Solve by Power Series

(1) <u>Cauchy Thm:</u>

· Consider $\frac{dy}{dx} = f(x, y)$.

where f is analytic on $G \subseteq \mathbb{R}^2$.

$\Rightarrow$ From uniqueness and existence Thm. the solution $y(x)$ is locally unique.

Next, we want to prove: $y(x)$ is locally analytic as well.

i.e. on neighbour of x_0, $y(x) = \sum a_n (x-x_0)^n$.

· Suppose $f(x,y)$ analytic in $R: |x-x_0| < \alpha$, $|y-y_0| < \beta$.

i.e. $f(x,y) = \sum a_{ij} (x-x_0)^i (y-y_0)^j$. Let $a < \alpha$, $b < \beta$.

restrict on $R_0: |x-x_0| < a$, $|y-y_0| < b$.

since $\sum a_{ij} a^i b^j$ converges $\therefore \exists M > 0$. s.t.

$a_{ij} a^i b^j \leq M$, $\forall i, j \geq 1$.

$\Rightarrow$ Construct: $F(x,y) = \frac{M}{(1-\frac{x-x_0}{a})(1-\frac{y-y_0}{b})} = \sum A_{ij} (x-x_0)^i (y-y_0)^j$

where we have $A_{ij} > a_{ij}$, i.e. F "dominates" f

<u>Lemma</u>. Consider $\frac{dy}{dx} = F(x, y)$, on R_0, $y(x_0) = y_0$.

in $|x-x_0| < \ell = a(1-e^{-\frac{b}{2aM}})$, we have an analytic solution $y(x)$.

Pf: It's easy to check by separating the variables. Caution about the domain.

$\Rightarrow$ **Cauchy Thm:**

If $f(x,y)$ is analytic on R. then the I.V.P. above has a unique analytic solution $y(x)$ on $|x-x_0|<e$.

Pf: Suppose $f(x,y) = \sum_{i,j\geq 0} a_{ij}(x-x_0)^i(y-y_0)^j$.

$y(x) = y_0 + \sum_{n=1}^{\infty} C_n (x-x_0)^n$.

Next. we will prove $y(x)$ converges. which means it's exactly a solution.

Firstly. solve for C_n:

$C_1 = a_{00}$. $C_2 = \frac{a_{10}+a_{11}a_{00}}{2}$ ... $C_n = P_n(\{a_{ij}\}_{i+j\leq n-1})$

where P_n is a polynomial with positive coefficients

From it: we obtain $\{C_n\}$ is determined uniquely.

Secondly. note that: $\frac{dy}{dx} = F(x,y)$. $y(x_0)=y_0$.

has a analytic solution $\tilde{y} = y_0 + \sum_{n\geq 1} \tilde{C}_n (x-x_0)^n$

Similarly $\tilde{C}_n = P_n(\{A_{ij}\}_{i+j\leq n-1})$

we have $\tilde{C}_n \geq |C_n|$. from prop of P_n.

$\therefore$ $y(x)$ converges! It's unique solution!

Remark: i) For $f(x,\eta)$ is non analytic. $\eta(x)$ may have no form of $\sum C_n(x-x_0)^n$. e.g. $\frac{d\eta}{dx} = \frac{\eta - x}{x}$. $\eta_0 = 0$.

or it may not converges. e.g. $x^2 \frac{d\eta}{dx} = \eta - x$, $\eta_0 = 0$.

ii) For vector equations:

$$\frac{d\eta_k}{dx} = f_k(x, \eta_1 \cdots \eta_n).\quad \eta_k(0) = 0.\quad 1 \le k \le n.$$

If f_k is analytic in $|x - x_0| \le a$. $|\eta_k - \eta_k(0)| \le b$.

Construct $$F(x, \vec{\eta}) = \frac{M}{(1 - \frac{x - x_0}{a}) \prod_1^n (1 - \frac{\eta_k - \eta_k(0)}{b})}$$

We confirm it has unique analytic solution $\vec{\eta}(x)$. by similar argument.

iii) In the case $f(x, \eta, \lambda) = \frac{d\eta}{dx}$. If $f(x, \eta, \lambda)$ is analytic on x, η, λ. So the solution $\eta(x, \lambda)$ will!

(2) In linear homogenous equation with 2 order:

· Consider $A(x)\eta'' + B(x)\eta' + C(x)\eta = 0$

To reduce it:

If in $|x - x_0| < \delta$. $A(x_0) \ne 0$. Let $p(x) = B(x)/A(x)$. $q(x) = C(x)/A(x)$.

we have: $\eta'' + p(x)\eta' + q(x)\eta = 0$. (E)

let $\eta_1 = \eta$. $\eta_2 = \eta'$. we can convert it to the vector case. $\Rightarrow$ the solution of (E) is locally unique and analytic.

Procedure: Suppose $y(x) = \sum_{n\geq 1} a_n (x-x_0)^n + y_0$

put it into the equation for solving $\{a_n\}$.

Then we will generate a recursion formula.

(3) Legendre Functions:

· Def: $P_n(x) = \frac{1}{2^n} \sum_{0}^{[\frac{n}{2}]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k}$.

It's from Legendre equation:

$(1-x^2)y'' - 2xy' + (n+1)ny = 0.$

properties: i) $P_{2n}(x) = P_{2n}(-x)$, $P_{2n+1}(x) = -P_{2n+1}(-x)$

ii) $P_n(x) = \frac{1}{2^n} \frac{1}{n!} \frac{d^n}{dx^n}[(x^2-1)^n]$

$\Rightarrow P_n(1) = 1$. $P_n(-1) = (-1)^n$.

iii) $\langle P_n, P_m \rangle_{L^2[-1,1]} = \begin{cases} 0, & n \neq m \\ \sigma_n = \frac{2}{2n+1}, & n = m \end{cases}$

The generating Func:

· $G(x,t) = (1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n$

Pf: Check $\frac{\partial^n}{\partial t^n} G(x,t)\big|_{t=0} = P_n(x) \cdot n!$

$\frac{\partial^n}{\partial t^n}(1-2xt+t^2)^{-\frac{1}{2}} = \frac{\partial^n}{\partial t^n} \sum \binom{-\frac{1}{2}}{k} (t^2-2xt)^k$

$= \sum \binom{-\frac{1}{2}}{k} \frac{\partial^n}{\partial t^n}[t^k (t-2x)^k]$. Let $t=0$. only $k \leq [\frac{n}{2}]$ terms!

(4) General Method:

· $A(x)y'' + B(x)y' + C(x)y = 0$. where we will suppose:

$A(x) = (x-x_0)^k A_0$. $k \geq 1$. $B(x_0)$. $C(x_0) \neq 0$.

Actually. the series of solutions aren't the usual form $\sum_{n \in \mathbb{Z}^+} C_n (x-x_0)^n + y_0$, but the general power series: $\sum_{n \geq 0} C_n (x-x_0)^{n+e}$ ($C_0 \neq 0$)

Next. we will solve a special case:

$(x-x_0)^2 P(x) y'' + (x-x_0) Q(x) y' + R(x) y = 0$. (E).

where P, Q, R are polynomials. $P(x_0) \neq 0$. $x-x_0 \parallel RQ$.

we call x_0 regular singularity.

<u>Thm.</u> For (E). If x_0 is a regular singularity.

Then it has convergent solution:

$y(x) = \sum_{n \geq 0} C_n (x-x_0)^{n+e}$. $C_0 \neq 0$.

<u>Pf.</u> Rewrite (E): (divide $P(x)$, expand in series)

$(x-x_0)^2 y'' + (x-x_0) \sum_{n \geq 0} a_n (x-x_0)^n y' + \sum b_n (x-x_0)^n y = 0$ (E')

put $y(x) = \sum_{n \geq 0} C_n (x-x_0)^{n+e}$ in (E').

We obtain recursion formulas.

From $C_0 \neq 0$. we have: $e(e-1) + a_0 e + b_0 = 0$

Solve for e_1. e_2. Next. prove it's convergent!

<u>Remark:</u> We usually choose $|e_1| \geq |e_2|$ to generate a solution. For the other l.i. solution. by Wronsky!