,r) \\ u = g & \text{on } \partial B(0,r) \end{cases} \quad \text{and } \tilde{u} \in C^\infty(B(0,r))$$

③ Energy Method:

· It's a technique involving L^2-norm. e.g. if we want to evalue the fluctuation of u. then consider $\int_U |Du|^2\, dx$.

Dirichlet's Principle:

Def: $I[w] = \int_U \frac{1}{2}|Dw|^2 - wf\, dx$ is the energy functional w.r.t Poisson equation:

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases} \quad u \in A. \text{ admissible set.}$$

where $A = \{ w \in C^2(\bar{U}) \mid w = g \text{ on } \partial U \}$.

Thm. $u \in C^2(\bar{U})$. u solves the Poisson equation

$\Leftrightarrow I[u] = \min_{w \in A} I[w]$.

Remark: It's characterization of the solution

Pf: ($\Rightarrow$). $\int_U (-\Delta u - f)(u-w)\,dx = 0$. for $w \in A$.

Integration by part. Note that $u-w$ cancels the boundary value.

$\therefore \int_U Du \cdot D(u-w) - f(u-w) = 0$. check by Am-Gm.

($\Leftarrow$). Fix $v \in C_c^\infty(U)$. $i(\tau) = I(u+\tau v)$. $\tau \in \mathbb{R}$.

$\therefore i'(\tau)|_{\tau=0} = 0$. i.e. $\int (-\Delta u - f) v = 0$. $\forall v \in C_c^\infty(U)$

(3) Heat Equation:

· $u_t - \Delta_x u = f$. $u : \bar{U} \times [0, +\infty) \longrightarrow \mathbb{R}$. $U \underset{open}{\subseteq} \mathbb{R}^n$. $f : U \times \bar{\mathbb{R}}_+ \longrightarrow \mathbb{R}$.

Physical Intepretion:

It describes the evolution in time of density $u(x,t)$.
of some quantity. e.g. heat. chemical concentration.
For $V \subset U$. the change of total quantity in V is:

$\frac{d}{dt} \int_V u\,dx = - \int_{\partial V} \vec{F} \cdot \vec{\nu}\,dS$. - net flux on ∂V. ($\vec{F}$ is density)

Then $u_t = -\mathrm{Div}(\vec{F})$. $\vec{F} = -a Du$. $\therefore u_t = a \Delta u$.

① Fundamental Solution:

· consider $u_t - \Delta u = 0$. (Heat equation)

Firstly, find something have invariant property:
Note that if $u(x,t)$ satisfies the equation. then
so $u(\lambda x, \lambda^2 t)$. The scalaring ratio $\frac{|x|^2}{t}$ is important.

Suppose $u(x,t) = v(\frac{x}{t^{\beta}})/t^{\alpha}$. $(x \in \mathbb{R}^n, t>0)$

which is from $u(x,t) = \lambda^{\alpha} u(\lambda^{\beta}x, \lambda t)$. Let $\lambda = t^{-1}$. and.

$v(\eta) = u(\eta, 1)$. i.e. invariant under $u(x,t) \mapsto \lambda^{\alpha} u(\lambda^{\beta} x, \lambda t)$

invert into the equation: $(\eta = x/t^{\beta})$

$$\alpha t^{-(\alpha+1)} v(\eta) + \beta t^{-(\alpha+1)} \eta \cdot Dv(\eta) + t^{-(\alpha+2\beta)} \Delta v(\eta) = 0.$$

Let $\beta = \frac{1}{2}$. Cancel $t^{-(\alpha+1)}$. Let $v(\eta) = w(|\eta|)$.

$$\therefore \alpha w + \frac{r}{2} w' + w'' + \frac{n-1}{r} w' = 0$$

Let $\alpha = \frac{n}{2}$ $\therefore (r^{n-1} w')' + \frac{1}{2}(r^n w)' = 0.$

Let $w \in S(\mathbb{R}^1)$. $\therefore r^{n-1} w' + \frac{1}{2} r^n w = 0$. $\therefore w = b e^{-\frac{r^2}{4}}$

<u>Def:</u> $\phi(\vec{x},t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & , \vec{x} \in \mathbb{R}^n, t>0 \\ 0 & , \vec{x} \in \mathbb{R}^n, t<0 \end{cases}$

<u>Remark:</u> $\begin{cases} \phi_t - \Delta \phi = 0, & \mathbb{R}^n \times (0,+\infty) \\ \phi(\vec{x},0) = \delta_0(\vec{x}). & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$

Note that $\int_{\mathbb{R}^n} \phi(\vec{x},t)\, d\vec{x} = 1.$ $\forall t>0.$

$$\therefore \lim_{t \to 0^+} \int_{\mathbb{R}^n} \phi(\vec{x},t)\, d\vec{x} = 1 = \int_{\mathbb{R}^n} \phi(\vec{x},0)\, d\vec{x}.$$

② <u>Initial-Value Problem:</u>

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Thm. If $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. $u(x,t) = \Phi(\cdot,t) * g(x)$

Then $u \in C^\infty(\mathbb{R}^n \times (0,+\infty))$. satisfies.

i) $u_t - \Delta u \equiv 0 \quad (x \in \mathbb{R}^n,\ t>0)$. ii) $\lim\limits_{\substack{(x,t)\to(x_0,0)\\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = g(x_0)$.

Pf: 1') Check u is bounded (point-wise). so well-def

and $u \in C^\infty(\mathbb{R}^n \times [\delta,\infty))$. $\forall \delta > 0$.

Besides. $u_t - \Delta u = \int_{\mathbb{R}^n} (\Phi_t - \Delta \Phi) u = 0$.

2') $|u(x,t) - g(x_0)| \le \int_{B(x_0,\delta)} \Phi(x-\eta,t) |g(\eta) - g(x_0)| \, d\eta$

$+ \int_{\mathbb{R}^n / B(x_0,\delta)} \Phi(x-\eta,t) |g(\eta) - g(x_0)| \, d\eta \triangleq I + J$.

Similarly. let δ be small enough. s.t.

$I \le \|g\|_\infty \varepsilon$. $J \le 2\|g\|_\infty \int_{\mathbb{R}^n/B(x_0,\delta)} \Phi(x_0-\eta,t)\, d\eta$. $(|\eta - x| > \frac{1}{2}|\eta - x_0|)$

Remark: If $g \in C(\mathbb{R}^n) \cap B(\mathbb{R}^n)$. $g \ge 0$. $g \not\equiv 0$. then

$u(\vec{x},t) > 0$ on $\forall (\vec{x},t) \in \mathbb{R}^n \times (0,+\infty)$. i.e.

If initial temperature ≥ 0. and >0 on somewhere

$\subseteq \mathbb{R}^n \times \{0\}$. Then $T > 0$. on everywhere. every later time.

iii) Nonhomogeneous Problem:

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{0\} \end{cases} \Rightarrow \quad (a) \begin{cases} u_t - \Delta u = 0 \\ u = g \end{cases} \text{ Solution } u_a$$

We can separate it into two group of equations

$$(b) \begin{cases} u_t - \Delta u = f \\ u \equiv 0 \end{cases} \text{ Solution } u_b$$

Let $u(x,t) = u_a(x,t) + u_b(x,t)$. obtain the solution of original equation. ($u_a = \phi * g$. known).

For Equation (b):

We wanna to remove f on the initial value:

$u = f$ on $\mathbb{R}^n \times \{t=0\}$. Since $u_t - \Delta u = f$ is about the rate of change. Apply integration to accumulate it. Note that $u(x,t;t-s) = \int_{\mathbb{R}^n} \phi(x-\eta, s) f(\eta, t-s) d\eta$

Solve: $\begin{cases} u_t(\cdot\,; t-s) - \Delta u(\cdot\,; t-s) = 0. & \mathbb{R}^n \times (s, +\infty) \\ u(\cdot\,; t-s) = f(\cdot\,; t-s). & \mathbb{R}^n \times \{s\}. \end{cases}$ $(t-s>0)$

Claim: $\int_0^t u(x,t;t-s) ds = u(x,t)$ solve (b).

<u>Thm.</u> If $f \in C_c^{(2,1)}(\mathbb{R}^n \times [0,+\infty))$

Then. $u \in C^{(2,1)}(\mathbb{R}^n \times [0,+\infty))$. satisfies.

i) $u_t - \Delta u = f$. $x \in \mathbb{R}^n$. $t>0$

ii) $\lim_{\substack{(x,t)\to(x_0,0) \\ t>0}} u(x,t) = 0$. $\forall x_0 \in \mathbb{R}^n$.

<u>Pf:</u> 1°) Check by definition of differentiation

Apply Dominated Converge Thm.

2°) $u_t - \Delta u = (\int_\varepsilon^t + \int_0^\varepsilon) \int_{\mathbb{R}^n} \phi(-\frac{\partial}{\partial s} - \Delta_\eta) f(x-\eta; t-s) d\eta ds$

$+ \int_{\mathbb{R}^n} \phi(\eta, t) f(x-\eta, 0) d\eta ds \stackrel{\Delta}{=} I_\varepsilon + J_\varepsilon + K.$

$|J_\varepsilon| \leq (\|\Delta f\|_\infty + \|\frac{\partial f}{\partial s}\|_\infty) \int_0^\varepsilon \int_{\mathbb{R}^n} \phi = C\varepsilon.$

$I_\varepsilon = \int_{\mathbb{R}^n} \Phi(\eta,\varepsilon) f(x-\eta, t-\varepsilon)\, d\eta - K.$

By Integration by part to convert $\frac{\partial}{\partial s} \cdot \Delta\eta$ to $\Phi(\eta,s)$.

$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \Phi(\eta,\varepsilon) f(x-\eta, t-\varepsilon)\, d\eta = f(x,t)$. by Dominated converge.

3°) $\|u(x,t)\|_\infty \le \|f\|_\infty \left| \int_0^t \int_{\mathbb{R}^n} \Phi \right| \le tC \to 0 \ (t\to 0)$.

③ Properties of Solution:

i) Mean Value Formula:

For $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$. open. bounded. Fix $T>0$.

Def: Parabolic Cylinder: $U_T = U \times (0,T]$.

Parabolic boundary: $\Gamma_T = \overline{U}_T - U_T$. (no Top).

Heat ball: $E(x,t;r) = \{(\eta,s) \in \mathbb{R}^{n+1} \mid s \le t.\ \Phi(x-\eta, t-s) \ge \frac{1}{r^n}\}$.

Thm. $u \in C^{(2,1)}(U_T)$ solve heat equation $u_t = \Delta u$.

Then $u(x,t) = \frac{1}{4r^n} \int_{E(x,t;r)} u(\eta,s) \frac{|x-\eta|^2}{(t-s)^2}\, d\eta\, ds$.

for $\forall\ E(x,t;r) \subseteq U_T$.

Pf: By shift, consider $x=t=0$. $\phi(r) = \int_{E(0,0,1)} u(r\eta, r^2 s) \frac{|\eta|^2}{s^2}\, d\eta\, ds$.

Check $\phi'(r) = 0$.

(Translation and dilation is for differentiation)

$\phi'(r) = \int_{E(1)} \sum \eta_i \frac{|\eta|^2}{|s|^2} u_{\eta_i} + 2r u_s \frac{|\eta|^2}{s}\, d\eta\, ds$

$= \frac{1}{r^{n+1}} \int_{E(r)} \sum \square + \square \ d\eta\, ds \triangleq A + B.$

Introduce $\psi(\eta,s) = \ln \phi(\eta,-s) + n\log r$. $\psi = 0$ on $\partial E(r)$.

which guarantees Integration by part will cancel boundary.

Note that $\psi_{\eta_i} = \frac{\eta_i}{2s}$, $\psi_s = -\frac{n}{2s} - \frac{|\eta|^2}{4s^2}$.

In B = Remove $\frac{\partial}{\partial s}$, $\frac{\partial}{\partial \eta_i}$ on u, for u_{η_i}, $u_s = \Delta u$.

i.e. we only need $(u_{\eta_i})_i^n$ in the equation.

Check $\lim_{t\to 0} \phi(t) = u(0,0) \lim_{t\to 0} \left(\frac{1}{t^n}\int_{E(t)} \frac{|\eta|^2}{s^2}\,dy\,ds\right) = 4u(0,0)$.

ii) Strong maximum Principle:

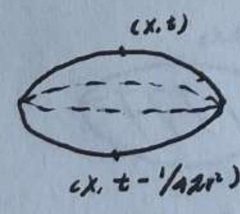
Thm. If $u \in C^{(2,1)}(U_T) \cap C(\overline{U_T})$ solve $u_t = \Delta u$ in U_T

Then i) $\max_{\overline{U_T}} u(x,t) = \max_{\Gamma_T} u(x,t)$

ii) For U is connected, exists $(x_0,t_0) \in U_T$

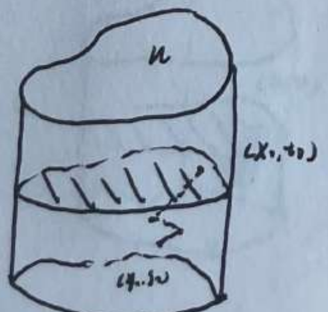
s.t. $u(x_0,t_0) = \max_{\overline{U_T}} u$. Then $u \equiv C$ in $\overline{U}_{t_0}$

Pf:



Heat Ball

$E(x,t;r)$



If $\exists (x_0,t_0)$ attain the maximum in the interior. Then by mean Value Thm.

$u \equiv M$ in $E(x_0,t_0;r) \subseteq U_T$

For $\forall \eta_0 \in U$, $t_0 > s_0 \geq 0$. We can find several line segments connect (x_0,t_0) and (η_0,s_0). for guarantee any line segment $\subseteq U_{t_0}$ (since U_T° open)

Denote $r_0 = \min\{s \geq s_0 \mid u = M,\ (x,t) \text{ in } L\}$

L is union of line segments.

prove: $r_0 = s_0$. by continuity.

For $\forall \eta_0 \in U$, $s_0 = t_0$, fix t, reduce to Laplace.

Remark: i) If $u(x,t)$ attains extremum in the interior

Then $u \equiv$ const in the earlier time.

ii) Within any U_T. It has infinite propagation

Speed again. for $\begin{cases} u_t = \Delta u & \text{in } U_T \\ u=0 & \text{on } \partial U \times [0,T] \\ u=g & \text{on } U \times \{t=0\} \end{cases}$

$g \geq 0$. positive somewhere on U.

Thm. $g \in C(\Gamma_T)$. $f \in C(U_T)$. Then exists at most one solution

$u \in C^{(2,1)}(U_T) \cap C(\bar{U}_T)$ of $\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T \end{cases}$

Thm. (For $U = \mathbb{R}^n$)

If $u \in C^{(2,1)}(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves

$\begin{cases} u_t = \Delta u & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$ satisfies $u \leq Ae^{a|x|^2}$

$\forall x \in \mathbb{R}^n$. $t \in [0,T]$. for some $A, a > 0$. Then.

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g(x).$$

Pf: 1°) Suppose $4aT < 1$. then $\exists \varepsilon > 0$. s.t. $\frac{1}{4(T+\varepsilon)} > a$.

Def: $v(x,t) = u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{|x-y|^2/4(T+\varepsilon-t)}$. $\mu > 0$.

$v_t - \Delta v = 0$. (the last term is from fundamental solution.)

2°) In $\mathbb{R}^n \times \{0\}$:

$v(x,0) \leq u(x,0) = g(x)$.

3°) In $U_T = \partial B(y,r) \times (0,T]$.

$v(x,t) \leq Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{\frac{n}{2}}} e^{r^2/4(T+\varepsilon)} \longrightarrow -\infty \ (r \to \infty)$

Since $\max_{\overline{U}_T} V(x,t) = \max_{\Gamma_T} V(x,t)$

$\therefore$ For large r. $\max_{\overline{U}_T} V \le \sup_{\mathbb{R}^n} g(x)$.

4°) From above. $V \le \sup g(x)$. Let $\mu \to 0$.

obtain $u(x,t) \le \sup g(x)$.

generally. separate $[0,T]$ into $([T_k, T_{k+1}])_0^{N-1}$

$T_0 = 0$. $T_N = T$. $|T_k - T_{k+1}| < \frac{1}{4a}$.

Thm. (Uniqueness)

Suppose $g \in C(\mathbb{R}^n)$. $f \in C(\mathbb{R}^n \times [0,T])$. Then exists at most one $u(x,t) \in C^{(2,1)}(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$.

of $\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,T] \\ u = g & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$ satisfies $|u| \le A e^{a|x|^2}$.

Remark: For $f = g = 0$. then $u \equiv 0$ is one solution.

by Thm. Other solution will grow rapidly.

iii) Regularity:

Def: $C(x,t;r) = \{(y,s) \mid |x-y| \le r,\ t - r^2 \le s \le t\}$.

which is a cylinder.

Thm. $u \in C^{(2,1)}(U_T)$ solve $u_t - \Delta u = 0$ in U_T.

Then $u \in C^\infty(U_T)$

Pf. 1°) For $(x_0, t_0) \in U_T$. Consider $C = C(x_0, t_0; r) \subseteq U_T$.

$C' = C(x_0, t_0; \frac{3r}{4})$. $C'' = C(x_0, t_0; \frac{r}{2})$.

Construct: ζ smooth. $|\zeta| \le 1$. $\zeta \equiv 1$ in C'.

$\zeta \equiv 0$ in $\overline{\mathbb{R}^n \times [0, T] - C}$.

prove: u is smooth in C''.

2°) Suppose u is smooth firstly. $v = \zeta u$.

$v = 0$ on $\mathbb{R}^n \times \{0\}$. $v_t - \Delta v = \zeta_t u - 2D\zeta \cdot Du - u\Delta\zeta \triangleq \tilde{f}$. in $\mathbb{R}^n \times (0, t_0)$.

$\tilde{v} = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) \tilde{f}(y, s)\, dy\, ds$ solve $\begin{cases} v_t - \Delta v = \tilde{f} \\ v = 0. \end{cases}$

Since $|v|, |\tilde{v}| \le C$ (cpt supp of f, v) $\therefore \tilde{v} = v$.

$\therefore$ For $(x, t) \in C''$. $u(x, t) = \int_C \Phi\, [(\zeta_s - \Delta\zeta) u - 2D\zeta \cdot Du]\, dy\, ds$.

Remove $\frac{\partial}{\partial s}$, Δ out of u. $\therefore u \triangleq \int_C K(x, t, y, s)\, u\, dy\, ds$.

3°) Replace u by $u_\varepsilon = u * \eta_\varepsilon \in C^\infty$.

obtain the formula in 2°). $\varepsilon \to 0$.

$K = 0$ in C'. K is smooth on $C - C'$. $\therefore u \in C^\infty(C'')$.

iv) Local Estimate:

Thm. There exists C_{kl} const. w.r.t k, l. s.t.

$$\max_{C(x, t, \frac{r}{2})} |D_x^k D_t^l u| \le \frac{C_{kl}}{r^{k+2l+n+2}} \|u\|_{L^1(C(x, t, r))}$$

for $\forall$ $C(x, t, \frac{r}{2}) \subseteq C(x, t, r) \subseteq U_T$. u solves heat

equation in U_T.

Pf: 1°) $\forall (x,t)$ in $C(x,t;\frac{1}{2})$.

$$|u(x,t)| = \left|\int_{C(x,t,1)} K(x,t,y,s)\,u(y,s)\,dy\,ds\right| \le C_{kl}\,\|u\|_{L^1(C(x,t,1))}$$

2°) For $C(x,t,\frac{r}{2})$. Define $v(x,t) = u(rx, r^2t)$.

Note that $D_x^k D_t^l v = r^{2l+k} D_x^k D_t^l u$. $\|v\|_{L^1} = \frac{1}{r^{n+2}}\|u\|_{L^1(C(\cdots))}$

Remark: $u(\vec{x},t)$ will be analytic on $\vec{x}$. But not in the variable t.

④ Energy Method:

- Consider $e(t) = \int_U w^2(x,t)\,dx$. $w_t = \Delta w$.

$\therefore \dot{e}(t) = -\int_U |Dw|^2 dx \le 0$. $e(t)$ is the energy functional.

Backward Uniqueness:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T \\ u = g & \text{on } \partial U \times [0,T] \end{cases} \qquad \begin{cases} \tilde{u}_t - \Delta \tilde{u} = 0 & \text{in } U_T \\ \tilde{u} = g & \text{on } \partial U \times [0,T] \end{cases}$$

If $u(x,T) = \tilde{u}(x,T)$, $u, \tilde{u} \in C^2(\bar{U}_T)$. Then $u \equiv \tilde{u}$ within $\bar{U}_T$

Pf. $w = u - \tilde{u}$. Consider $e(t) = \int_U w^2 dx$

1°) $e'^2 \le e\,e''$.

Let $f(t) = \log e(t)$. $\therefore f'' \ge 0$. Convex.

2') By contradiction. $\exists [t_1, t_2] \subseteq [0, T]$. s.t. $\forall t \in [t_1, t_2)$

$e(t) > 0$. $e(t_2) = 0$. We have $f((1-\lambda)t_1 + \lambda t_2) \le (1-\lambda) f(t_1) + \lambda f(t_2)$.

But $e(t) \equiv 0$. on $[t_1, t_2]$. contradiction!.

(4) Wave Equation:

. $u_{tt} - \Delta u = f(x,t)$. $u(x,t): \bar{U} \times [0, +\infty) \longrightarrow \mathbb{R}$. $f: U \times [0,\infty) \to \mathbb{R}$.

Physical Intepletion:

$u(x,t)$ represents the displacement in some direction

of point $\vec{x}$ at time t.

For $V \subset U$. $\frac{d^2}{dt^2} \int_V u \, dx = \int_V u_{tt} \, dx = - \int_{\partial V} \vec{F} \cdot \vec{\nu} \, ds$.

$\vec{F}$ is the force acting on V through ∂V. Suppose $m = 1$.

i.e. this means: $ma = F$.

$\therefore u_{tt} = -\mathrm{Div}(F)$. $F = F(Du) \simeq -a Du$ $\therefore u_{tt} - a \Delta u = 0$.

Remark: The initial value condition should contain:

displacement u and velocity u_t at $t=0$.

① Spherical Means:

i) For $n=1$:

(a) d'Alembert's Formula:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g & \text{on } \mathbb{R} \times [t=0]. \\ u_t = h & \end{cases}$$

Note that $u_{tt} - u_{xx} = (\frac{\partial}{\partial t} + \frac{\partial}{\partial x})(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}) u = 0$

Let $v(x,t) = u_t - u_x$. Then it becomes two group of transport equations. We obtain:

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

Thm. $g \in C^2(\mathbb{R})$. $h \in C^1(\mathbb{R})$. Then $u(x,t) \in C^2(\mathbb{R} \times [0,\infty))$

satisfies $u_{tt} - u_{xx} = 0$. $\lim\limits_{\substack{(x,t)\to(x_0,0)\\ t>0}} u(x,t) = g(x_0)$. $\lim\limits_{\substack{(x,t)\to(x_0,0)\\ t>0}} u_t(x,t) = h(x_0)$

Remark: u will not attain instantaneous smoothness as the heat equation does.

(b) Reflection Method:

· consider the equation on the $\mathbb{R}^+ \times [0,\infty)$.
$$\begin{cases} u_{tt} - u_{xx} = 0. & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = g,\ u_t = h. & \text{on } \mathbb{R}^+ \times \{0\} \\ u = 0 & \text{on } \{x=0\} \times (0,\infty) \end{cases}$$

and $g(0) = h(0) = 0$. ($u, u_t = 0$ on boundary).

Extend g, h, u on $\mathbb{R}$. by odd reflection:

$$\tilde{u}(x,t) = \begin{cases} u(x,t). & \mathbb{R}_+^2. \\ -u(-x,t). & \mathbb{R}_- \times \mathbb{R}_+. \end{cases} \qquad \tilde{g}(x) = \begin{cases} g(x) & (x \geq 0) \\ -g(-x) & (x \leq 0) \end{cases}$$

$$\tilde{h}(x) = \begin{cases} h(x). & x \geq 0 \\ -h(-x). & x \leq 0. \end{cases} \Rightarrow \begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ \tilde{u} = \tilde{g}.\ \tilde{u}_t = \tilde{h}. & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$

We obtain: $\tilde{u}(x,t) = \frac{1}{2}[\tilde{g}(x+t) + \tilde{g}(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(x)\,dx$

i.e. $u(x,t) = \begin{cases} \frac{1}{2}[g(x+t)+g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy. & x \geq t \geq 0 \\ \frac{1}{2}[g(x+t)-g(t-x)] + \frac{1}{2}\int_{t-x}^{x+t} h(y)\,dy. & 0 \leq x \leq t. \end{cases}$

<u>Remark</u>: Consider t is fixed. For $x \to t^+$ and t^-:

$$u \in C^2(\mathbb{R}\times[0,\infty)) \iff g''(0) = 0.$$

(c). <u>Spherical means</u>:

suppose $n, m \geq 2$. $u \in C^m(\mathbb{R}^n\times[0,\infty))$ solve:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n\times(0,\infty) \\ u = g.\ u_t = h. & \text{on } \mathbb{R}^n\times\{0\}. \end{cases}$$

<u>Notation</u>: $U(x;r,t) = ⨍_{\partial B(x,r)} u(y,t)\,dS(y)$. similarly.

$G(x,r) = ⨍_{\partial B(x,r)} g(y)\,dS(y)$. $H(x,r) = ⨍_{\partial B(x,r)} h(y)\,dy$.

<u>Lemma</u>. Fix $x \in \mathbb{R}^n$. if u solves the equation. then

$U \in C^m(\overline{\mathbb{R}_+}\times[0,\infty))$, satisfies:

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0 & \text{in } \mathbb{R}_+\times(0,\infty) \\ U = G.\ U_t = H & \text{on } \mathbb{R}_+\times\{0\} \end{cases}$$

<u>Remark</u>: Note that integration for mean value diminishes the dimension of variables (to r,t) after fix the center point $\vec{x}$.

<u>Pf</u>: 1°) For U_r. U_{rr}:

$$U_r = \frac{r}{n}⨍_{B(x,r)} \Delta u(y,t)\,dy. \quad U_r \to 0\ (r\to 0^+)$$

$$U_{rr} = ⨍_{\partial B(x,r)} \Delta u\,dS + \left(\frac{1}{n}-1\right)⨍_{B(x,r)} \Delta u\,dy.$$

$$\longrightarrow \frac{1}{n}\Delta u(x,t) \quad (r\to 0^+)$$

Similarly, we can verify $U \in C^m(\bar{\mathbb{R}}_+ \times [0,\infty))$

2) For U_{tt}:

Using $u_{tt} = \Delta u$. $\therefore r^{n-1} U_r = \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt}$

$\therefore (r^{n-1} U_r)_r = \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt} = r^{n-1} U_{tt}$.

ii) <u>For $n = 3, 2$:</u>

· The idea is using the spherical mean to reduce the equation into one dimension.

<u>(a). $n=3$:</u>

Set $\tilde{U} = rU$. $\tilde{G} = rG$. $\tilde{H} = rH$.

$$\therefore \begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ \tilde{U} = \tilde{G}.\ \tilde{U}_t = \tilde{H} & \text{on } \mathbb{R}_+ \times \{t=0\}. \end{cases} \qquad \tilde{U} = 0 \text{ on } \{0\} \times (0,\infty)$$

$$\therefore \tilde{U}(x,r,t) = \frac{1}{2}[\tilde{G}(r+t) - \tilde{G}(t-r)] + \frac{1}{2}\int_{t-r}^{r+t} \tilde{H}(y)dy.$$

for $0 \le r \le t$. since $u(x,t) = \lim_{r\to 0^+} \frac{\tilde{U}(x,r,t)}{r}$

$$\therefore u(x,t) = ⨍_{\partial B(x,t)} th(y) + g(y) + Dg(y)\cdot(y-x)\, dS(y).$$

which is called Kirchhoff's Formula.

<u>(b) $n=2$:</u>

Let $\bar{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t)$. by method of descent. we can reduce it to $n=3$ case.

Similarly. $\bar{g}(x_1, x_2, x_3) = g(x_1, x_2)$. $\bar{h}(x_1, x_2, x_3) = h(x_1, x_2)$.

$$\therefore \bar{u}(\bar{x}, t) = u(x,t) = \frac{\partial}{\partial t}\left(t ⨍_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S}\right) + t ⨍_{\partial \bar{B}(\bar{x},t)} \bar{h}\, d\bar{S}$$

$\bar{x} = (x, 0)$. $d\bar{S} = (1 + |D(t^2 - |\eta - x|^2)^{\frac{1}{2}}|^2)^{\frac{1}{2}} d\eta$.

$m(\partial \bar{B}(\bar{x}, t)) = 4\pi t^2$. $\partial \bar{B}(\bar{x}, t) = B(x, t)$

$$\therefore u(x,t) = \frac{1}{2} ⨍_{B(x,t)} \frac{t g(\eta) + t^2 h(\eta) + t\, Dg(\eta)\cdot(\eta - x)}{(t^2 - |\eta - x|^2)^{\frac{1}{2}}} d\eta.$$

$x \in \mathbb{R}^2$. $t > 0$. which is called Poisson's Formula.

iii) n is odd:

Lemma. (a). $\left(\frac{d^2}{dr^2}\right)\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^k \left(r^{2k}\frac{d\phi}{dr}\right)$

(b). $\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \sum_{j=0}^{k-1} \beta_j^k r^{j+1} \frac{d^j\phi}{dr^j}$. where.

$$\beta_0^k = (2k-1)!!$$

Pf: By induction:

(a). $\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^k\left(r^{2k+1}\phi\right) =$

$$\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}(2k+1)\phi + r^{2k-1}\cdot r\phi(r)\right)$$

(b) $\left(\frac{1}{r}\frac{d}{dr}\right)^k\left(r^{2k+1}\phi(r)\right) =$

$$\left(\frac{1}{r}\frac{d}{dr}\right)\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}(r^2\phi(r))\right)$$

Denote the differential operators in (a). (b). by D_a. D_b. Let $\tilde{u} = D_{(a)}(u)$. $\tilde{G} = D_a(G)$. $\tilde{H} = D_a(H)$.

where $n = 2k+1$.

check: $\begin{cases} \tilde{u}_{tt} - \tilde{u}_{rr} = 0 & \text{in } \mathbb{R}^+ \times (0,\infty) \\ \tilde{u} = \tilde{g} \quad \tilde{u}_t = \tilde{h} & \text{on } \mathbb{R}^+ \times \{0\} \\ \tilde{u} = 0 & \text{on } \{0\} \times (0,\infty) \end{cases}$

By $U_{tt} = U_{rr} + \frac{n-1}{r} U_r$. with lemma.

Note that $\tilde{u}(r,t) = \sum_{0}^{k-1} \beta_j^k r^{j+1} \frac{\partial^j}{\partial r^j} U(x,r,t)$.

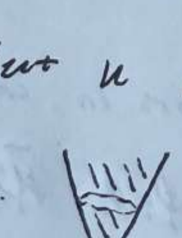
$\therefore$ $u(x,t) = \lim_{r \to 0} U(r,x,t) = \lim_{r \to 0} \tilde{u} / \beta_0^k r$

Remark: For u, we only need the information of g, h on $\partial B(x,r)$. And $u(x,t)$ won't be smooth as g. Since the formula involve the derivates of g rather than g.

iv) n is even:

By method of descent again. let $\bar{u}(x, x_{n+1}, t) = u(x,t)$. similarly for $\bar{g}$, $\bar{h}$. Insert them into $n =$ odd case.

Huygens' Principle:

Note that the formula in $n =$ even involve all the information of g, h in $B(x,t)$. i.e. they will affect u within all of cone $= \{(y,t) \mid t>0$ $|y-x|<t\}$.  . But in $n =$ odd, they will only affect the boundary $\partial C = \{(\eta, t) \mid t>0,\ |x-\eta| = t\}$.

(propagates along ∂C)

③ Nonhomogeneous Case:

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \Rightarrow \quad (a) \begin{cases} u_{tt} - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u=0,\ u_t = 0 & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \quad (b) \begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g,\ u_t = h. & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

We separate it into two groups of equations.

We have solved (b). Next, consider (a):

For $0 < s < t$, consider in $[0,t]$. $f\Delta t$ is the change of velocity u_t make by the force f. (Δt is neighbour of s).

$$\therefore \begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ u=0 \quad u_t = f(x, t-s)\Delta t & \text{on } \mathbb{R}^n \times \{s\}. \end{cases} \quad \text{is obtained by (a).}$$

suppose $u(x,t;t-s)\Delta t$ is its solution.

$$\therefore u(x,t) = \lim_{\substack{\Delta t \to 0}} \sum_{s \in [0,t]} u(x,t;t-s_i)\Delta t = \int_0^t u(x,t;t-s)\,ds.$$

<u>Thm.</u> For $n \geq 2$, $f \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n \times [0,\infty))$. Then

$u \in C^2(\mathbb{R}^n \times [0,\infty))$. solves the equation.

<u>Cor.</u> $g \in C^{[\frac{n}{2}]+2}$, $f, h \in C^{[\frac{n}{2}]+1}$. Then we have:

$u = u_a + u_b \in C^2$. Solve the nonhomogeneous. equation.

③ Energy Method:

i) Uniqueness:

For $\begin{cases} u_{tt} - \Delta u = 0 & \text{in } U_T \\ u = g & \text{on } \Gamma_T \\ u_t = h & \text{on } U \times \{t=0\}. \end{cases}$ There exists

at most one solution $u \in C^2(\bar{U}_T)$

Pf: Let $w = u - \tilde{u}$. where $u, \tilde{u}$ are 2 different solutions.

Define: Energy Function:

$$E(t) = \frac{1}{2} \int_U w_t(x,t)^2 + |Dw(x,t)|^2 \, dx. \quad 0 \le t \le T.$$

$E'(t) = 0$ by $w_{tt} = \Delta w$. Since $w_t . Dw \equiv 0$ on boundary.

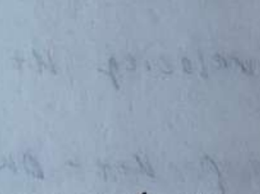
$\therefore E(t) = E(0) = 0. \quad \therefore w \equiv 0.$

Remark: We will put the change (derivate) of u into L^2-norm to construct its "energy".

ii) Domain of Dependence:

Def: Backwards wave cone with apex (x_0, t_0):

$K(x_0, t_0) = \{(x,t) \mid 0 \le t \le t_0, \ |x - x_0| \le t_0 - t\}.$



Thm. (Finite Propagation Speed)

If $u \in C^2$ solves $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$. s.t.

$u \equiv u_t = 0$ on $B(x_0, t_0) \times \{t=0\}$. Then $u \equiv 0$ in $K(x_0, t_0)$

Remark: Any disturbance outside $B(x_0, t_0)$

has no effect within $K(x_0, t_0)$. So

it won't propagate everywhere instanetously.

Pf: $E(t) = \frac{1}{2} \int_{B(x_0, t_0 - t)} u_t^2 + |Du|^2 \, dx \quad \forall \, 0 \le t \le t_0$.

$$E' = \int_{\partial B(x_0, t_0 - t)} \frac{\partial u}{\partial \nu} u_t - \frac{1}{2} u_t^2 - \frac{1}{2} |Du|^2 \le 0 \quad (\text{by AM-GM})$$

$\therefore E(t) \le E(0) = 0$. i.e. $u \equiv 0$ in $K(x_0, t_0)$