Markovian Loops.

(1) Rooted Loops:

Def: i) $L_{r,t} = \Gamma_t \cap \{\gamma(0) = \gamma(t)\}$. $L_r = \bigcup_{t>0} L_{r,t}$.

ii) For $\gamma \in L_r$. $N(\gamma) = n$ with jump times $(T_k(\gamma))_1^n$. Set $Z_k(\gamma) = \gamma(T_k)$. and extend it to $(Z_k)_{k\in\mathbb{Z}}$ by periodicity.

γ is trivial loop if $n = 0$.

γ is pointed loop if $n = 0$ or $\zeta(\gamma) = T_n(\gamma)$.

iii) ~~Set~~ measure μ_r on L_r:

$$\mu_r(B) = \sum_{x\in E} \int_0^\infty \mathbb{P}^t_{x,x}(B)\lambda_x\, dt/t$$

prop. μ_r is σ-finite measure. if $|E| < \infty$

Pf: 1) $\mu_r(\zeta > 0) = \int_0^\infty \sum_{x\in E} \mathbb{P}_x(X_t = x)/t\, dt$

$$\geq \int_0^\infty dt/t = \infty$$

2) $\mu_r(\zeta \geq n) = \sum_E \int_n^\infty r_t(x,x)\lambda_x\, dt/t$

$$\leq \frac{1}{n} \sum_E g(x,x)\lambda_x \overset{|E|<\infty}{<} \infty$$

prop. i) For $0 < t_1 < \cdots < t_k < t$. $x_i \in E$. $\mu_r(X_{t_i} = x_i,\ \zeta \in t + dt)$

$$= \begin{cases} r_{t_2-t_1}(x_1,x_2)\lambda_{x_2} \cdots r_{t_k-t_{k-1}}\lambda_k\ r_{t_1+t-t_k}(x_k,x_1)\lambda_{x_1}\frac{dt}{t}. & k>1 \\ r_t(x_1,x_1)\lambda_{x_1}\, dt/t & . k=1 \end{cases}$$

ii) $\mu_r(N=n,\ Z_i = x_i,\ 0 \le i \le n-1,\ T_i \in t_i + dt_i,\ 1 \le i \le n,\ \zeta \in dt + t)$

$$= \begin{cases} p_{x_0 x_1} \cdots p_{x_{n-1} x_0} \, I_{\{0 < t_1 < \cdots < t_n \le t\}} \dfrac{e^{-t}}{t} \prod_1^n dt_i \cdot dt, & n > 1 \\ e^{-t} dt / t, & n = 1 \end{cases}$$

Pf: Use the explicit expression of $\mathbb{P}_{x,y}$ before.

Cor. $\mu_r(N=n,\ Z_i = x_i,\ 0 \le i \le n-1) = \frac{1}{n} p_{x_0 x_1} \cdots p_{x_{n-1} x_0}$.

Cor. $\mu_r(N=n) = \frac{1}{n} \operatorname{Tr}(P^n)$.

Pf: $LHS = \sum_{x_0} \sum_{x_1 \sim x_{n-1}} \mu_r(N=n,\ Z_0 = x_0, \cdots Z_{n-1} = x_{n-1})$

$= \sum_{x_0} \frac{1}{n} \langle P^n I_{x_0}, I_{x_0} \rangle = \frac{1}{n} \operatorname{Tr}(P^n)$.

Cor. $\mu_r(N > 1) = -\log |I - P| < \infty$

Pf: By $\sum \frac{P^n}{n} = -\log(I-P)$

$\Rightarrow LHS = -\operatorname{Tr}(\log(I-P))$

$= -\sum_1^{|E|} \log(1 - y_i)$

$= -\log |I - P|$.

where (y_i) is eigenvalue of P.

prop. (Shift-invariant of discrete and conti. loops)

i) $\mu_r(N > 1,\ (Z_{k+m})_{m \in \mathbb{Z}} \in \cdot) = \mu_r(N > 1,\ (Z_m) \in \cdot)$

ii) $\theta_v \circ \mu_r = \mu_r$.

Pf: i) prove: $m_r(N=n,\ (Z_{k+m})_{m\le n} \in \cdot) =$

$m_r(N=n,\ (Z_m)_{m\le n} \in \cdot)$. $\forall k \in \mathbb{Z}$

Since (Z_k) has periodic n. on $[N=n]$.

It follows from loop structure.

ii) prove: $\theta_v \circ (\sum_E \mathbb{P}^t_{x,x} \lambda_x) = \sum_E \mathbb{P}^t_{x,x} \lambda_x$

WLOG. Set $0<v<t$. Since $\theta_{kt}(y) = y$

$\Leftrightarrow$ prove: $\sum_E \mathbb{P}^t_{x,x}(X_{t_i}=x_i,\ 1\le i\le n)\lambda_x =$

$\sum_E \mathbb{P}^t_{x,x}(X_{v+t_i}=x_i \quad 1\le i\le k,$

$X_{v+t_i-t}=x_i,\ k+1\le i\le n)\lambda_x$

where $t_{k+1} > t-v$, $t_k \le t-v$.

It also follows from loop structure.

Prop. (Time-reversal invariant of loops)

i) $m_r(N>1,\ (Z_m)\in\cdot) = m_r(N>1,\ (Z_{-m})\in\cdot)$

ii) $V \circ m_r = m_r$.

Pf: Similar as above. By sym of C_{xy} and r_t.

(2) Pointed Loops:

Def: i) $L_p = L_r \cap \{y \text{ is trivial or } T_n(y) = \zeta(y)\}$

ii) Durations of jump: $\sigma_0(y) = T_1(y) + \zeta(y) - T_n(y)$.

$\sigma_k(y) = T_{k+1}(y) - T_k(y)$, $k \le n-1$.

iii) Set measure μ_p on L_p by:

$\mu_p (N = n,\ z_i = x_i,\ 0 \le i \le n-1,\ \sigma_k \in s_k + ds_k,\ 0 \le k \le n-1) =$

$$\begin{cases} \frac{1}{n} p_{x_0 x_1} \cdots p_{x_{n-1} x_0}\, e^{-\sum_0^{n-1} s_k} \prod_0^{n-1} ds_i & \text{if } n > 1,\ s_i > 0. \\ e^{-t}\, dt / t & \text{if } n = 1. \end{cases}$$

Rmk: $\gamma \in L_r \Rightarrow \theta_{T_m}(\gamma) \in L_p$, for some $m \le N(\gamma) = n$.

So: $\theta_{T_m} \circ (I_{\{N=n\}} \mu_r)$ is measure on $L_p \cap \{N=n\}$

prop. i) $\theta_{T_m} \circ (I_{\{N=n\}} \mu_r) (z_i = x_i,\ \sigma_i \in s_i + ds_i,\ 0 \le i \le n-1)$

$$= p_{x_0 x_1} \cdots p_{x_{n-1} x_0} \frac{s_{n-m}}{\sum_0^{n-1} s_i} \cdot e^{-\sum_0^{n-1} s_i} \prod_0^{n-1} ds_i, \quad \forall m \le n.$$

ii) $\theta_{T_m} \circ (I_{\{N=n\}} \mu_r) = n \cdot \frac{\sigma_{n-m}}{\sum_0^{n} \sigma_i} \mu_p \cdot I_{\{N=n\}}$.

iii) $F : L_r \to \mathbb{R}_+$ bdd. measurable. st. $F \circ \theta_v = F$, $\forall v \in \mathbb{R}'$.

$$\Rightarrow \int_{\{N=n\}} F\, d\mu_r = \int_{\{N=n\}} F\, d\mu_p.$$

Pf: i) ii) by extending γ_k periodically and using explicit form of μ_r.

iii) $LHS = \int_{\{N=n\}} \sum_1^n F \circ \theta_{T_m}\, d\mu_r / n$

$$= \int_{\{N=n\}} F \sum_1^n d (\theta_{T_m} \circ d\mu_r I_{\{N=n\}}) / n$$

$$\overset{ii)}{=} \int_{\{N=n\}} F \cdot \sum_1^n \frac{\sigma_{n-m}}{\sum \sigma_i}\, d\mu_p = RHS.$$

(3) Restriction:

Def: $L_{r,u} = \{ \gamma \in L_r \mid \gamma \subset U \subset E \}$.

prop. If $U \subseteq E$ connected. $\mu_{r,U}$, $\mu_{p,U}$ are 2 measure defined as μ_r, μ_p by replacing E with U endowed with weight C_{xy} and $\tilde{k}_x = k_x + \sum_{y \in E/U} C_{xy}$. Then:

$I_{L_{r,U}} \mu_r = \mu_{r,U}$. $\quad I_{L_{r,U}} \mu_p = \mu_{p,U}$.

Rmk: If U isn't connected. Then apply on different components.

Pf: Note $\lambda_x = \tilde{\lambda}_x$ when $x \in U$.

(4) Local Time:

Def: local time of $\gamma \in L_r$ at $x \in E$ is:

$$L_x(\gamma) = \int_0^{\zeta(\gamma)} I_{\{X_s(\gamma) = x\}} ds / \lambda_x.$$

Rmk: Note $L_x(\gamma) \circ \theta_s = L_x$. $L_x(\gamma) \circ V = L_x(\gamma)$

$\Rightarrow$ We can use μ_p or μ_r indifferently when calculating expectation of $L_x(\gamma)$.

prop. (Laplace Transf.)

i) $\forall v \geq 0$. $\int_{\{N=0\}} (1 - e^{-vL_x}) d\mu_r = \log(1 + \frac{v}{\lambda_x})$

ii) $V: E \to \mathbb{R}^+$. $\int (1 - e^{-\sum_E V_x L_x}) d\mu_r =$

$\log |I + GV| = \log |I + V^{\frac{1}{2}} G V^{\frac{1}{2}}|$

$= -\log \frac{|G_V|}{|G|}$. $\quad G_V = (V - L)^{-1}$

Cor. $\int (1-e^{-vL_x})\, d\mu_r = \log(1+v\, g(x,x))$. $\forall v \geq 0$.

prop. (With restriction)

$V: U \to \mathbb{R}^+$. Then we have:

$$\int_{\{\gamma \subseteq U\}} (1-e^{-\sum_E V(x)L_x})\, d\mu_r = \log |I + G^U V|$$

$$= \log |I + V^{\frac{1}{2}} G^U V^{\frac{1}{2}}|$$

$$= -\log \frac{|G_{U,V}|}{|G^U|}$$

where. $G_{U,V} = (V - L_U)^{-1}$. $L_U f(x) = \sum_{y \in U} c_{xy} f_y - \lambda_x f(x)$

$G^U =: (g_U(x,y))_{U \times U}$.

prop. (Laplace Transf. of L_∞^z)

$$V: E \to \mathbb{R}^+. \Rightarrow (\mathbb{E}_{x,y}(e^{-\sum_E V(z) L_\infty^z}))_{E \times E} = G_V \stackrel{\Delta}{=} (V - L)^{-1}$$

(5) Unrooted Loops:

Def: i) $L^* =: L_r/\sim$. where $\gamma \sim \gamma'$ if $\gamma = \theta_v(\gamma')$,

for some $v \in \mathbb{R}'$. Set π^* is its canonical

map. equip L^* with σ-algebra $\mathcal{L}^* =:$

$\{B \subseteq L^* \mid \pi^{*-1}(B)$ is measurable in $L_r\}$.

ii) Loop measure μ^* on L^* is $\mu^* = \pi^* \circ \mu_r$.

Rmk: μ^* is also a σ-finite measure if

$|E| < \infty$: $\mu^*(\zeta \geq a) = \mu_r(\zeta \geq a) < \infty$.

for $\forall a > 0$.

① Def: (Unit Weight)

Measurable func. $T: L_r \to \mathbb{R}_{\geq 0}$ is unit weight

if $\int_0^{\zeta(y)} T(\theta_v(y))\, dv = 1$. $\forall y \in L_r$.

e.g. $T = \zeta^{-1}$. trivial case.

Lemma. For T is unit weight. $F \geq 0$ measurable on L^*. Then. We have:

$$\int_{L^*} F\, dm^* = \sum_E \int_{L_r} F \circ \tau^*(y)\, T(y)\, \lambda_x\, d\mathbb{P}_{x,x}(y).$$

Pf: $RHS \overset{\text{Fubini}}{=} \int_0^\infty \sum_E \lambda_x \mathbb{E}_{x,x}^t (F \circ \tau^* T)\, dt$

With $\sum \lambda_x \mathbb{E}_{x,x}^t (F \circ \tau^* T) =$

$\frac{1}{t} \int_0^t \sum \lambda_x \mathbb{E}_{x,x}^t ((F \circ \tau^* \circ \theta_v)(T \circ \theta_v))\, dv$

$= \frac{1}{t} \sum \lambda_x \mathbb{E}_{x,x}^t (F \circ \tau^* \cdot \int_0^t T \circ \theta_v\, dv)$

$= \frac{1}{t} \sum_E \lambda_x \mathbb{E}_{x,x}^t (F \circ \tau^*)$

Rmk: It offers a method of calculating $\int_{L^*} F\, dm^*$.

Cor. $I_{\{x \in y^*\}}\, dm^* = \tau^* \circ (\frac{1}{L_x}\, d\mathbb{P}_{x,x})$

$= d\, \tau^* \circ \mathbb{P}_{x,x} / L_x$

Pf: $T(y) = \begin{cases} \zeta(y)^{-1}. & \text{if } y(t) \neq x.\ \forall t \\ I_{\{y(0)=x\}} / \int_0^{\zeta(y)} I_{\{y(s)=x\}}\, ds. & \text{otherwise} \end{cases}$

$\Rightarrow T(\gamma)$ is an unit weight

(Only check $\exists t$. s.t. $\gamma(t) = x$ case)

Apply the Lemma above.

__Cor.__ $\int_{\{\gamma^* \ni x\}} L_x \, d\mu^* = g(x,x)$. $\forall x \in E$.

__Pf:__ $\int_{L_r} d\mathbb{P}_{xx} = g(x,x)$.