Isomorphism Theorems.

(1) GFF:

Consider GFF $(\varphi)_{x\in E}$ with law $\mathbb{P}^G$. s.t.

$\mathbb{E}^G(\varphi_x\varphi_y) = g(x,y)$. $\mathbb{E}^G(\varphi_x) = 0$.

For $\emptyset \neq K \subseteq E$. $U = E/K$. Set $\mathbb{P}^{G,U}$ is law

s.t. $\mathbb{E}^{G,U}(\varphi_x\varphi_y) = g_U(x,y)$. (So $\varphi = 0$ on K)

Lemma. For $h(x) = \mathbb{E}_x(H_K < \infty, \varphi_{X_{H_K}}) \in \sigma(\varphi_y, y\in K)$

Write $\varphi_x = \psi_x + h_x$. Then:

i) ψ is indept of $\sigma(\varphi_y, y\in K)$ under $\mathbb{P}^G$.

ii) $\psi \sim \varphi$. under $\mathbb{P}^{G,U}$.

iii) $(\varphi_x)_{x\in K}$ has density:

$$\frac{1}{(2\pi)^{|K|/2}\sqrt{\det_{K\times K} G}}\, e^{-\frac{1}{2}\mathcal{E}^*(\varphi,\varphi)} \prod_{x\in K} d\varphi_x.$$

Pf: i) we have proven it before.

ii) Consider $(\psi_x)_{x\in U}$ under $\mathbb{P}^G$

Since $\psi = 0$ on K under $\mathbb{P}^{G,U}$

$$\mathbb{E}^G(F(\psi_x)_{x\in U}) \propto \int_{\mathbb{R}^E} F(\varphi_x - h_x)_{x\in U} \cdot e^{-\frac{1}{2}\mathcal{E}(\varphi,\varphi)} \prod_{x\in E} d\varphi_x. \quad \text{for } F\in C_b.$$

Note $\mathcal{E}(\varphi,\varphi) = \mathcal{E}(\varphi-h,\varphi-h) + \mathcal{E}^*(\varphi|_K,\varphi|_K)$

Let $\varphi'_x = \varphi_x - h_x$. by change of variables:

LHS $\propto \int_{\mathbb{R}^U} F((\varphi_x)_{x\in U}) \cdot e^{-\frac{1}{2}\mathcal{E}(\varphi_U,\varphi_U)} \prod_{x\in U} d\varphi_x$.

iii) Similar as ii): Replace $F(\varphi)_{x\in U}$ by $H((\varphi_x)_{x\in K})$.

<u>Rmk</u>: We can obtain conditional dist. of GFF:

$$\mathbb{P}^G((\varphi_x)_{x\in E} \mid (\varphi_x)_{x\in K}) = \tilde{\mathbb{P}}^G((\tilde{\varphi}_x + h_x)_{x\in E})$$

where $\tilde{\varphi}_x \sim$ centered GFF under $\tilde{\mathbb{P}}^G$ with covariance $(g_U(x,y))_{U\times U}$ (so $\tilde{\varphi} = 0$ on K) and $\tilde{\mathbb{P}}^G$ doesn't act on h.

<u>Cor.</u> For $\emptyset \neq K \subset E$, A is increasing event, $\alpha \in \mathbb{R}^1$

$$\Rightarrow \mathbb{P}(A \mid \varphi|_K = \alpha) \le \mathbb{P}(A \mid \varphi|_K \ge \alpha)$$

<u>Pf</u>: On $\{\varphi|_K \ge \alpha\}$. We have:

$$h_x = \sum_K \varphi_y \, \mathbb{P}_x(H_K < \infty,\ X_{H_K} = y)$$

$$\ge \alpha\, \mathbb{P}_x(H_K < \infty) \overset{\Delta}{=} m_x(\alpha)$$

$$\mathbb{P}^G(A \mid \varphi|_K = \alpha)\cdot I_{\{\varphi|_K \ge \alpha\}} =$$

$$\tilde{\mathbb{P}}^G(A((\tilde{\varphi}_x + m_x(\alpha))_E))\, I_{\{\varphi|_K \ge \alpha\}} \overset{(A\uparrow)}{\le}$$

$$\tilde{\mathbb{P}}^G(A((\tilde{\varphi}_x + h_x)_E))\, I_{\{\varphi|_K \ge \alpha\}} =$$

$$\mathbb{P}^G(A \mid \varphi|_K)\cdot I_{\{\varphi|_K \ge \alpha\}}.$$

Integrating both side with $\mathbb{P}^G(\cdot \mid \varphi|_K \ge \alpha)$

(2) Path measure $\mathbb{P}_{x,\eta}$:

Def: i) $\mathcal{I}_t$ = space of right-conti. functions $[0,t] \to E$. with finite jumps.

ii) $\mathcal{I} = \bigcup_{t>0} \mathcal{I}_t$ E-valued trajectories with finite durations.

iii) Set $\zeta : \mathcal{I} \to \mathbb{R}'$. st. $\zeta(\gamma) = t$. if $\gamma \in \mathcal{I}_t$.

iv) Endow $\mathcal{I}_1 \times \mathbb{R}_+$ with σ-algebra: $\sigma(X_s,\ 0 \le s \le 1) \otimes \mathcal{B}_{\mathbb{R}_+}$.

$\Rightarrow$ Endow $\mathcal{I}$ with σ-algebra by:

$\phi : (w,t) \in \mathcal{I}_1 \times (0,\infty) \mapsto w(\frac{\cdot}{t}) \in \mathcal{I}$.

a bijection

v) For $(X_t)_{t\ge 0}$ canonical map. D_E is space of right-conti function with finite jumps taking values in $E \cup \{\Delta\}$.

$$\begin{aligned} \psi : D_E \cap \{X_t = \eta\} &\to \mathcal{I}_t \\ (X_t)_{t\ge 0} &\mapsto (X_s)_{s\le t} \end{aligned}$$

Set measure $\mathbb{P}^t_{x,\eta} = \psi \circ (I_{\{X_t=\eta\}} \mathbb{P}_x / \lambda_\eta)$

$$\mathbb{P}_{x,\eta} = \int_0^\infty \mathbb{P}^t_{x,\eta}(\cdot)\, dt$$

defined on $\mathcal{I}_t$ and $\mathcal{I}$.

Rmk: Total mass:

$$\mathbb{P}^t_{x,\eta}(I_t) = \mathbb{P}_x(X_t=\eta)/\lambda_\eta = r_t(x,\eta)$$

$$\mathbb{P}_{x,\eta}(I) = \int_0^\infty r_t(x,\eta) = g(x,\eta)$$

prop. i) For $0<t_1<\cdots<t_n$, $x_1\cdots x_n\in E$. We have:

$\mathbb{P}_{x,\eta}(X_{t_i}=x_i,\ 1\le i\le n) = \mathbb{P}_x(X_{t_i}=x_i,\ 1\le i\le n)\, g(x_n,\eta)$

$= r_{t_1}(x,x_1)\cdots r_{t_n-t_{n-1}}(x_{n-1},x_n)\, g(x_n,\eta)\cdot \prod_1^n \lambda_{x_i}$.

Rmk: It claims relation of $\mathbb{P}_x$ and $\mathbb{P}_{x,\eta}$.

ii) If $K\subseteq E$, $B\in\sigma(X_{s\wedge H_K},\ s\ge 0)$. Then:

$P_{x,\eta}(B,\ H_K\le\zeta) = \mathbb{E}_x(B\cap\{H_K<\infty\},\ g(X_{H_K},\eta))$

Pf: i) $LHS = \int_{t_n}^\infty \mathbb{P}_x(X_{t_i}=x_i,\ 1\le i\le n,\ X_t=\eta)/\lambda_\eta\, dt$.

ii) easy to check by Markov prop and Fubini.

Def: i) For $y\in I$, $N(y)$ is number of jumps of y before $\zeta(y)$. If $N(y)=n$. Then $\exists(T_i(y))_1^n$, st. $0<T_1(y)<T_2(y)\cdots<T_n(y)<\zeta(y)$ jump times of y.

ii) $v: y\in I\mapsto \check{y}\in I$ is time reversal, st. $y(0)=\check{y}(\zeta)$, $y(\zeta)=\check{y}(0)$, $\check{y}(s)=\lim_{\varepsilon\downarrow 0} y(\zeta-s-\varepsilon)$

iii) Let measure $\mathbb{P}_{\eta,x} = v\circ\mathbb{P}_{x,\eta}$.

Lemma: $\mathbb{P}_{x,y}(N=n,\ X_{T_1}=x_1, \cdots, X_{T_n}=x_n,\ T_1 \in t_1+dt_1,$

$\cdots, T_n \in t_n + dt_n,\ \zeta \in t+dt) =$

$$\frac{c_{xx_1}\cdots c_{x_{n-1}y}}{\lambda_x \lambda_{x_1}\cdots\lambda_{x_{n-1}}\lambda_y}\ \delta_{x_n,y}\ I_{\{0<t_1<\cdots<t_n<t\}}\ e^{-t}\prod_1^n dt_i \cdot dt.$$

Pf: $LHS = \mathbb{P}^t_{x,y}(N=n,\ X_{T_i}=x_i,\ T_i \in t_i+dt_i)\,dt$

$= \mathbb{P}_x(X_\cdot \text{ has } n \text{ jumps in } [0,t],\ X_{T_i}=x_i,$

$T_i \in t_i + dt_i)\ \delta_{x_n=y}\ \lambda_y^{-1}\ dt$

$= \mathbb{P}_x(Z_i = x_i,\ 1\le i\le n)\ \mathbb{P}_x(T_n < t < T_{n+1}, \cdots)$

$\cdot\ \delta_{x_n,y}\ /\ \lambda_y \cdot dt$

Rmk: $\mathbb{P}_{x,y}(N=0,\ \zeta \in t+dt) = e^{-t}\delta_{x,y}\ dt$

prop: For $V: E \to \mathbb{R}^1$, $g^y(\cdot) = g(y,\cdot)$, we have:

i) $\mathbb{E}_{x,y}\left(\left(\int_0^\infty V(X_s)ds\right)^n\right) = n!\,((GV)^n g^y)(x)$

ii) $\mathbb{E}_{x,y}\left(\prod_1^n L_\infty^{x_i}\right) = \sum_{\sigma\in S_n} g(x, x_{\sigma(1)}) \cdots g(x_{\sigma(n)}, y)$

S_n is set of permutations of $\{1,2,\cdots,n\}$.

iii) If $\|G|V|\|_\infty < 1$. Then:

$$\mathbb{E}_{x,y}\left(e^{\sum_z V(z) L_\infty^z}\right) = ((I-GV)^{-1} g^y)(x)$$

Pf: **Lemma:** $\mathbb{E}_{x,y}\left(\prod_1^n \int_0^\infty V_i(X_s)ds\right) =$

$$\sum_{\sigma\in S_n} (GV_{\sigma(1)} \cdots GV_{\sigma(n)}\, g^y)(x)$$

Pf: use explicit form of $\mathbb{P}_{x,y}$.

$\Rightarrow$ i) set $V_i = V$. ii) set $V_i = I_{x_i} / \lambda_{x_i}$.

iii) Note: $LHS = \mathbb{E}_{x,y}\left(\sum_{n \geq 0} \frac{1}{n!} \left(\int_0^\infty \frac{V}{\lambda}(X_s) ds\right)^n\right)$

(3) Isomorphism Thms:

Thm. (Dynkin's)

$\forall$ F bdd measurable on $\mathbb{R}^E$. we have:

$\mathbb{E}_{x,y} \otimes \mathbb{E}^G\left(F\left(L_\infty^z + \frac{1}{2}\varphi_z^2\right)_{z \in E}\right) = \mathbb{E}^G\left(\varphi_x \varphi_y F\left(\frac{1}{2}\varphi_x^2\right)_{x \in E}\right)$.

i.e. $\left(L_\infty^z + \frac{1}{2}\varphi_z^2\right)_{z \in E}$ under $\mathbb{P}_{x,y} \otimes \mathbb{P}^G$. $\sim$

$(\varphi_z^2/2)_{z \in E}$ under $\varphi_x \varphi_y \mathbb{P}^G$.

Pf: prove: $\mathbb{E}_{x,y} \otimes \mathbb{E}^G\left(e^{\sum_E V(z)\left(L_\infty^z + \frac{1}{2}\varphi_z^2\right)}\right) =$

$\mathbb{E}^G\left(\varphi_x \varphi_y\, e^{\sum_E \frac{1}{2} V(z) \varphi_z^2}\right)$.

for $\|V\|_\infty$ small.

$\Rightarrow$ Then check ch.f.'s are identical.

1°) We have $\mathbb{E}_{x,y}\left(e^{\sum_E V(z) L_\infty^z}\right) = \left((I - GV)^{-1} g^y\right)(x)$

follows from prop. above.

2°) Set $\mathcal{E}_V(\varphi, \varphi) = \mathcal{E}(\varphi, \varphi) - \sum_E V(z) \varphi_z^2$

it's positive definite for V small enough.

Besides $\mathcal{E}_V(\varphi, \varphi) = \langle (-L - V)\varphi, \varphi \rangle$.

Set $\mathbb{P}^{G,V} = \left((2\pi)^{|E|/2} \sqrt{|G|}\, \mathbb{E}^G\left(e^{\frac{1}{2}\sum_E V(z)\varphi_z^2}\right)\right)^{-1}$.

$e^{-\frac{1}{2}\mathcal{E}_V(\varphi, \varphi)} \prod_E d\varphi_x$. p.m. on $\mathbb{R}^E$.

where $\varphi \sim$ centered GFF under $\mathbb{P}^{h,V}$

with covariance $(\langle (-L-V)^{-1} I_x, I_y \rangle)_{E\times E}$

$$\Rightarrow \mathbb{E}^h (\varphi_x \varphi_y e^{\frac{1}{2}\sum_E V(z)\varphi_z^2})$$

$$= \mathbb{E}^{h,V} (\varphi_x \varphi_y) \cdot \mathbb{E}^h (e^{\frac{1}{2}\sum V(z)\varphi_z^2})$$

$$= ((-L-V)^{-1} I_x)(y) \cdot \mathbb{E}^h (e^{\frac{1}{2}\sum V(z)\varphi_z^2})$$

$$= ((I-hV)^{-1} h I_y)(x) \cdot \square = LHS.$$

<u>Thm.</u> (Eisenbaum's)

$\forall F$ bdd measurable on $\mathbb{R}^E$. Then

$$\mathbb{E}_x \otimes \mathbb{E}^h (F(L_\infty^z + \frac{(\varphi_z+s)^2}{2})_{z\in E}) = \mathbb{E}^h ((1+\frac{\varphi_x}{s}) F(\frac{(\varphi_z+s)^2}{2})_{z\in E})$$

i.e. $(L_\infty^z + \frac{1}{2}(\varphi_z+s)^2)_{z\in E}$ under $\mathbb{P}_x \otimes \mathbb{P}^h \sim$

$(\frac{1}{2}(\varphi_z+s)^2)_{z\in E}$ under $(1+\frac{\varphi_x}{s}) \mathbb{P}^h$.

<u>Lemma.</u> For $V_i : E \to \mathbb{R}^+$. $n \geq 0$. $1\leq i\leq n$.

i) $\mathbb{E}_x (\prod_1^n V_i(X_s) ds) = \sum_{\sigma\in S_n} (Q V_{\sigma(1)} \cdots Q V_{\sigma(n)} 1^n)(x)$

ii) $\mathbb{E}_x (\prod_1^n L_\infty^{x_i}) = \sum_{\sigma\in S_n} g(x, x_{\sigma(1)}) \cdots g(x_{\sigma(n-1)}, x_{\sigma(n)})$

iii) When $\| h|V| \|_\infty < 1$, we have:

$$\mathbb{E}_x (e^{\sum_E V(z) L_\infty^z}) = ((I-hV)^{-1} I_E)(x)$$

<u>Lemma.</u> For (X,Y) 2-dim Gaussian vector. Then

$\mathbb{E}(X e^{sY}) / s\mathbb{E}(e^{sY}) = \mathbb{E}(XY)$ if $s\neq 0$

<u>Pf:</u> By $\frac{\partial}{\partial t} \mathbb{E}(e^{tX+sY})|_{t=0} = \frac{\partial}{\partial t} e^{\frac{1}{2}\mathbb{E}((tX+sY)^2)} |_{t=0}$

Pf: Prove: $\mathbb{E}_x \otimes \mathbb{E}_h (e^{\sum_z V(z) (L_\infty^z + \frac{(\varphi_z+s)^2}{2})}) =$

$\mathbb{E}_h ((1+\frac{\varphi_x}{s}) e^{\sum V(z) \frac{(\varphi_z+s)^2}{2}})$ for V small.

Similar as above with Lemmas.

Combined with:

$\mathbb{E}^h ((1+\frac{\varphi_x}{s}) e^{\frac{1}{2}\sum V(z) (\varphi_z+s)^2}) / \mathbb{E}^h (e^{\sum V(z) \frac{(\varphi_z+s)^2}{2}})$

$= 1 + \frac{1}{s} \cdot \mathbb{E}^{h,V} (\varphi_x e^{s \sum V(z) \varphi_z}) / \mathbb{E}^{h,V} (e^{s \sum V(z) \varphi_z})$

$\overset{\text{Lem.}}{=} 1 + \mathbb{E}^{h,V} (\varphi_x \sum V_z \varphi_z)$

$= 1 + ((I - hV)^{-1} hV) (x) = \mathbb{E}_x (e^{\sum V_z L_\infty^z})$

Rmk: Note that in the two isomorphism.

Thm. We obtain:

$\varphi_x \varphi_y \mathbb{P}^h$, $(1+\frac{\varphi_x}{s}) \mathbb{P}^h$ are both positive.

(4) Generalized Ray-Knight Thm.

For $x_0 \in E$. Set $U = \bar{E}/\{x_0\}$. Assume $k_x = 0$, $\forall x \neq x_0$.

$\mathbb{P}^{h,U}$ is law of centered GFF φ with cov. g_U.

Thm. (First Ray-Knight)

$\forall x \in E$, $s \neq 0$. We have: $(\frac{1}{2}(\varphi_z+s)^2)_{z\in E}$ under

$(1+\frac{\varphi_x}{s}) \mathbb{P}^{h,U} \sim (L_{H_{x_0}}^z + \frac{1}{2}(\varphi_z+s)^2)_{z\in \bar{E}}$ under $\mathbb{P}_x \otimes \mathbb{P}^{h,U}$

Pf: Replace $\bar{E}$ with U, g with g_U in

Eisenbaum isomorphism. Thm.