Qualitative Analysis

(1) Phase Plane:

① . suppose a point M is at the place of its ordinate $\vec{x} = (x_1 \cdots x_n)$. with speed $\vec{v}(\vec{x})$ at the time t. Then it satisfies: $\frac{d\vec{x}}{dt} = \vec{v}(\vec{x})$ (E)

Given $\vec{x}(t_0) = \vec{x}_0$. we may solve the I.V.P.

$\vec{x} = \vec{\varphi}(t, t_0, \vec{x}_0)$ as unique solution.

<u>Def</u>: We call the domain of $\vec{x}$: phase plane . P

the domain of $(t, \vec{x})$: extended phase plane . P'

<u>Remark</u>: $\vec{x}(t) \in P$. but $(t, \vec{x}) \in P'$.

② <u>Properties</u>:

i) For $\vec{x}(t) = \vec{\varphi}(t, t_0, x_0)$ is a solution

Then $\vec{x}(t) = \vec{\varphi}(t+c, t_0, x_0)$ is another solution.

ii) If the I.V.P has unique solution $\vec{x} = \varphi(t, t_0, x_0)$

Then the locuses in P won't intersect mutually

<u>Pf</u>: If $\exists\ \varphi_1, \varphi_2$ the locuses of point M. st.

$\varphi_1(t_1) = \varphi_2(t_2)$. Then $\varphi_1(t + t_2 - t_1) = \varphi_2(t)$ by uniqueness!

iii) $\varphi(t_2, \varphi(t_1, x_0)) = \varphi(t_1 + t_2, x_0)$. If exists unique solution

Pf: Convert to I.V.P: $x(t_0) = \varphi(t_1, x_0)$

$\therefore$ $\varphi(t, t_0, \varphi(t_1, x_0))$ is solution.

Since $\varphi(t+t_1, t_0, x_0)$ is another solution. By uniqueness.

$\therefore \varphi(t, t_0, \varphi(t_1, x_0)) = \varphi(t+t_1, t_0, x_0)$.

iv) If exists unique solution and $\exists t_0, T > 0$. s.t.

$x(t_0+T) = x(t_0)$. Then $\forall t$. $x(t+T) = x(t)$.

Pf: $x(t+T)$. $x(t)$ is solution of (E) with $x(t_0) = x_0$.

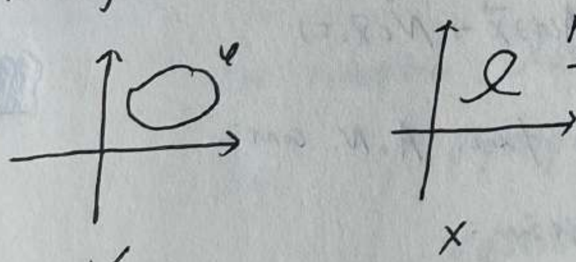
By uniqueness. $x(t+T) = x(t)$.

v) φ can't arrive a singularity in finite time

if it sets up from a nonsingular point.

vi) The only case of φ is intersected with

itself is φ is simply closed curve.



✓ X

Pf: If not. the tangent of curve won't be same at same point $x(t) = x(t_0)$!

Remark: For nonhomogeneous case, it will happen.

(2) Stability of Solutions:

. For $\frac{d\vec{x}}{dt} = \vec{f}(t, \vec{x})$. $\vec{f}$ satisfies Lipschitz condition on $\vec{x}$. conti. on $\mathbb{R} \times G$. (E)

· **Def:** A solution $\vec{x} = \vec{\varphi}(t)$ of (E) on $[t_0, +\infty)$

i) It's stable. if:

$\forall \varepsilon > 0$. $\exists \delta(\varepsilon) > 0$. s.t. $|\vec{x}_0 - \vec{\varphi}(t_0)| < \delta$.

Then for solution $\vec{x}$ of (E) with $x(t_0) = x_0$

satisfies : $|\vec{x}(t_0, x_0, t) - \vec{\varphi}(t)| < \varepsilon$. $\forall t \geq t_0$.

ii) It's asymptotically stable. if:

$\forall \varepsilon > 0$. $\exists \delta$. s.t. $|x_0 - \varphi(t_0)| < \delta$.

Then $\lim\limits_{t \to \infty} (x(t_0, x_0, t) - \varphi(t)) = 0$.

Remark: We call the domain D of x_0. which satisfies ii) : asymptotic stable domain.

① Linear Approximation:

· suppose: $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}, t) \simeq A(t)\vec{x} + \vec{N}(\vec{x}, t)$

where $A(t)$ is a matrix func. A. N. conti.

Under some special condition:

It connects to its linear part: $\frac{d\vec{x}}{dt} = A(t)\vec{x}$. (E)

Thm: If $A(t) = A \in M^{n\times n}(\mathbb{R})$. φ is solution of (E).

Then the stability of φ is determined by $\{\lambda_k\}_1^n$. eigenvalues of A.

(Consider the structure of solution!)

Thm. $\vec{X}(t) \equiv 0$ is a solution of (E). If

it's asymptotically stable on D

Then $D = \mathbb{R}^n$.

Pf: For $\frac{d\vec{X}}{dt} = A(t)\vec{X}$. general solution:

$\vec{\varphi}(t) = \phi(t)\vec{C}$. Since $\vec{X_0} = \phi(t_0)\vec{C}$

$\therefore \vec{C} = \phi^{-1}(t_0)\vec{X_0}$ $\therefore \vec{\varphi}(t) = \phi(t)\phi^{-1}(t_0)\vec{X_0}$

Since $\exists \delta > 0$. s.t. $\| \phi(t)\phi^{-1}(t_0)\vec{X_0} \| \to 0$. $(t \to \infty)$. If $\|X_0\| < \delta$.

i.e. $\sup\limits_{\|X_0\| < \delta} | \phi(t)\phi^{-1}(t_0)\vec{X_0} | \to 0 \ (t \to \infty)$

$\therefore \| \phi(t)\phi^{-1}(t_0)X_0 \| \le \|X_0\| \cdot \frac{2}{\delta} \| \phi(t)\phi^{-1}(t_0) \frac{\delta}{2} \frac{X_0}{\|X_0\|} \|$

$\le \frac{2\|X_0\|}{\delta} \sup\limits_{\|X'\| \le \delta} \| \phi(t)\phi^{-1}(t_0)X' \| \to 0$. $(t \to \infty)$

For $\forall X_0 \in \mathbb{R}^n$. $\therefore D = \mathbb{R}^n$.

Remark: Stability of $\frac{d\vec{X}}{dt} = A(t)\vec{X}$.

For solution $\vec{X}(t) = \phi(t)\phi^{-1}(t_0)\vec{X_0}$

Then $\begin{cases} \| \phi(t)\phi^{-1}(t_0) \| \le M < \infty. \Rightarrow \text{Stable} \\ \| \phi(t) \| \to 0. \ (t \to \infty) \Rightarrow \text{Asy-stable}. \end{cases}$

② Second Method

of Liapounov:

. Consider $\frac{d\vec{X}}{dt} = \vec{f}(\vec{X})$. satisfies the existence

and Uniqueness Thm.

If exist $V(\vec{x}): \mathbb{R}^n \to \mathbb{R}$. def on $|\vec{x}| \le M$.

satisfies: $V(\vec{0}) = 0$. $V(\vec{x}) > 0$. $\forall \vec{x} \ne 0$.

Then. $\begin{cases} \text{If } \dfrac{dV(\vec{x}(t))}{dt} = \sum \dfrac{\partial V}{\partial x_i} f_i > 0.\ \forall \vec{x} \ne 0 \Rightarrow \vec{x}(t) = 0 \text{ isn't stable} \\ \text{If } \dfrac{dV(\vec{x}(t))}{dt} = \sum \dfrac{\partial V}{\partial x_i} f_i \le 0.\ \forall \vec{x} \ne 0 \Rightarrow \vec{x}(t) = 0 \text{ is stable.} \\ \text{If } \dfrac{dV(\vec{x}(t))}{dt} = \sum \dfrac{\partial V}{\partial x_i} f_i < 0.\ \forall \vec{x} \ne 0 \Rightarrow \vec{x}(t) = 0 \text{ is asy-stable} \end{cases}$

Remark: Reverse the sign of inequality $V(\vec{x}) > 0$ and $\{$

it still holds!

e.g. $V(x, y) = \frac{1}{2}(x^2 + y^2)$. a common V-Function.

satisfies: $V \ge 0$. $V(0) = 0$.

Others: $V = (x+y)^2$. $(x+y)^2 + y^2$. More generally.

$\lambda x^2 + \mu y^2$, $\lambda x^4 + \mu y^2$. $(\lambda, \mu > 0)$. for offseting

(3) Dynamic System

in Dimension 2:

Consider $\begin{cases} \dfrac{dx}{dt} = X(x, y) \\ \dfrac{dy}{dt} = Y(x, y) \end{cases}$ conti. on $\mathbb{R}^2$. Satisfy uniqueness condition.

① Equilibrium points:

· For $\dfrac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

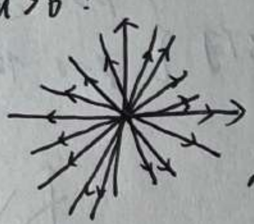
we only consider $|A| \neq 0$. Its unique equilibrium point is $(0,0)$.

Remark: If $|A| = 0$. Then $A\begin{pmatrix} x \\ y \end{pmatrix} = 0$ has infinite solutions (so that equilibrium points) on $a'x + b'y = 0$.

Under rotations and dilations:

1°) $A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. $\lambda\mu \neq 0$. $\Rightarrow$ solution $y = C|x|^{\mu/\lambda}$. $x = 0$.

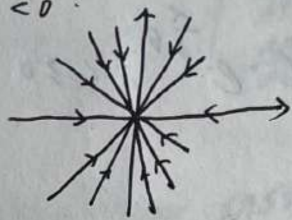
$\lambda = \mu > 0$:



$(0,0)$ is a source

不稳定星型结点

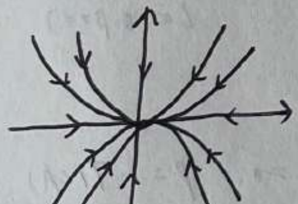
$\lambda = \mu < 0$:



$(0,0)$ is a sink.

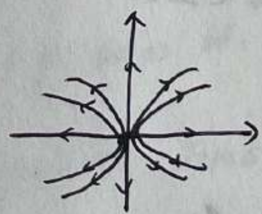
渐近稳定星型结点.

$\lambda \neq \mu$. $\lambda\mu > 0$:



$|\frac{\mu}{\lambda}| > 1$. $\mu < \lambda < 0$

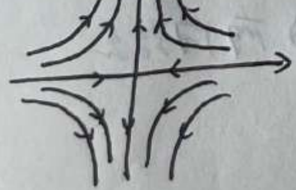
稳定的两向结点



$|\frac{\mu}{\lambda}| < 1$. $\lambda > \mu > 0$

不稳定的两向结点

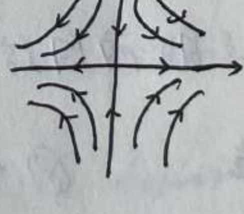
$\lambda \neq \mu$, $\lambda\mu < 0$:



$\mu > 0 > \lambda$

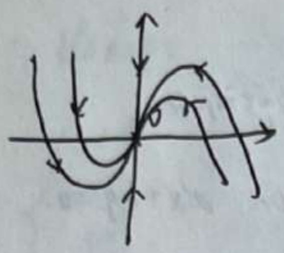
鞍点

(不稳定)



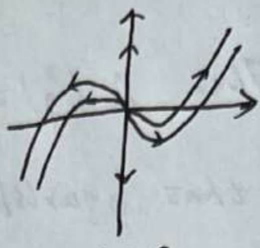
$\lambda > 0 > \mu$

2°) $A=\begin{pmatrix}\lambda & 0\\ 1 & \lambda\end{pmatrix}$. $y=cx+\frac{x}{\lambda}\ln|x|$. $x=0$.



$\lambda<0$

稳定的单向结点

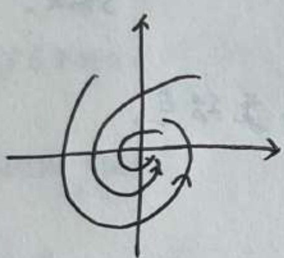


$\lambda>0$

不稳定的单向结点.

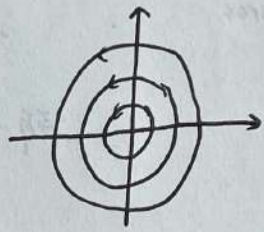
3°) $A=\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}$ Let $x=r\cos\theta$. $y=r\sin\theta$.

$\Rightarrow\ r=Ce^{\frac{\alpha}{\beta}\theta}$, $C\geq 0$. β决定盘旋方向 $\begin{cases}\beta>0 & \text{逆时针}\\ \beta<0 & \text{顺时针.}\end{cases}$

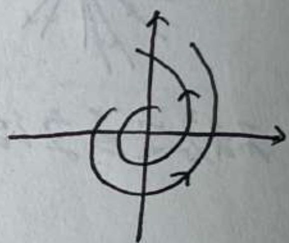


$\alpha\beta<0$. 稳定焦点

($\alpha<0, \beta>0$)



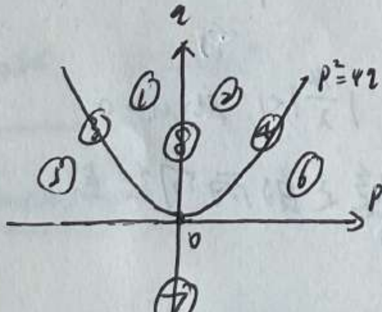
$\alpha=0$. $\beta>0$. 中心点



$\alpha\beta>0$. 不稳定焦点.

($\alpha>0, \beta>0$)

Thm.



$\lambda^2+p\lambda+q=0$. $p=-Tr(A)$. $q=|A|$.

①不稳定焦点 ②稳定焦点

③不稳定单向/星形结点

④稳定单向/星形结点

⑤不稳定的两向结点 ⑥稳定的两向结点.

⑦鞍点 ⑧中心.

② Procedure of drawing a Phase Plane:

Thm. For $\begin{cases} \frac{dx}{dt} = ax+by+\varphi(x,y) \\ \frac{dy}{dt} = cx+dy+\psi(x,y) \end{cases}$, $\varphi, \psi \in C'$.

Suppose $(0,0)$ is the unique equilibrium point.

If $\varphi(x,y), \psi(x,y) = o(r^{1+\varepsilon})$, $\exists \varepsilon>0$. when $r=\sqrt{x^2+y^2} \to 0$.

Then its structure is same as $\begin{cases} \frac{dx}{dt} = ax+by \\ \frac{dy}{dt} = cx+dy. \end{cases}$

$\Rightarrow$ 1°) Check whether the nonlinear system can be reduced to linear system

2°) If it can. Then calculate p, q. figure out what kind of e.p. $(0,0)$ is.

3°) If the locus will tend to some direction. then let $y=kx$. solve the limit tangent k!

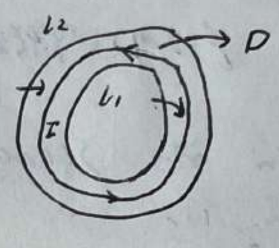
③ Limit Cycle:

· A isolated closed orbit in an annulus domain is called a limit cycle of the system.

If $t \to \infty$. the orbits outside the domain tend to it. then it's stable. If $t \to -\infty$. they tend to it. then it's unstable.

Thm. (Poincaré - Bendixson)

If Domain D are contoured by two simply closed curves which 're not orbits of the system. Besides. $\bar{D}$ doesn't contain equilibrium point. The orbits start from L_1, L_2 won't leave $\bar{D}$. Then in D, there exists at least one limit cycle I.



Remark: The interpretion:

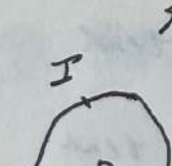
If there doesn't exist sink or source in D. there 're flows from outside D into D. Then there exists a circulation

Thm. If $P(x,y), Q(x,y) \in C^1(D)$. D is simply connected. $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \neq 0$, $\forall (x,y) \in D$. Then,

For the system: $\begin{cases} \frac{dx}{dt} = P(x,y) \\ \frac{dy}{dt} = Q(x,y) \end{cases}$ there no closed orbit exists.

Pf: If $x(t), y(t)$ is the periodic solution of system.

s.t. $x(0) = x(T)$, $y(0) = y(T)$, generate a closed orbit I.



Then $\oint_I P\,dy - Q\,dx = \int_0^T PQ - QP = 0$

$= \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \neq 0$. Contradict!

→ Direct method for limit cycles

① Let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ in the system

Then: $\begin{cases} \frac{dr}{dt} = f(r,\theta) \\ \frac{d\theta}{dt} = g(r,\theta) \end{cases}$

Find $r = r_0$ s.t. $f(r_0,\theta) = 0$. Then $r = r_0$ is a closed orbit.

② Liapunov:

$\frac{dv}{dt} = f(x,y)$. Find the energy balance curve function.