Potential Theory

(1) Markov Chain X_t:

Suppose $E \neq \emptyset$. countable set.

Denote: i) $c_{xy} = c_{yx} \geq 0$. $c_{xx} = 0$. $\forall x, y \in E$. weight.

$k_x \geq 0$. $(\exists x \in E.\ k_x > 0)$. killing measure.

$$\lambda_x = \sum_{y \in E} c_{xy} + k_x. \quad \forall x \in E$$

ii) Dirichlet form of $f : E \to \mathbb{R}'$:

$$\mathcal{E}(f, f) = \frac{1}{2} \sum_{x,y \in E} c_{xy} (f(x) - f(y))^2 + \sum_E k_x f^2(x)$$

Rmk: By $\mathcal{E}(I_x, I_y) = -c_{xy}$. $\forall x \neq y \in E$.

$$\mathcal{E}(I_x, I_x) = \sum_{y \in E} c_{xy} + k_x = \lambda_x.$$

So: $\mathcal{E} \overset{\text{corrsp}}{\Longleftrightarrow} (c_{xy}) \cup (k_x)$.

iii) $(\cdot, \cdot)_\lambda$ is inner product in $L^2(d\lambda)$.

def by: $(f, g)_\lambda = \sum_E f(x) g(x) \lambda_x$.

iv) Set sub-Markovian trans. prob. on $E \cup \{\Delta\}$. Δ is coffin state:

$$P_{xy} = c_{xy} / \lambda_x. \quad P_{x\Delta} = k_x / \lambda_x. \quad P_{\Delta\Delta} = 1.$$

Rmk: (P_{xy}) is λ-reversible:

$$\lambda_x P_{xy} = c_{xy} = \lambda_y P_{yx}. \quad \forall x, y \in E$$

$\Rightarrow$ Set (Z_n) is the DTMC correspond (P_{xy}) on $E \cup \{\Delta\}$

And introduce CTMC $(X_t)_{t\geq 0}$ with embedded chain (Z_n) and holding time of $\exp(1)$.

Def: i) Sub-Markovian trans semi-group of X_t defined for $f: E \to \mathbb{R}^1$ is R_t:

$$R_t f(x) = \mathbb{E}_x(f(X_t))$$
$$= \sum_{n\geq 0} e^{-t} \frac{t^n}{n!} \mathbb{E}_x(f(Z_n))$$
$$= \sum_{n\geq 0} e^{-t} \frac{t^n}{n!} P^n f(x) = e^{t(P-I)} f(x)$$

Rmk: P and R_t are both bdd self-adjoint w.r.t $(\cdot,\cdot)_\lambda$ by λ-reversible.

ii) Transition density: $r_t(x,y) = (R_t I_y)(x)/\lambda_y$

Rmk: i) $r_t(x,y) = r_t(y,x)$ by self-adjoint of R_t. $\forall x,y \in E$

ii) $r_{t+s}(x,y) = \sum_{z\in E} r_t(x,z) r_s(z,y) \lambda_z$.

iii) $r_t(x,y) > 0$. $\forall t>0$. $x,y \in E$.

iii) Green function $g(x,y)$ is defined by:

$$g(x,y) = \int_0^\infty r_t(x,y)\, dt = \mathbb{E}_x\left(\int_0^\infty I_{\{X_t=y\}}\, dt\right)/\lambda_y$$

Lemma. $g(x,y) \in (0,\infty)$. symmetric if $|E|<\infty$.

Pf: 1) $\exists N\geq 0$. $\varepsilon>0$. s.t.

$$\min_E \mathbb{P}_x(Z_n = \Delta \text{ for some } n \leq N) \geq \varepsilon.$$

$\Rightarrow P^{kN} I_E(x) = P_x(Z_n \neq \Delta,\ \text{for } 0 \le n \le kN)$

$\overset{mp}{\le} (1-\varepsilon)^k,\ \text{for } k \ge 1.$

2) By interpolation : $\sup_{x \in E} (P^n I_E)(x) \le c e^{-c'n}$

So : $g(x,y) \le \frac{1}{\lambda_y} \int_0^\infty R_t I_E(x)\, dt$

$\le \frac{1}{\lambda_y} \int_0^\infty \sum_1^\infty e^{-t} \frac{t^n}{n!} \sup_x P^n I_E(x)\, dt$

$\le \frac{c}{\lambda_y} / (1-e^{-c'}) < \infty.$

3) Positive and symmetric are from $r_t(x,y)$.

(2) <u>Potential:</u>

Follows also hold for discrete def and RW (Z_n).

① <u>Def:</u> i) Potential operator :

$Gf(x) = \sum_E g(x,y) f(y) \lambda_y$ for $f: E \to \mathbb{R}^1$.

the potential of func. f.

ii) $G\nu(x) = \sum_{y \in E} g(x,y) \nu(y)$. for $\nu : E \to \mathbb{R}^1$.

the potential of measure ν.

iii) Dual bracket between func. and measure

on E : $\langle \nu, f \rangle = \sum_E \nu(x) f(x)$.

<u>Prop.</u> i) $G = (I-P)^{-1}$ ii) $Gk = 1$. $k = x \mapsto k_x$.

iii) $G = (-L)^{-1}$ where $Lf(x) = \sum_y c_{xy} f(y) - \lambda_x f(x)$

iv) $\Sigma(G\nu, f) = \langle \nu, f \rangle$ for measure ν. func. f.

v) $\exists\, c>0$. s.t. $\mathcal{E}(f,f) \geq c\|f\|^2_{L^2(\lambda)}$, $\forall$ func. f.

vi) $E(\nu,\mu) =: \langle \nu, G\mu\rangle = \sum_{x,y} \nu_x\, g(x,y)\, \mu_y$ defines a positive definite, sym bilinear form for measures.

<u>Pf</u>: i) By DCT. exchange $\sum_0^\infty$ and $\int_0^\infty$.

Note that $\int_0^\infty \sum_{n\geq 0} e^{-t}\frac{t^n}{n!}|P^n f(x)|\,dt < \infty$.

iii) $-L = \lambda(I-P) \Rightarrow (-L)^{-1} = (I-P)^{-1}\lambda^{-1} = G\lambda^{-1} = G$

iv) By $\mathcal{E}(f,g) = \sum_{x,y} f(x)\, g(y)\, \mathcal{E}(I_x, I_y)$

$= \langle f, -Lg\rangle = \langle -Lf, g\rangle$.

v) $f(x) = \langle I_x, f\rangle = \mathcal{E}(GI_x, f)$

$\overset{\text{Cauchy}}{\leq} \mathcal{E}(GI_x, GI_x)^{\frac{1}{2}}\, \mathcal{E}(f,f)^{\frac{1}{2}}$.

ii) $G(\mathbb{1}, f) = \sum k_x f(x) = \langle k, f\rangle = \mathcal{E}(Gk, f)$

$\Rightarrow \mathbb{1} = Gk$.

vi) Symmetric is from def of $g(x,y)$

$E(\nu,\nu) = \langle \nu, G\nu\rangle = \mathcal{E}(G\nu, G\nu) \geq 0$.

with: $E(\nu,\nu) = 0 \iff G\nu = 0 \iff \nu = (-L)G\nu = 0$.

<u>Rmk</u>: i) k is called equilibrim measure of $\overline{E}$.

ii) $E(\nu,\nu)$ is energy of ν.

<u>Def</u>: $r_{t,u}(x,y) = \mathbb{P}_x(X_t = y,\ t < T_u)/\lambda_y$. $g_u(x,y) = \int_0^\infty r_{t,u}(x,y)\,dt$

trans. density and green func. outside u.

Rmk: If $U \subseteq E$. connected. Then we can define $(c_{xy})_{U\times U}$. and $\tilde{K}_x = K_x + \sum_{E/U} c_{xy}$ the weight and killing measure for it.

If U isn't connected. Then apply to each component of U.

prop. i) $g_U(x,y) = g_U(y,x)$. $\forall\, x,y \in E$

ii) $g(x,y) = g_U(x,y) + \mathbb{E}_x(H_A < \infty;\ g(X_{H_A},y))$

where $A = E/U$. $H_A = \inf\{t \geq 0 \mid X_t \in A\}$.

iii) (Hunt's switching id.)

$\mathbb{E}_x(H_A < \infty;\ g(X_{H_A},y)) = \mathbb{E}_y(H_A < \infty;\ g(X_{H_A},x))$

Pf: i) By the Rmk. above

ii) $g(x,y) = g_U(x,y) + \mathbb{E}_x(T_U < \infty.\ (\int_0^\infty I_{\{X_t = y\}})\circ\theta_{T_U})/\lambda_y$

$\overset{\text{Smp}}{\underset{T_U = H_A}{=}}$ RHS

iii) By ii): $g.\ g_U$ are symmetric.

Cor. i) For x or $y \notin U$. $\Rightarrow g(x,y) = \mathbb{E}_x(H_A < \infty.\ g(X_{H_A},y))$

ii) $\mathbb{P}_x(H_{\{x_0\}} < \infty) = g(x_0,x_0)/g(x,x_0)$

iii) $g_U(x,y) = g(x,y) - g(x,x_0)g(x_0,y)/g(x_0,x_0)$.

for $U = E/\{x_0\}$. $x,y \in E$.

Pf: ii) Apply on $A = \{x_0\}$. $U = \overline{E} / \{x_0\}$.

② Capacity:

Def: i) For measure ν on E. $\mathbb{P}_\nu = \sum_E \nu_x \mathbb{P}_x$.

ii) For $\widetilde{H}_A = \inf\{t > 0 \mid X_t \in K, \exists s \in (0,t). \text{ s.t. } X_s \neq X_0\}$. $A \subseteq \overline{E}$.

iii) Equilibrim measure of A is $e_A(x) = \mathbb{P}_x(\widetilde{H}_A = \infty) \cdot \lambda_x \cdot I_A(x)$. $x \in E$.

iv) capacity of A is $\text{cap}(A) = \sum_{x \in A} e_A(x)$

Rmk: i) $e_A(x)$ supports on $\partial^{int} A$. actually.

ii) For $A = \overline{E}$. $e_{\overline{E}}(x) = k_x$.

Lemma. i) $\text{cap}(K_1 \cup K_2) \leq \text{cap}(K_1) + \text{cap}(K_2)$.

ii) $\text{cap}(K_1) \leq \text{cap}(K_2)$. if $K_1 \subseteq K_2$.

Pf: $e_{K_1 \cup K_2}(x) = \lambda_x \mathbb{P}_x(\widetilde{H}_{K_1 \cup K_2} = \infty)$

$\leq \lambda_x \mathbb{P}_x(\widetilde{H}_{K_i} = \infty)$. $\forall x \in K_i$. $i = 1.2$.

Prop. For $A \subseteq E$. $L_A = \sup\{t > 0 \mid X_t \in A\}$ last visit time.

i) $\mathbb{P}_x(L_A > 0, X_{L_A^-} = \eta) = g(x,\eta) e_A(\eta)$

So $h_A(x) =: \mathbb{P}_x(L_A > 0) = \mathbb{P}_x(H_A < \infty) = G e_A(x)$

$= \sum_{\eta \in E} g(x,\eta) e_A(\eta)$.

ii) $A \neq \emptyset \Rightarrow e_A$ is the unique measure ν supports on A. s.t. $h\nu = 1$. on A.

iii) $\mathbb{P}_{e_B}(H_A < \infty,\ X_{H_A} = x) = \mathbb{P}_{e_B}(L_A > 0,\ X_{L_A^-} = x)$

$= e_A(x)$. i.e. entrance dist in $A \overset{\mathbb{P}_{e_B}}{\sim}$ last exit dist. of $A \overset{\mathbb{P}_{e_B}}{\sim} e_A$. $A \subseteq B \subseteq E$

<u>Pf</u>: i) $LHS = \mathbb{P}_x(\bigcup_{n \geq 0} \{Z_n = y,\ \forall k > n.\ Z_k \notin A\})$

$\overset{MP}{=} \sum \mathbb{P}_x(Z_n = y)\, \mathbb{P}_y(\forall k > 0.\ Z_k \notin A)$.

With: $\mathbb{E}_x(\int_0^\infty I_{\{X_t = y\}}\, dt) =$

$\mathbb{E}_x(\sum_{n \geq 0} (T_{n+1} - T_n)\, I_{\{Z_n = y\}})$

$= \sum_{n \geq 0} \mathbb{E}_x(I_{\{Z_n = y\}})$. by $\mathbb{E}(\tau) = 1$.

ii) By i) $he_A = 1$ on A. if ν is another.

Set $\mu = e_A - \nu$. $\Rightarrow (h\mu, \mu) = 0$

$\Rightarrow \mathcal{E}(\mu, \mu) = 0 \Rightarrow \mu = 0$.

iii) The first follows from Hunt switching id.

With: $\mathbb{P}_{e_B}(L_A > 0,\ X_{L_A^-} = y) \overset{i)}{=}$

$\sum_{x \in E} e_B(x)\, g(x, y)\, e_A(y) \overset{A \subseteq B}{=} \mathbb{P}_y(H_A < \infty)$

<u>Cor.</u> For (Z_n) SRW in $E = \mathbb{Z}^d$. (i.e. $c_{xy} = 1/2d$. $k_x \equiv 0$) : $\mathbb{P}_x(H_K < T_D) = \sum_{y \in K} g_D(x, y)\, \mathbb{P}_y(\widetilde{H}_K > T_D)$

for $K \subseteq D$. H_K. T_D. $\widetilde{H}_K$ are stopping time of Z.

Cor. i) $\mathrm{cap}(x) = 1/g(0)$. $\mathbb{P}_x(H_y < \infty) = g(x,y)/g(0)$.

ii) For $x \neq y$. $\mathrm{cap}(\{x,y\}) = 2/(g(0) + g(y,x))$

Pf: By $\mathbb{P}_x(H_A < \infty) = \sum g(x,y) e_A(y)$.

Set $A = \{x\}$. or $\{x,y\}$.

Prop. (Variational Characterization)

$\emptyset \neq A \subseteq E$. $\tilde{e}_A = e_A / \mathrm{cap}(A)$. normalized e.m.

i) $\mathrm{cap}(A) = 1/\inf\{E(\nu,\nu) \mid \nu \text{ is p.m. supp on } A\}$.

which is attained by $\tilde{e}_A$ uniquely.

ii) $\mathrm{cap}(A) = \inf\{\mathcal{E}(f,f) \mid f \geq 1 \text{ on } A\}$.

which is attained by h_A uniquely.

Pf: i) $E(\nu,\nu) = E(\tilde{e}_A, \tilde{e}_A) + \square$.

Note $E(\nu - \tilde{e}_A, \tilde{e}_A) = \sum_{x \in E} (\nu_x - \tilde{e}_A(x))(\sum_y g(x,y)\tilde{e}_A(y))$

$= (1-1)/\mathrm{cap}(A). = 0.$

$E(\tilde{e}_A, \tilde{e}_A) = \sum_{x,y} e_A(x) g(x,y) e_A(y) / \mathrm{cap}(A)^2$

$= e_A(A)/\mathrm{cap}^2(A) = 1/\mathrm{cap}(A)$

ii) Note $h_A = h_{e_A} = 1$ on A

Check: $\mathcal{E}(f - h_A, h_A) = 0$. again.

Prop. (Converse of above)

For $K \subset\subset E$. $\Sigma^{\uparrow} = \{\varphi \text{ supp on } K \mid G\varphi(x) \leq 1 \text{ on } K\}$.

$\Sigma^{\downarrow} = \{\varphi \text{ supp on } K \mid G\varphi(x) \geq 1 \text{ on } K\}$. family of func.

$\Rightarrow \mathrm{cap}(K) = \max_{\Sigma^{\uparrow}} \sum_K \varphi(x) = \min_{\Sigma^{\downarrow}} \sum_K \varphi(x)$.

Pf: $\sum_{x\in K} \tilde{e}_K(x)\, G\varphi(x) = \frac{1}{\mathrm{cap}(K)} \sum_{y\in K} \varphi(y)\cdot G e_K(y)$

$$= \frac{1}{\mathrm{cap}(K)} \sum_{K} \varphi(y)$$

$$\begin{cases} \le 1. & \text{if } \varphi \in \Sigma^{+} \\ \ge 1. & \text{if } \varphi \in \Sigma^{\downarrow} \end{cases}$$

With $e_K \in \Sigma^{\downarrow} \cap \Sigma^{+}$. $Ge_K = 1$ on K.

Cor. Consider $E = \mathbb{Z}^d$. $K \subset\subset \mathbb{Z}^d$. Then

$$|K| / \sup_{x\in K} G I_K(x) \le \mathrm{cap}(K) \le |K| / \inf_{x\in K} G I_K(x)$$

Cor. Using $g(x,y) \sim (1+|x-y|)^{2-d}$ in SRW case. $\Rightarrow \mathrm{cap}(B(R)) \sim R^{d-2}$.

③ Decomposition:

Def: For $U \subseteq E$. $K = E/U$. Set $\mathcal{F} = \{f: E \to \mathbb{R}'\}$.

i) $\mathcal{F}_U = \{\varphi: E \to \mathbb{R}' \mid \varphi(x) = 0$ on $K\}$.

ii) $\mathcal{H}_U = \{\varphi: E \to \mathbb{R}' \mid Ph(x) = h(x)$ on $U\}$. the space of harmonic func.

iii) $\mathcal{G}_K = \{G\nu \mid \nu$ is measure supp on $K\}$. the space of potential of measure supp on K.

prop. i) $\mathcal{H}_U = \mathcal{G}_K$

ii) $\mathcal{F} = \mathcal{F}_U \oplus \mathcal{H}_U$. $\mathcal{H}_U \perp \mathcal{F}_U$ w.r.t $\Sigma(\cdot,\cdot)$.

Pf: i) check by $h = G(-L)h$

ii) $\mathcal{H}_n \perp \mathcal{Z}_n$ is easy to check by i).

For $f \in \mathcal{G}$. Set $h(x) = \mathbb{E}_x(H_K < \infty, f(X_{H_K}))$

$\Rightarrow h \in \mathcal{H}_n$. Set $\varphi(x) = f(x) - h(x)$

$\Rightarrow \varphi \in \mathcal{Z}_n$.

Rmk: Set $A = K$ in $g_n(x, y) + \mathbb{E}_x(H_A < \infty : \cdots) = g(x, y)$

$\Rightarrow$ For $h = GV \in \mathcal{H}_n$. we have:

$h(x) = \mathbb{E}_x(H_K < \infty, h(X_{H_K}))$. $\forall x \in E$.

Def: For $K \subseteq E$. $f : K \to \mathbb{R}^1$.

Trace form on K is $\mathcal{E}^*$. St. $\mathcal{E}^*(f, f) =$

$\mathcal{E}(\tilde{f}, \tilde{f})$. where $\tilde{f}(x) = \mathbb{E}_x(H_K < \infty. f(X_{H_K}))$

Rmk: i) Extend to bilinear form $\mathcal{E}^*(f, g) =$

$\mathcal{E}(\tilde{f}, \tilde{g})$. $\Rightarrow$ It's symmetric

ii) $\mathcal{E}^*(f, f) = \inf\{\mathcal{E}(g, g) \mid g : E \to \mathbb{R}^1.\ g = f$ on $K\}$. follows from by orthogonal decomposition: $g = \varphi + \tilde{f}$. $\varphi \in \mathcal{Z}_n$.

prop. $K \neq \emptyset \subseteq \bar{E}$. nonempty subset of $\bar{E}$. Next restrict in K:

For $c^*_{xy} = \lambda_x \mathbb{P}_x(\tilde{H}_K < \infty, X_{\tilde{H}_K} = y)$. $x \neq y$ in K.

$k^*_x = \lambda_x \mathbb{P}_x(\tilde{H}_K = \infty)$. $\lambda^*_x = \lambda_x(1 - \mathbb{P}_x(\tilde{H}_K < \infty, X_{\tilde{H}_K} = x))$

with Dirichlet form $\tilde{\mathcal{E}}^*(f, f) = \frac{1}{2} \sum_{x, y \in K} c^*_{xy}(f(x) - f(y))^2$

$+ \sum k^*_x f^2(x)$ for $f : K \to \mathbb{R}^1$. and green func. g^*.

$\Rightarrow \tilde{\mathcal{E}}^* = \mathcal{E}^*$ trace form. $g^*(x, y) = g(x, y)$. for $x, y \in K$.

Pf: 1°) $c(x_\eta^*, k_x^*, \lambda_x^*)$ satisfies the same property as $c(x_\eta, k_x, \lambda_x)$.

2°) By unique determination.

Check values of $\mathcal{E}^*(I_x, I_\eta)$

3°) For $\psi_x(\cdot) = g(x, \cdot)|_K$, $\psi_x^* = g^*(x, \cdot)$

Note that $\overline{\psi_x} = h I_x$.

$\Rightarrow \mathcal{E}^*(\psi_x, I_\eta) = \mathcal{E}^*(\psi_x^*, I_\eta)$, $\forall \eta \in K$.

So $g = g^*$ on K.

Rmk: For $K \subseteq K' \subseteq E$. Then: trace form on K of $\mathcal{E}$ = trace form on K' of $\mathcal{E}$.

$\Rightarrow$ It's tower property of traces.

④ Feynman-Kac Formula:

Thm: For $V, f : E \to \mathbb{R}'$. Set $Vf(x) = V(x)f(x)$.

$$\Rightarrow \mathbb{E}_x\left(f(X_t)\, e^{\int_0^t V(X_s)\,ds}\right) = \left(e^{t(P-I+V)} f\right)(x).$$

Pf: Identical as pf of Diffusion version.

⑤ Local Times:

Def: Local time of X at site $x \in E$ before time t is $L_t^x = \int_0^t I_{\{X_s = x\}}\, ds / \lambda_x$.

Rmk: $t \mapsto L_t^x$ is conti. $\uparrow$. With finite limit L_∞^x.

Prop. i) $\mathbb{E}_x(L_\infty^y) = g(x,y)$. $\bar{\mathbb{E}}_x(L_{T_n}^y) = g_n(x,y)$.

ii) $\sum_{E\cup\{\Delta\}} V(x) L_t^x = \int_0^t V/\lambda (X_s)\, ds$

iii) $L_t^x \circ \theta_s + L_s^x = L_{t+s}^x$.

Pf: ii) RHS $= \sum_{E\cup\{\Delta\}} V(x) \int_0^t I\{X_s = x\}\, ds/\lambda_x$.

(3) Variable jumping rate:

① Def: i) $L_t = \sum_{E\cup\{\Delta\}} L_t^x = \int_0^t \lambda_{X_s}^{-1}\, ds$.

ii) $\tau_u = \inf\{t \geq 0 \mid L_t \geq u\}$. $\bar{X}_u =: X_{\tau_u}$.

the time-change process with local time: $\bar{L}_u^x$.

Prop. i) $\bar{X}_u$ is Markov chain with semigroup $\bar{R}_t$:

$$\bar{R}_t f(x) =: \bar{\mathbb{E}}_x(f(\bar{X}_t)) = e^{tL} f(x)$$

ii) $X_t = \bar{X}_{L_t}$. $L_t^x = \bar{L}_{L_t}^x$. $L_\infty^x = \bar{L}_\infty^x$.

Rmk: i) means $\bar{X}.$ has embedded chain Z_n. and jumping rates (λ_x)

Thm. (Feynman - Kac Formula)

$$\mathbb{E}_x(f(\bar{X}_u)\, e^{\int_0^u V(\bar{X}_v)\, dv}) = e^{u(L+V)} f(x)$$

③ Trace Form:

Def: i) $\bar{L}^k_u = \sum_{K\cup\{\Delta\}} L^x_u = \int_0^u I_{\{\bar{X}_v \in K\cup\{\Delta\}\}} dv$

ii) $\bar{\tau}^k_v = \inf\{ u \geq 0 \mid \bar{L}^k_u > v\}$

Set $\bar{X}^k_v = \bar{X}_{\bar{\tau}^k_v}$. trace process of $\bar{X}$.

prop. Under $\mathbb{P}_x$, $x \in K\cup\{\Delta\}$. $\bar{X}^k_v$ is Markov Chain on K with coffin state Δ.

St. i) Its Semigroup $\bar{R}^k_t$ satisfies:

$$\bar{R}^k_t f(x) = e^{tL^*} f(x). \quad f: K \to \mathbb{R}'.$$

ii) Its generator $\mathcal{L}$ satisfies:

$$(-\mathcal{L})^{-1} = (g(x,y)|_K)_{K\times K} = (g^*)_{K\times K}.$$

Rmk: Note that $\bar{X}^k_{\bar{L}^k_v} = \bar{X}_v$

$\bar{L}^k_v \uparrow$ on v when $\bar{X}_v \in K\cup\{\Delta\}$

$\Rightarrow \bar{X}^k_v$ only takes value in $K\cup\{\Delta\}$.