Discrete Loop Soup.

(1) Definitions:

Def: i) $\widetilde{\mathcal{I}}_n : \{0, 1, \cdots n\} \to E$. $\widetilde{\mathcal{I}} = \cup \widetilde{\mathcal{I}}_n$

ii) $\widetilde{L}_{r,n} = \widetilde{\mathcal{I}}_n \cap \{ l(0) = l(n),\ l(i) \neq l(i+1) \}$

space of rooted (based) loops of length n. $\widetilde{L}_r = \bigcup_{n \in \mathbb{Z}^+} \widetilde{L}_{r,n}$.

iii) $\widetilde{L}^* = \widetilde{L}_r / \sim$. where $l \sim l'$ if $\exists n \in \mathbb{Z}$. s.t. $\theta_n(l) = l'$.

Rmk: We endow them with σ-algebra generated by finite loops.

Def: (Measure)

i) $\widetilde{\mu}_r(l) = \frac{1}{n} P_{x_0 x_1} \cdots P_{x_n x_0}$ for $\forall l = (x_0, x_1 \cdots x_n) \in \widetilde{L}_r$.

ii) $\widetilde{\mu}^* = \widetilde{\pi}^* \circ \widetilde{\mu}_r$ $\quad \widetilde{\pi}^* : \widetilde{L}_r \to \widetilde{L}^*$ canonical.

iii) Set $j_l(y) = \# \{ i \in \{1, \cdots \widetilde{\zeta}(l)\} \mid x_i = y \}$ for $l = (x_0 \cdots x_{\widetilde{\zeta}(l)})$. $\widetilde{\mu}_x(l) =: j_l(x)^{-1} P_{x_0 x_1} \cdots P_{x_n x_0}$

Rmk: For $\mathcal{M}_{D,x}$ is set of unrooted loops visit point x in D. Then:

$\widetilde{\mu}^* |_{\mathcal{M}_{D,x}} = L \circ \widetilde{\mu}_x$. where L is loop-erasure.

Def: i) $\widetilde{\mathcal{N}} = \mathcal{I}\, W = \sum_{\mathcal{I}} \delta_{\ell_i^*}$ point measure on $(\widetilde{L}^*, \widetilde{\mathcal{L}}^*)$. (W is σ-finite. $|\mathcal{I}| \leq \mathbb{S}'$).

ii) $\widetilde{\mathbb{P}}_\alpha \in \mathcal{N}$ on $(\widetilde{L}^*, \widetilde{\mathcal{L}}^*)$ is law of PPP with intensity $\tau \widetilde{\mu}^*$. for $\tau > 0$.

iii) Set $\mathcal{L}_\alpha$ is loop-soup $\sim \widetilde{\mathbb{P}}_\alpha$.

(2) Properties of measure:

prop. $F \subset\subset E$. E is transient graph. Then:

$$\widetilde{\mu}^*(\ell^* \cap F \neq \emptyset) = \log \det (h|_{F \times F})$$

Pf: 1') $\widetilde{\mu}^*(\ell^* \subset R) = \widetilde{\mu}_r(\ell \subset R)$

$$= \sum_{k \geq 2} \widetilde{\mu}_r(\ell \subset R.\ \widetilde{J}(\ell) = k)$$

$$= \sum_{k \geq 2} \frac{Tr(P|_{R \times R})^k}{k}$$

2') $LHS = \lim_{n \to \infty} \widetilde{\mu}_r(\ell \cap F \neq \emptyset.\ \ell \subset B_n)$

$$= \lim_{n \to \infty} \widetilde{\mu}_r(\ell \subset B_n) - \mu_r(\ell \subset B_n / F)$$

$$= \lim_{n \to \infty} \log \frac{\det(I - P|_{B_n/F \times B_n/F})}{\det(I - P|_{B_n \times B_n})}$$

where $B_n \subset\subset E$. $B_n \uparrow E$.

3') With Jacobi's equality:

$$\frac{\det(I - P|_{(B_n/F)^2})}{\det(I - P|_{B_n^2})} = \det((I - P|_{B_n \times B_n})^{-1}|_{F \times F}) \xrightarrow{n \to \infty} \det(h|_{F \times F})$$

Cor. For $\{x_i\}_1^n \subset E$. we have:

$$\tilde{m}^*(x_i \in \ell^*.\ \forall 1 \le i \le n) = \sum_{\substack{A \subset \{x_i\}_1^n \\ A \ne \emptyset}} (-1)^{|A|+1} \log \det (h|_{A \times A})$$

Def: i) For $\ell \in \tilde{L}_r$. multiplicity of ℓ is:

$m(\ell) = \max\{k \mid \ell = (\ell_1, \ell_2 \cdots \ell_k).\ \ell_i \in \tilde{L}_r.$ and $\ell_1 = \ell_2 \cdots = \ell_k\}$.

Rmk: If $\tilde{J}(\ell^*) = n$. Then: we have.

$\tilde{m}^*(\ell^*) = \frac{n}{m(\ell)} \cdot \tilde{m}_r(\ell)$ for the loop $\ell = (z_1 \cdots z_n)$

ii) For $S_1, S_2 \subset E$. $S_1 \cap S_2 = \emptyset$. Set map:

$$L(S_1, S_2): \begin{array}{ccc} \tilde{L}^* \cap (\ell^* \text{ visits } S_1, S_2) & \longrightarrow & P(\tilde{L}_r) \\ \ell^* & \longmapsto & R \end{array}$$

R is subset of $\tilde{L}_r$ containing ℓ' satisfies:

a) $\ell \sim \ell'$ in $\tilde{L}^*$

b) $\ell' = (z_1, \cdots z_n)$. s.t. $z_1 \in S_1$

c) $\exists\, i$. s.t. $z_i \in S_2$ and $z_j \notin S_1 \cup S_2$. $\forall j > i$.

Rmk: $L(S_1, S_2)(\ell^*) \ne \emptyset \iff \ell$ visits S_1, S_2.

iii) For $\ell' = (z_1, \cdots z_n) \in L(S_1, S_2)(\ell^*)$. Set $z_0 = 1$

$z_{2k+1} = \inf\{j > z_{2k} \mid z_j \in S_2\}$. $z_{2k+2} = \inf\{j > z_{2k+1} \mid z_j \in S_1\}$

and set $\inf\{\alpha\} = n+1$.

iv) set $k(l)$ satisfies $\tau_{2k(l')-1} = n$, $\tau_{2k(l')} = n+1$

<u>prop.</u> $\forall\, l^* \in \tilde{L}^*$ visits S_1, S_2, $\tilde{\zeta}(l^*) = n$, discrete loop

$$\Rightarrow \tilde{m}^*(l^*) = \frac{n}{k(l)} \cdot \sum_{l' \in L(S_1,S_2)(l)} \tilde{m}_r(l')$$

<u>Pf</u>: It follows $k(l) = m(l) \cdot \# L(S_1,S_2)(l)$

(It's guaranteed by c) in Def ii))

<u>prop.</u> For $E = \mathbb{Z}^d$, $d \geq 3$, $k_x = 0$.

i) For $\lambda > 1$, $\exists\, c = c(d,\lambda) < \infty$, s.t. $\forall N \geq 1$, $M \geq \lambda N$

and $K \subset B(0,N)$, $\tilde{m}^*(K \overset{l^*}{\longleftrightarrow} \partial B(0,M)) \leq c \cdot \mathrm{cap}(K) \cdot M^{2-d}$

ii) Set $F(d,n) = I_{\{d=3\}} \cdot n + I_{\{d=4\}} \cdot \frac{n^2}{\log n} + I_{\{d\geq 5\}} \cdot n^2$

For $\lambda > 1$, $\exists\, c = c(d,\lambda) < \infty$, s.t. $\forall N \geq 1$, $M \geq \lambda N$

and $K \subset B(0,N)$, we have:

$\tilde{m}^*(K \overset{l^*}{\longleftrightarrow} \partial B(0,M),\ \mathrm{cap}(l) > F(d,n)) \geq c \cdot \mathrm{cap}(K) \cdot M^{2-d}$.

<u>Pf</u>: i) Set $\tau_0 = H_K$, $\tau_{2k+1} = \inf\{n > \tau_{2k} \mid Z_n \in \partial B(0,M)\}$

$$\tau_{2k+2} = \inf\{n > \tau_{2k+1} \mid Z_n \in K\}.$$

$$\Rightarrow \tilde{m}^*(K \overset{l^*}{\longleftrightarrow} \partial B(0,M)) = \sum_{n\geq 1} \frac{1}{n} \sum_{x \in \partial K} \mathbb{P}^x(X_{\tau_{2n}} = x)$$

(count $m(l) = n$, $n \geq 1$, with $\tilde{m}_r(\cdot) = \square$)

By Markov prop. and Harnack inequi.:

$$\mathbb{P}^x(X_{\tau_{2n+2}} = x) \underset{d,\lambda}{\lesssim} \mathbb{P}^0(X_{\tau_2} = x) \cdot (\max_{\partial K} \mathbb{P}^x(\tau_2 < \infty))^n$$

$$\leq c \cdot \mathbb{P}^0(X_{\tau_2} = x) \cdot \epsilon^n, \quad \epsilon < 1.$$

So: $\widetilde{M}^*(k \overset{l^*}{\longleftrightarrow} \partial B(0,m)) \leq C \cdot \mathbb{P}^0(Z_2 < \infty)$

By last exit id. we have:

$\mathbb{P}^z(H_K < \infty) \lesssim M^{2-d}\,\mathrm{cap}(K)$. $\forall z \in \partial B(0,m)$

ii) Check: $\mathbb{P}^\eta(\mathrm{cap}(\{Z_0, \cdots Z_{H(K)}\}) > c F(\lambda, n))$

$\geq c = c(d, \lambda) > 0$.

(3) <u>Decomposition of Loops</u>

<u>in Excursion:</u>

Next, we set $E = \mathbb{Z}^d$. $d \geq 3$. $k_x = 0$. X_n is a SRW on $\mathbb{Z}^d$. $\emptyset = A \cap B$. $A, B \subset \mathbb{Z}^d$.

<u>Def</u>: $\mathcal{L}_{A,B} = \{ l^* \in \widetilde{L}^* \mid l^* \cap A,\ l^* \cap B \neq \emptyset \}$.

<u>Lemma.</u> There $\exists\ c = c(d)$. $C = C(d) < \infty$. s.t.

$\forall n \geq 1$. $m > 2n$. $A \subset B(0,n)$. $X \notin B(0,m)$.

$c\,\widetilde{e}_A(\eta) \leq \mathbb{P}_x(X_{H_A} = \eta \mid H_A < \infty) \leq C\,\widetilde{e}_A(\eta)$

<u>Lemma</u> $\dot{\mathcal{L}}^\alpha_{A,B}$ is a PPP on $\dot{\mathcal{L}}_{A,B} \overset{\Delta}{=} L(A,B)(\mathcal{L}_{A,B})$ with intensity $\alpha \dot{\mu}_{A,B}$. where $\dot{\mu}_{A,B}(l)$

$= \frac{1}{k(l)} \mathbb{P}_{x_0}((X_0, \cdots X_{n-1}) = l,\ X_n = x_0)$ for l ($n = |l|$)

$\in \dot{\mathcal{L}}(A,B)$. Then $\pi^*(\dot{\mathcal{L}}^\alpha_{A,B})$ is PPP on

$\mathcal{L}_{A,B}$ with intensity $\alpha\, I_{\mathcal{L}_{A,B}} \cdot \mu$

<u>Pf</u>: By Transf prop. of PPPs.

<u>Def</u>: i) $\dot{\mathcal{I}}^{\alpha,j}_{A,B}$ is restriction of $\dot{\mathcal{I}}^{\alpha}_{A,B}$ on $\dot{\mathcal{L}}^{j}_{A,B}$

$=: \{ l \in \mathcal{I}_{A,B} \mid k(l) = j \}$.

<u>Rmk</u>: i) $(\dot{\mathcal{I}}^{\alpha,j}_{A,B})_{j\geq 1}$ are indept PPPs with

intensity $(\alpha I_{\dot{\mathcal{L}}^{j}_{A,B}} \dot{\mu}_{A,B})_{j\geq 1}$

ii) $\dot{\mathcal{I}}^{\alpha}_{A,B} = \sum\limits_{j\geq 1} \dot{\mathcal{I}}^{\alpha,j}_{A,B}$.

ii) For $l^* \in \mathcal{L}_{A,B}$, $l' \in L(A,B)(l^*)$.

set $\phi_1(l') = 1$, $\psi_1(l') = \inf\{ j > 1 \mid X_j \in B \}$.

$\phi_k(l') = \inf\{ j > \psi_{k-1}(l') \mid X_j \in A \}$.

$\psi_k(l') = \inf\{ j > \phi_{k}(l') \mid X_j \in B \}$.

iii) For $l' \in \dot{\mathcal{L}}^{j}_{A,B}$, $l' = (X_1 \cdots X_n)$.

set $\tilde{\phi}_i(l') = X_{\phi_i(l')}$, $\tilde{\psi}_i(l') = X_{\psi_i(l')}$, $1 \leq i \leq j$.

excursion from A to B:

$\overrightarrow{W}_i(l') = (X_{\phi_i(l')} \cdots X_{\psi_i(l')})$, $1 \leq i \leq j$

and excursion from B to A:

$\overleftarrow{W}_i(l') = (X_{\psi_i(l')}, \cdots X_{\phi_{i+1}(l')})$, $1 \leq i \leq j-1$.

$\overleftarrow{W}_j(l') = (X_{\psi_j(l')} \cdots X_n, X_1)$

iv) $\Sigma^{\alpha,j}_{A,B} = \{ ((\tilde{\phi}_1(l'), \tilde{\psi}_1(l')), \cdots (\tilde{\phi}_j(l'), \tilde{\psi}_j(l'))) $

for $l' \in \dot{\mathcal{I}}^{\alpha,j}_{A,B} \}$

$$\overrightarrow{\Sigma}^{\tau,j}_{A,B} = \{ (\overrightarrow{W}_1(l'), \cdots \overrightarrow{W}_j(l')). \ l' \in \dot{\mathcal{L}}^{\tau,j}_{A,B} \}$$

$$\overleftarrow{\Sigma}^{\tau,j}_{A,B} = \{ (\overleftarrow{W}_1(l'), \cdots \overleftarrow{W}_j(l')) \ l' \in \dot{\mathcal{L}}^{\tau,j}_{A,B} \}.$$

prop. For path $w = (X_0, X_1 \cdots)$. Set $\tau_0(w) = 0$.

$\tau_{2j+1}(w) = \inf\{ k > \tau_{2j}(w) \mid X_k \in B \}$.

$\tau_{2j+2}(w) = \inf\{ k > \tau_{2j+1}(w) \mid X_k \in A \}$.

Then. for $\forall \tau > 0$. $j \geq 1$.

i) Intensity of $\Sigma^{\tau,j}_{A,B}$ is

$$((a_1, b_1) \cdots (a_j, b_j)) \longmapsto \frac{\tau}{j} \mathbb{P}_{a_1} \left(\begin{matrix} \tau_{2j} < \infty, & X_{\tau_{2j}} = a_1 \\ X_{\tau_{2(i-1)}} = a_i, & X_{\tau_{2i-1}} = b_i \end{matrix} \right)$$

ii) Conditional on $\Sigma^{\tau,j}_{A,B} = \{ ((a_{i1}, b_{i1}) \cdots (a_{ij}, b_{ij})) \}_{i=1}^{n}$

$\overrightarrow{\Sigma}^{\tau,j}_{A,B}$, $\overleftarrow{\Sigma}^{\tau,j}_{A,B}$ are indept and sampled as

products of bridge measures $\mathbb{P}^{B}_{a_{ik}, b_{ik}}$. resp.

$\mathbb{P}^{A}_{b_{ik}, a_{i(k+1)}}$. where $\mathbb{P}^{m}_{x,\eta}(\cdot) = \mathbb{P}_x(\cdot \mid X_{H_m} = \eta)$

Rmk: It means loops from $\dot{\mathcal{L}}^{\tau}_{A,B}$ can be sampled by:

i) First sample number and its starting and ending locations of all the loops in $\dot{\mathcal{L}}^{\alpha}_{A,B}$

(It's achieved by sampling indeptly $(\Sigma^{\tau,j}_{A,B})_j$)

iii) Sample indept RW bridges from $\mathbb{P}^{B}_{\cdot\cdot}$ and $\mathbb{P}^{A}_{\cdot\cdot}$.

Pf: Note that present of MC only depends on the nearest past and future.

$$\Rightarrow \dot{m}_{A,B}(l) = \frac{1}{j} \mathbb{P}_{x_1}\left(\begin{matrix} Z_{2j} < \infty, & X_{Z_{2j}} = x_1, & 1 \le i \le j-1 \\ X_{Z(i-1)} = \tilde{\varphi}_i(l), & X_{Z_{i-1}} = \tilde{\gamma}_i(l) \end{matrix} \right)$$

$$\cdot \prod_1^j \mathbb{P}^B_{\tilde{\varphi}_i(l), \tilde{\gamma}_i(l)}(\overrightarrow{W}_i(l)) \prod_1^j \mathbb{P}^A_{\tilde{\gamma}_i(l), \tilde{\varphi}_{i+1}(l)}(\overleftarrow{V}_i(l))$$

prop. (Large deviation)

For $Z^{T}_{A,B}$ total number of excursions from A to B of loops from $\dot{\mathcal{L}}^{\tau}_{A,B}$. If $\sup_{y \in B} \mathbb{P}_y(H_A < \infty) \le \frac{1}{2e}$

Then: $\mathbb{P}(Z^{\tau}_{A,B} \ge k) \le e^{\tau - k}$.

Pf: Set $Z^{\tau,j}_{A,B}$ is number of loops in $\dot{\mathcal{L}}^{\tau,j}_{A,B}$

$$Z^{\tau}_{A,B} \overset{d}{\sim} \sum_{j \ge 1} j\, Z^{\tau,j}_{A,B}.$$

By prop is above. $Z^{\tau,j}_{A,B} \sim \mathrm{Poi}(\lambda_j)$

$$\lambda_j = \frac{\tau}{j} \sum_{x \in A} \mathbb{P}_x(Z_{2j} < \infty, X_{Z_{2j}} = x)$$

$$\le \frac{\tau}{j} \sup_A \mathbb{P}_x(Z_{2j} < \infty) \overset{\text{MP}}{\le} \frac{\tau}{j}\left(\sup_A \mathbb{P}_x(Z_2 < \infty)\right)^j$$

$$\overset{\text{MP}}{\le} \frac{\tau}{j}\left(\sup_B \mathbb{P}_y(H_A < \infty)\right)^j \le \frac{\tau}{j}(2e)^{-j}$$

combine: $\mathbb{E}(e^{Z^{\tau}_{A,B}}) = \prod_{j \ge 1} \mathbb{E}(e^{j Z^{\tau,j}_{A,B}})$,

with exponential Chebyshev inequi.

(4) Properties of Loop soups:

① Def: For $x \in E$. $\mathcal{L}_D^x$ is set of loops in $\mathcal{L}$ going through point x. in D

prop. $\widetilde{\mathbb{P}}_1(\mathcal{L}_D^x = \emptyset) = e^{-\widetilde{m}^*(M_{D,x})} = 1/h_D(x,x)$

Pf: Define u is the sum of $(2d)^{-|\gamma|}$ over all rooted loops from x to x in D visiting x only once

$\Rightarrow \widetilde{m}^*(M_{D,x}) = \sum_{j\geq 1} u^j/j = -\ln(1-u)$

Besides: $h_D(x,x) = 1 + \sum_{j\geq 1} u^j = 1/(1-u)$

Cor. If $\mathcal{L}_D$ is set of loops in $\mathcal{L}$ staying in D. Then: $\widetilde{\mathbb{P}}_1(\mathcal{L}_D = \emptyset)$

$= 1/\det h_D$.

Pf: $LHS = \prod_{k\geq 0} \widetilde{\mathbb{P}}_1($No loop in $D/\{x_0 \cdots x_k\}$ goes through $x_{k+1})$

$= \prod_{k\geq 0} h^{-1}_{D/\{x_0\cdots x_k\}}(x_{k+1}, x_{k+1}) = 1/|h_D|$

where define $x_0 = \emptyset$.

Cor. $\widetilde{m}^*(M_{D,x})$ is finite if $|D| < \infty$.

Pf: $\sum_D \widetilde{m}^*(M_{D,x}) < \infty$. if $|D| < \infty$.

Cor. $\widetilde{\mathbb{P}}_\alpha(\mathcal{L}_D^x = \emptyset) = (1/h_D(x,x))^\alpha$

$\widetilde{\mathbb{P}}_\alpha(\mathcal{L}_D = \emptyset) = (\det h_D)^{-\alpha}$

Pf: Note for $\mathcal{L} \sim PPP(\alpha m)$

$\mathcal{L}' \sim PPP(\alpha' m)$. indept loop soups.

$\mathcal{L} + \mathcal{L}' \sim PPP((\alpha+\alpha')m)$.

Consider $\alpha \in \mathbb{Z}^+ \Rightarrow \mathbb{Q}^+ \Rightarrow \mathbb{R}^+$.

Prop. For $N = \#\mathcal{L}_D^x$. We choose uniformly at random an order of these N rooted loops $\lambda_1, \cdots \lambda_N$

Set $\widetilde{\lambda} = \lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_N$. concatenation of them

If ℓ is a rooted loop from x to x in D.

Then $\widetilde{\mathbb{P}}_1(\widetilde{\lambda} = \ell) = (2d)^{-\widetilde{\zeta}(\ell)} / h_D(x,x)$

Pf: **Lemma.** $\forall k \in \mathbb{Z}^+$. $\sum_{r \geq 1} \sum_{\sum_1^r j_i = k} 1/r! \prod_1^r j_i = 1$.

Note: Fix $(\ell^i)_1^r$ $\ell^1 \oplus \cdots \ell^r = \ell$.

$\widetilde{\mathbb{P}}_1(\lambda_1 = \ell^1, \cdots \lambda_N = \ell^N, N(M_{D,x}) = r)$

$= \frac{1}{r!} \cdot \prod_1^r \widetilde{m}^*(\ell^i) \cdot e^{-\widetilde{m}^*(M_{D,x})}$

$= (2d)^{-\widetilde{\zeta}(\ell)} \cdot \frac{1}{r!} \cdot \frac{1}{\prod_1^r j_i} \cdot 1/h_D(x,x)$

Sum over $r \geq 1$ and $\sum_{i=1}^r j_i = j_\ell(x)$

Rmk: $\widetilde{\lambda} \overset{L}{\sim} \lambda$. the loop from x to x that we erase before last visit to x.

Besides. the constructions are similar!

By the prop and remark above. We can recover the whole process of Wilson's algorithm in $D = \{x_1, \cdots x_n\}$ with root x_0 by sampling unrooted loop soups when $\tau = 1$ and indept UST.

: 1) Sample N_1 loops in $\mathcal{L}_D^{\eta_1}$. order it uniformly at random and concatate it. we obtain a loop η_1.

2) Jump η_1 to η_2.

3) Sample N_2 loops in $\mathcal{L}^{\eta_2}_{D/\{\eta_1\}}$ and repeat 1)

4) Proceed until reach the root x_0.

$\Rightarrow$ We recover a whole loop from $(\eta_1 \cdots \eta_{s+1})$

prop. (Converse)

Start from Wilson's algorithm:

to read off rooted loops λ_i from η_i to η_i in $D / \{\eta_1 \cdots \eta_{i-1}\}$ (non-loop-erased)

i) For each i indeptly. if λ_i returns k times to η_i. choose to split it into r smaller loops. with $j_1 \cdots j_r$ times return to η_i. resp. s.t. $\sum_1^r j_i = k$. with prob: $\tilde{\mathbb{P}}_1(\lambda_1 = \ell^1 \cdots \lambda_r = \ell^r$ $N(\mu_{D,\eta_i})) / \tilde{\mathbb{P}}_1(\tilde{\lambda} = \ell) = 1/ r! \prod_1^r j_i$.

ii) Then we obtain a point process of rooted loop in D from $(\lambda_i)_1^s$.

$\Rightarrow$ This process is indept of UST $\mathcal{T}$ consted by Wilson's algorithm with law of loop-soups in D. with $\tau = 1$.

③ <u>Occupation Field:</u>

i) <u>Def</u>: $V = (V(x))_{x \in D}$ is occupation field of $\mathcal{L}_D$ loop soups with $\tau = 1$. in D. i.e. $V(x)$ is total number of visit to x by all loops in $\mathcal{L}_{D,1}$

<u>prop.</u> $V \sim \tilde{V} = V - 1$

<u>Pf</u>: By prop. above. we know it's identical dist. as Occupation field of Wilson's algorithm.

<u>Cor.</u> For $\tau > 0$. V_τ is occupation field of loop soup with intensity $\alpha \tilde{\mu}^*$.

$$\Rightarrow \mathbb{E}\left(\prod_1^n (1 + k(x_j))^{-V_\tau(x_j)}\right) = \left(\frac{1 - \bar{\Delta}_D 1}{1 - \bar{\Delta}_{D,k} 1}\right)^\tau$$

<u>Pf</u>: By infinite divisibility.

ii) **Def:** (Occupation Time on edge)

Set $T = (T(e))_{e \in E(D)}$ is occupation field on $E(D)$ i.e. $T(e)$ is total number of times visit to edge e by all loops in $\mathcal{L}_{D,1}$

Rmk: Set $S(x) = \frac{1}{2} \sum_{e \in E(x)} T(e)$. $\Rightarrow S = \mathcal{V}_{1/2}$

$$\mathbb{E}\left(\prod_{1}^{n} (1 + k(x_i))^{-S(x_i)} \right) = \left(\frac{1 - \bar{A}_{D1}}{1 - \bar{A}_{D,K1}} \right)^{1/2}$$

Denote: i) $t = (t(e))_{e \in E(D)} \in \mathbb{R}^{E(D)}$. $|t| = \sum_{E(D)} t(e)$

ii) $s = (s(x))_{x \in D} \in \mathbb{R}^{D}$. $s(x) = \frac{1}{2} \sum_{E(x)} t(e)$.

iii) Set $\mathcal{P}(2n) = (2n)! / 2^n \cdot n!$ is the number of possible pairs of $\{1, 2, \cdots 2n\}$.

prop. (Law of occupation field T)

$$\widetilde{\mathbb{P}}_1 (T = t) = |G_D|^{-\frac{1}{2}} \cdot (2d)^{-|t|} \cdot \prod_{x \in D} \mathcal{P}(2s(x)) \cdot \prod_{E(D)} 1/t(e)!$$

cor. Condition on $T = t$. law of loop soup is: at each site x. choose indeptly a pair of $2s(x)$ adjacent pipes uniformly among all choices.

Rmk: When knowing T. the missing info. on loop-soup is how to connect all these jumps along edges to each other.