Poisson Gas of Markovian Loops

Def: i) $\Omega = \{ w = \sum_I \delta_{\gamma_i^*}$ point measure on $(L^*, \mathcal{L}^*)$ |

w is σ-finite, $|I| \leq \aleph_0$ }

with canonical σ-algebra.

ii) Set $\mathbb{P}_\alpha \in \Omega$, a poisson point measure on $(L^*, \mathcal{L}^*)$, with intensity $\alpha \mu^*$, $\alpha > 0$.

We call it Poisson gas of Markovian loops at level α.

(1) Occupation Field:

Def: For $w \in \Omega$, $x \in E$. The occupation field of w at x is $\mathcal{L}_x(w) = \langle w, L_x \rangle = \sum_I L_x(\gamma_i^*)$

Thm: (Laplace Transf.)

For $V : E \to \mathbb{R}^+$. $\mathbb{E}_\alpha(e^{-\sum_E V(x) \mathcal{L}_x}) =$

$\det(I + GV)^{-\alpha} = \det(I + V^{\frac{1}{2}} G V^{\frac{1}{2}})^{-\alpha} = \left(\frac{|G_V|}{|G|}\right)^{\alpha}$

Cor. For $v \geq 0$, $x \in E$. $\mathbb{E}_\alpha(e^{-v\mathcal{L}_x}) =$

$(1 + v g(x,x))^{-\alpha}$.

Cor. $\mathcal{L}_x \sim \text{Gamma}(\alpha, g(x,x))$ and

$\mathcal{L}_x < \infty$, $\mathbb{P}_\alpha$-a.s. $\forall x$.

Pf: To use ch.f of PPPs. Set $\phi : L^* \to \mathbb{R}^+$

s.t. $\langle w, \phi \rangle = \sum_E V(x) \mathcal{L}_x(w)$

i.e. $\phi(\gamma^*) = \sum_E V(x) L_x(\gamma^*)$. for $\gamma^* \in L^*$.

Combine with id. of local time L_x.

Pf of Cor. $\mathbb{P}_\alpha(\mathcal{L}_x < \infty) = \lim_{v \to 0} \mathbb{E}_\alpha(e^{-v\mathcal{L}_x} ; \mathcal{L}_x < \infty)$

$= \lim_{v \to 0} (1 + v g(x,x))^{-\alpha} = 1$

Thm. (Connection with GFF)

$(\mathcal{L}_x)_{x \in E}$ under $\mathbb{P}_{\alpha = \frac{1}{2}}$ $\sim$ $(\frac{1}{2}\varphi_x^2)_{x \in E}$ under $\mathbb{P}^G$.

Pf: By ch.f's: $\mathbb{E}_{\frac{1}{2}}(e^{-\sum_E V(x)\mathcal{L}_x}) = |I + GV|^{-\frac{1}{2}}$

$= \mathbb{E}^G(e^{-\frac{1}{2}\sum V(x)\varphi_x^2})$.

Def: Occupation time of non-trivial loops is

$\hat{\mathcal{L}}_x(w) = \langle w 1_{\{N>1\}}, L_x \rangle = \sum_I 1_{\{N(\gamma_i^*)>1\}} L_x(\gamma_i^*)$.

Thm. For $V : E \to \mathbb{R}^+$. $\mathbb{E}_\alpha(e^{-\sum_E V(x)\hat{\mathcal{L}}_x}) = \left(\frac{|I - P^V|}{|I - P|}\right)^{-\alpha}$

where $P^V f(x) =: \sum_{y \in E} \frac{c_{xy}}{\lambda_x + V(x)} f(y)$. $f : E \to \mathbb{R}^1$.

Pf: Similarly. By exponential id. of local time.

$\int_{\{N>1\}} (1 - e^{-\sum_E V(x) L_x}) d\mu_r = -\log\frac{|G_V|}{|G|} + \log|\lambda| - \log|\lambda + V|$

(Note: $\int_{\{N=1\}} (1 - e^{-\sum_E V(x) L_x}) d\mu_r = \sum_E \int_{\{N=1\}} (1 - e^{-V(x)L_x}) d\mu_r$)

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Cor. For $v \geq 0$, $x \in E$, $\mathbb{E}_\alpha(e^{-v\hat{\mathcal{L}}_x}) = \left(\frac{1+vg(x,x)}{1+v/\lambda_x}\right)^{-\alpha}$

Cor. $\mathbb{P}_\alpha(\hat{\mathcal{L}}_x = 0) = (\lambda_x g(x,x))^{-\alpha}$ for $x \in E$

Pf: Set $v \to \infty$ in above.

Cor. $\mathbb{P}_\alpha(\tilde{\mathcal{L}}_x = 0,\ \forall x \in E) = |I-P|^\alpha$

Pf: Set $V(x) \uparrow \infty$, $\forall x \in E$.

Rmk: It's consistent with:

$$\mathbb{P}_\alpha(w(N>1)=0) = e^{-\alpha\, I_{\{N>1\}} \circ\, m r(dr)}$$

$$= e^{-\alpha\, mr(N>1)} = |I-P|^\alpha$$

(2) Representation Formula:

Def: i) For set S, $(D_i)_1^k$ is a pairing of S if $\sum_1^k D_i = S$, $|D_i| = 2$.

ii) Laplace Transf. of p.m. ν on $\mathbb{R}^+$ is

$$h(u) = \int_0^\infty e^{-su}\, d\nu(s), \quad \forall u \geq 0$$

iii) $$\overline{\mathbb{P}}^{h,h} = \frac{1}{Z_h} e^{-\frac{1}{2}\mathcal{E}(\varphi,\varphi)} \prod_E h\left(\frac{\varphi_x^2}{2}\right) d\varphi.$$

where Z_h is the normalization.

iv) $\langle F \rangle_h$ is expectation of $F : \mathbb{R}^E \to \mathbb{R}^1$ w.r.t. $\overline{\mathbb{P}}^{h,h}$.

v) Set $(\tilde{\Omega}, \tilde{\mathbb{P}}, \tilde{\mathcal{A}})$ is a prob. space endowed with collection of nonnegative r.v.'s $(V(x,\tilde{\omega}))_{x\in E}$ $\overset{\text{iid}}{\sim}$ ν-dist.

<u>Thm</u> (Symanzik's Representation Formula)

For $\forall k \geq 1$, $(z_i)_1^{2k} \subset E$, $\langle \prod_1^{2k} \varphi_{z_i} \rangle_\nu =$

$$\sum_{\substack{\text{pairing} \\ \text{of } \{1\cdots 2k\}}} \frac{\mathbb{E}_{x_1,\eta_1} \otimes \cdots \mathbb{E}_{x_k,\eta_k} \otimes \tilde{\mathbb{E}} \otimes \mathbb{E}_{\frac{1}{2}} \left(e^{-\sum_E V(x,\tilde{\omega})(\mathcal{L}_x(\omega) + \sum_1^k L^x_\infty(w_i))} \right)}{\tilde{\mathbb{E}} \otimes \mathbb{E}_{\frac{1}{2}} \left(e^{-\sum_E V(x,\tilde{\omega}) \mathcal{L}_x(\omega)} \right)}$$

where $\{x_i, \eta_i\} =: D_i$ form a pairing of $\{1\cdots 2k\}$.

<u>Rmk:</u> When $k=1$, we can easily obtain the equation by the isomorphism Thm before.

(3) <u>Connection with RIs:</u>

① <u>Lemmas:</u>

<u>Def:</u> For $U \subseteq E$, set $\mathcal{L}_x^U(\omega) = \langle 1_{\{y^* \subseteq U\}} \omega, L_x \rangle$
$= \sum_i 1_{\{y_i^* \subseteq U\}} L_x(y_i^*)$.

<u>prop.</u> $K \subseteq E$, $U = E \setminus K$, $V: E \to \mathbb{R}^+$.

$$\mathbb{E}_\alpha \left(e^{-\sum_E V_x (\mathcal{L}_x - \mathcal{L}_x^U)} \right) = \left(\frac{\det G_V}{\det G} \cdot \frac{\det G^U}{\det G_{U,V}} \right)^\alpha$$
$$= \left(\frac{\det_{K\times K} G_V}{\det_{K\times K} G} \right)^\alpha$$

<u>Pf:</u> By indept of PPP, since we know $\mathbb{E}(e^{-\langle V, \mathcal{L} \rangle})$

<u>prop.</u> If $K_1 \cap K_2 = \emptyset$. $U_i = E/K_i$. Then:

$$\mathbb{P}_\alpha(\text{no loop intersects both } K_1 . K_2) = \left(\frac{\det h}{\det h^{U_1}} \cdot \frac{\det h^{U_1 \cap U_2}}{\det h^{U_2}} \right)^{-\alpha}$$

$$= \left(\frac{\det_{K_1 \times K_1} h^{U_2}}{\det_{K_1 \times K_1} h} \right)^{\alpha}$$

<u>Cor.</u> For $x \in E$. $K \subseteq E$. $x \notin K$. Then:

$$\mathbb{P}_\alpha(\text{All loops going through } x \text{ don't intersect } K)$$

$$= \left(1 - \frac{\mathbb{E}_x(H_K < \infty .\ g(X_{H_K}, x))}{g(x,x)} \right)^{\alpha}.$$

<u>Pf:</u> Set $K_1 = \{x\}$. $K_2 = K$. $U = E/K$.

By decompose: $g = g_U + \mathbb{E}(\cdots)$

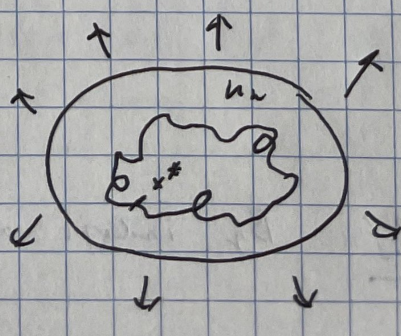
③ Next, we will recover random interlacement on $\mathbb{Z}^d$. $d \geq 3$. by letting loops go through ∞:

i) Consider $U_n \subset\subset \mathbb{Z}^d$. $\uparrow \mathbb{Z}^d$. Fix $x_* \in \mathbb{Z}^d$. endowed with weight $c_{x,\tilde{y}} = I_{\{|x-y|=1\}}/2d$. and $k_x^n = \sum_{y \in \mathbb{Z}^d/U_n} I_{\{|x-y|=1\}}/2d$. on $E = \mathbb{Z}^d$.

<u>Def:</u> μ_n is $\mu|_{\{\gamma \in U_n\}}$. $\mathbb{P}^{\alpha, n}$ for μ_n is law of Markovian loop at level α.

Next, 1) Set $n \to \infty$

2) Set $x^* \to \infty$.



Def: $J_{n,x^*}(\omega) =: \{ z \in K_n \mid \exists\, \gamma^* \in \mathrm{supp}(\omega),\ \text{s.t.}$ γ^* goes through x^* and z. $\}$ for $\omega \in \Omega_n$

Thm. For $u \geq 0$, $K \subset\subset \mathbb{Z}^d$, $\hat{r} =: u\, g(0)\, |x^*|^{2(d-2)} / c_d^2$

where $g(x) \sim c_d |x|^{-d+2}$, $(|x| \to \infty)$

We have: $\lim\limits_{x^* \to \infty} \lim\limits_{n \to \infty} \mathbb{P}^{\hat{r}, n}(J_{n,x^*} \cap K = \emptyset)$

$= \exp(-u\, \mathrm{cap}_{\mathbb{Z}^d}(K))$

Pf: By Lemmas in ①:

$$\mathbb{P}^{\hat{r},n}(\cdots) = \left(1 - \frac{E^{\mathbb{Z}^d}_{x^*}(H_K < T_{K_n},\ g_{K_n}(X_{H_K}, x^*))}{g_{K_n}(x^*, x^*)}\right)^{\hat{r}}$$

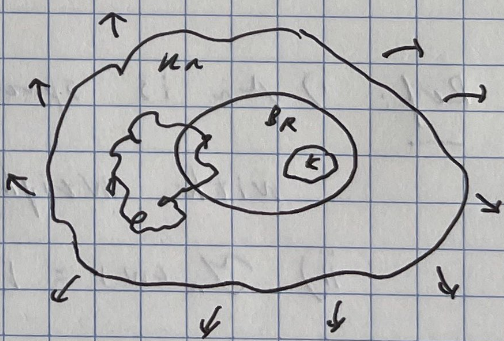
$$\xrightarrow{n \to \infty} \left(1 - \frac{E^{\mathbb{Z}^d}_{x^*}(H_K < \infty,\ g(X_{H_K}, x^*))}{g(x^*, x^*)}\right)^{\hat{r}}$$

$\Rightarrow$ Estimate the order

ii) Consider $K_n \subset\subset \mathbb{Z}^d$, $\uparrow \mathbb{Z}^d$. $B_R \hat{=} \{|x| \leq R\}$

Next. 1) Let $n \to \infty$

2) Let $R \to \infty$.



Def: $K_{n,R}(\omega) =: \{ z \in K_n \mid \exists\, \gamma^* \in \mathrm{supp}(\omega)$

goes through z and $B_R^c \}$. for $\omega \in \Omega_n$.

Thm. For $u \geq 0$, $K \subset\subset \mathbb{Z}^d$, $\hat{r} = u R^{d-2} / c_d$. Then:

$$\lim_{R \to \infty} \lim_{n \to \infty} \mathbb{P}^{n, \hat{r}}(K_{n,R} \cap K = \emptyset) = e^{-u\, \mathrm{cap}_{\mathbb{Z}^d}(K)}$$

Pf: By Cor. in ① above:

$$\mathbb{P}^{n,\hat{\ }}(\cdots) = \left(\frac{\det_{k\times k} G^{BR}}{\det_{k\times k} G^{U_n}} \right)^{\gamma}$$

similarly estimate the order as above

iii) Consider $\mathcal{I} \subset G$ is countable transient connected graph with $c_{x,y} > 0$, with SRW with jump rate $= 1$ on $\mathcal{I}$, induced by c.

Set: $U_n \subset\subset \mathcal{I}$, $U_n \uparrow \mathcal{I}$, $x_* \in \mathbb{Z}^d / \mathcal{I}$.

$E_n = U_n \cup \{x_*\}$ endowed with the weights obtained by collapsing U_n^c on x_*:

$\hat{c}_{x,y} = c_{x,y}$ if $x, y \in U_n$

$\hat{c}_{x_*,y} = \hat{c}_{y,x_*} = \sum_{x \in G/U_n} c_{x,y}$ when $y \in U_n$.

$\hat{k}_{x_*} = \lambda_n \xrightarrow{n\to\infty} \infty$. $\hat{k}_x = 0$. $\forall x \in E_n / \{x_*\}$.

Ref: i) $\mathcal{L}_n$ is space of unrooted loops on E_n endowed with weight c^n and killing k^n.

ii) $\mathcal{J}_n(w) = \{ z \in U_n \mid y^* \in \mathrm{supp}(w),\ \text{go through } x_*, z \}$.

Thm. For $u \geq 0$, $K \subset\subset G$, $\hat{\alpha} = u\lambda_n$. We have:

$$\lim_{n\to\infty} \mathbb{P}^{n,\hat{\ }}(\mathcal{J}_n \cap K = \emptyset) = e^{-u\,\mathrm{cap}_G(K)}.$$

$$\text{Pf: } \mathbb{P}^{n,\hat{\ }}(\cdots) = \left(1 - \frac{\hat{\mathbb{E}}_{x_*}(H_K < \infty,\ g_n(X_{H_K}, x_*))}{g_n(x_*, x_*)} \right)^{\alpha}$$