Linear Differential Equations

(1) General Case:

· Consider $\frac{d\eta_i}{dx} = \sum_{j=1}^{n} a_{ij}(x)\eta_j + f_i(x)$. $1 \le i \le n$. a_{ij}, f_i conti.

i.e. $A(x) = (a_{ij}(x))_{n\times n}$, $\vec{\eta} = (\eta_1 \cdots \eta_n)^T$. $\vec{f}(x) = (f_1(x) \cdots f_n(x))^T$.

We have: $\frac{d\vec{\eta}}{dx} = A(x)\vec{\eta} + \vec{f}(x)$.

(Since $A(x)\vec{\eta}$ satisfies Lipschitz condition. So if we suppose initial value $\vec{\eta}(x_0) = \vec{\eta}_0$, Then the solution is unique.)

① Homogeneous Case:

· Consider $\frac{d\vec{\eta}}{dx} = A(x)\vec{\eta}$. first. $(\vec{\eta} = (\eta_1 \cdots \eta_n))$

Properties: i) Linear combination of solutions is solution

ii) Suppose S is the set of solution. Then

$$S \cong \mathbb{R}^n.$$

Pf: Fix $x_0 \in (a,b)$. $\vec{\eta}_0 = \vec{\eta}(x_0)$

$H: \eta_0 \longmapsto \eta(x)$. $\mathbb{R}^n \to S$.

By Uniqueness and Existence.

H is bijective. linear. map.

Remark: $H^{-1}: S \to \mathbb{R}^n$. H^{-1} maps $\{(x, \vec{\eta}) \mid \vec{\eta} \in S\}$.

which is $n+1$ dimensional space to

$\Sigma_{x_0}: \{(x_0, \vec{\eta}(x_0)) \mid \vec{\eta} \in S\}$ n dimensional space. (i.e. Intersection of $x = x_0$ with (x, S))

Cor. If $\{\vec{\eta}_k(x_0)\}_1^n$ linearly indept. Then $\{\vec{\eta}_k(x)\}_1^n$ l.i.

Pf: $\sum c_k \vec{\eta}_k(x_0) = 0 \Rightarrow \forall c_k = 0$.

$\therefore H^{\wedge}(\sum c_k \vec{\eta}_k(x)) = 0 \quad \sum c_k \vec{\eta}_k(x) = 0 \Rightarrow \forall c_k = 0$.

Cor. On $a<x<b$. There're n l.i. solutions

Pf: $\{e_i\}_1^n$ basis of $\mathbb{R}^n$. Then $\varphi_k(x) = H(e_k)$. l.d.

Remark: The general solutions can be expressed:

$\vec{\eta} = \sum_1^n c_k \varphi_k(x)$. Note that $\frac{D(\eta_1, \eta_2 \cdots \eta_n)}{D(c_1, c_2 \cdots c_n)} \neq 0$.

$\therefore \{c_k\}_1^n$ indept. const.

A Tool: Wronsky Determinant:

It's used to check whether n solutions $\{\vec{\eta}_k(x)\}_1^n$ are l.i. or not. $W(x) = \det((\vec{\eta}_1, \vec{\eta}_2 \cdots \vec{\eta}_n))$

Lemma: (Liouville Formula)

$$W(x) = W(x_0)\, e^{\int_{x_0}^{x} tr[A(x)]\,dx}, \quad a<x<b$$

Pf: $\frac{dW}{dx} = \sum_i^n \begin{vmatrix} \eta_{11} & \cdots & \eta_{1n} \\ \vdots & & \\ \frac{d\eta_{i1}}{dx} & \cdots & \frac{d\eta_{in}}{dx} \\ \eta_{n1} & \cdots & \eta_{nn} \end{vmatrix}$. where $\vec{\eta}_k = \begin{pmatrix} \eta_{1k} \\ \vdots \\ \eta_{nk} \end{pmatrix}$

$$= \sum_{i=1}^n \begin{vmatrix} \eta_{11} & \cdots & \eta_{1n} \\ \vdots & & \vdots \\ \sum_{j=1}^n a_{ij}\eta_{j1} & \cdots & \sum_{j=1}^n a_{ij}\eta_{jn} \\ \vdots & & \\ \eta_{n1} & \cdots & \eta_{nn} \end{vmatrix} = \sum a_{ii} W(x)$$

$\therefore \frac{dW}{dx} = tr(A(x))\,W(x) \Rightarrow W(x) = W(x_0)\, e^{\int_{x_0}^{x} tr(A(x))\,dx}$

Thm. $\{\vec{\eta}_k(x)\}_1^n$ is l.i. $\Leftrightarrow$ $W(x) \neq 0$. $(a<x<b)$

Pf: It's easy to see: $\{\vec{\eta}_k(x_0)\}_1^n$ l.i.

Assume $\{\eta_k(x)\}_1^n$ are solutions of the equation

Cor. $\{\vec{\eta}_k(x)\}_1^n$ is l.d. $\Leftrightarrow$ $W(x) \equiv 0$. $(a<x<b)$

Pf: i.e. $\exists\, x_0 \in (a,b)$. $W(x_0)=0$.

Remark: For general function vectors. $W(x) \equiv 0 \not\Rightarrow$ l.d.

Let $\eta_k(x) = \begin{pmatrix} x^k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. $\{\eta_k(x)\}_1^n$ l.i. But $W(x) \equiv 0$

Define: $\phi(x)$ is the fundamental solution Matrix if

$\phi(x) = (\vec{\eta}_1(x) \cdots \vec{\eta}_n(x))$. $\{\vec{\eta}_k(x)\}_1^n$ is l.i. solution.

$\Rightarrow$ Then express solution of $\frac{d\vec{\eta}}{dx} = A(x)\vec{\eta}$: $\vec{\eta} = \phi(x)\vec{c}$.

$\to$ It absolutely determines a Homo-equation

If $\phi(x)$ is FSM of $\frac{d\vec{\eta}}{dx} = A(x)\vec{\eta}$, $\frac{d\vec{\eta}}{dx} = B(x)\vec{\eta}$ $\Rightarrow A(x) \equiv B(x)$

Thm. $\phi(x)$ is FSM $\Leftrightarrow$ $\phi'(x) = A(x)\phi(x)$. $\mathrm{Det}(\phi(x_0)) \neq 0$. $\exists x_0$.

Pf: Easy to check. $|\phi(x_0)| \neq 0$ guarantee l.i.

Cor. i) $C = (c_{ij})_{n\times n}$. $|C| \neq 0$. Then $\phi(x)C$ is FSM.

ii) Conversely. If $\phi(x)$, $\varphi(x)$ are FSM. Then

$\exists\, C \in \mathbb{R}_{n\times n}$ s.t. $\phi(x) = \varphi(x)C$

Pf: Only for ii): Suppose $\phi(x) = \varphi(x)C(x)$.

$\therefore C(x) = \varphi^{-1}(x)\phi(x)$. $\frac{d\phi(x)}{dx} = \frac{d\varphi(x)}{dx}C(x) + \varphi(x)\frac{dC(x)}{dx}$

$\Rightarrow \varphi(x)\frac{dC(x)}{dx} = 0_{n\times n}$ $\therefore \frac{dC(x)}{dx} = 0_{n\times n}$. $C(x) \equiv C \in \mathbb{R}_{n\times n}$.

Remark: 3) Since the multiplication of matrixes isn't commutative. $\varphi(x) = C\phi(x)$ may not be FSM.

$\because \varphi'(x) \overset{?}{=} A(x)C\phi(x) \iff C\phi'(x) \overset{?}{=} A(x)C\phi(x)$.

② Nonhomogenous Case.

$\cdot\ \frac{d\vec{\eta}}{dx} = A(x)\vec{\eta} + \vec{f}(x) \quad \cdots \ (E)$

Lemma. If $\phi(x)$ is FSM. $\varphi^*(x)$ is a particular solution. Then any solution of (E) has form:

$\varphi(x) = \phi(x)\vec{C} + \varphi^*(x)$

$\Rightarrow$ Actually. we can express $\varphi^*(x)$ by using $\phi(x)$:

Using Method of Variation on const.

$\varphi^*(x) = \phi(x)c(x)$, bring into (E):

$\phi(x)c'(x) = f(x) \quad \therefore c(x) = \int_{x_0}^{x} \phi^{-1}(s) f(s) ds$.

we obtain: $\varphi^*(x) = \phi(x)\int_{x_0}^{x} \phi^{-1}(s) f(s) ds$.

Thm. The general solutions of (E): $\phi(x)(C + \int_{x_0}^{x} \phi^{-1}(x) f(x) dx)$

Thm. $f(x) \not\equiv 0$. There exists k l.i. solutions of (E). where $1 \le k \le n+1$.

Pf: Check $\{\varphi^*(x)\} \cup \{\phi_i(x)\}_1^n$ l.i.

If $\exists\, \varphi(x) \ne \varphi^*(x), \phi_i(x)$. $\{\varphi(x), \varphi^*(x)\} \cup \{\phi_i(x)\}_1^n$ l.i.

Since $\{\varphi(x) - \varphi^*(x);\ \varphi^*(x) - \phi_n(x)\} \cup \{\phi_{i+1} - \phi_i\}_1^{n}$, $n+1$ l.d. solutions of $\frac{d\vec{\eta}}{dx} = A(x)\vec{\eta}$. Check there's contradiction!

③ <u>Inverse Problem</u>:

- For given n solutions $\vec{\varphi}_j(x)$. i.e. Can we find equations having FSM $(\vec{\varphi}_1(x), \cdots \vec{\varphi}_n(x)) = \phi(x)$

<u>Claim</u>: The form is $\begin{vmatrix} \eta_i' & \varphi_{i1}' & \cdots & \varphi_{in}' \\ \eta_1 & \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \vdots & & \vdots \\ \eta_n & \varphi_{n1} & \cdots & \varphi_{nn} \end{vmatrix} = 0$, $1 \le i \le n$

<u>Pf</u>: Suppose $\vec{\eta}\,' = A(x)\vec{\eta}$ is what we need.

i.e. $\eta_i' = \sum_{j=1}^{n} a_{ij}(x)\eta_j$. Since $(\vec{\varphi}_1 \cdots \vec{\varphi}_n)$ is FSM.

Then $\phi' = A(x)\phi(x)$ $\therefore A(x) = \phi'(x)\phi^{-1}(x)$

$\Rightarrow \vec{\eta}\,' = \phi'(x)\phi^{-1}(x)\vec{\eta}$.

$$\begin{vmatrix} \eta_i' & \varphi_{i1}' & \cdots & \varphi_{in}' \\ \eta_1 & \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & & & \vdots \\ \eta_n & \varphi_{n1} & & \varphi_{nn} \end{vmatrix} = \eta_i' \det(\phi(x)) - [\varphi_{i1}' \sum x_j A_{j1}(\phi) \cdots]$$

$$= \eta_i' \det(\phi(x)) - (\varphi_{i1}' \cdots \varphi_{in}') \begin{pmatrix} A_{11}(\phi) & \cdots & A_{n1}(\phi) \\ \vdots & & \\ A_{1n}(\phi) & \cdots & A_{nn}(\phi) \end{pmatrix} (x_1 \cdots x_n)^T$$

$$\text{Since } \begin{pmatrix} A_{11}(\phi) & \cdots & A_{n1}(\phi) \\ \vdots & & \\ A_{n1}(\phi) & \cdots & A_{nn}(\phi) \end{pmatrix} = \phi^* = |\phi(x)|\,\phi^{-1}(x)$$

$$\Rightarrow \begin{vmatrix} \eta_i' & \varphi_{i1}' & \cdots & \varphi_{in}' \\ \eta_1 & & \cdots & \\ \vdots & & & \\ \eta_n & & & \end{vmatrix} = |\phi(x)|\,(\eta_i' - \phi'\phi^{-1}\vec{\eta}) = 0 \quad 1 \le i \le n.$$

$\Rightarrow \vec{\eta}\,' = \phi'\phi^{-1}\vec{\eta}$.

(2) When $A(x) \equiv A$. $a_{ij} = \text{const.}$:

$\cdot \frac{d\vec{\eta}}{dx} = A\vec{\eta} + \vec{f}(x)$. $\vec{f}(x)$ conti. $a<x<b$.

① Homogenous Case:

$\frac{d\vec{\eta}}{dx} = A\vec{\eta}$. $\Rightarrow$ FSM: $e^{xA} = \phi(x)$ (easy to check)

② Nonhomogeneous Case:

Apply Thm from (1): $\eta = e^{xA}c + \int_{x_0}^{x} e^{(x-s)A}\vec{f}(s)ds$

$\Rightarrow$ From definition: $e^{xA} = \sum \frac{x^k A^k}{k!}$. How can we express the explicit form of e^{xA} ?

By Jordan Form:

$$e^{xA} = e^{xPJP^{-1}} = Pe^{xJ}P^{-1} = P\begin{pmatrix} e^{xJ_1} & & 0 \\ & \ddots & \\ 0 & & e^{xJ_m} \end{pmatrix}P^{-1}$$

$$\Rightarrow e^{xA}P = P\begin{pmatrix} e^{xJ_1} & & \\ & \ddots & \\ & & e^{xJ_m} \end{pmatrix} \text{ is FSM.}$$

i) A has distinct eigenvalues:

Then $e^{xA}P = P\begin{pmatrix} e^{\lambda_1 x} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n x} \end{pmatrix} = \phi(x) = (e^{\lambda_1 x}\vec{P_1}, \cdots e^{\lambda_n x}\vec{P_n})$

$P = (\vec{P_1} \cdots \vec{P_n})$. Let $x=0 \Rightarrow \phi(0) = P$. $\therefore e^{xA} = \phi(x)\phi^{-1}(0)$.

$\Rightarrow$ To find $\phi(x)$. It suffices to find $\{\vec{P_i}\}_1^n$

Thm. $\vec{\eta} = e^{\lambda x}\vec{r}$ is solution of $\vec{\eta}' = A\vec{\eta}$.

$\Leftrightarrow (A - \lambda I)\vec{r} = 0$.

Thm. If A has n distinct eigenvalues $\{\lambda_i\}_1^n$

Then $\phi(x) = (e^{\lambda_1 x}\vec{r_1} \cdots e^{\lambda_n x}\vec{r_n})$ is FSM of $\frac{d\vec{y}}{dx} = A\vec{y}$.

where λ_i is eigenvalue of A. $\vec{r_i}$ is correspond vector.

Pf: $|\phi(x)| \neq 0$. $\phi'(x) = A\phi(x)$. check!

Cor. If A can be diagonalized. Then find n l.i. eigenvalue vectors $\{r_i\}$. correspond $\{\lambda_i\}$. FSM:

$\phi(x) = (e^{\lambda_1 x}\vec{r_1} \cdots e^{\lambda_n x}\vec{r_n})$

(ii) If A has multiple eigenvalues:

· Suppose A has eigenvalues $\{\lambda_i\}_1^s$. multiple number $\{n_i\}_1^s$.

$\sum_1^s n_i = n$. The solution in FSM has form:

$$e^{\lambda_i x}(\vec{r_0} + x\vec{r_1} + \cdots \frac{x^{n_i-1}}{(n_i-1)!}\vec{r}_{n_i-1})$$

$$(\text{Since } e^{xJ_i} = e^{x\begin{pmatrix}\lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & & \lambda_i\end{pmatrix}} = e^{x\begin{pmatrix}\lambda_i & & \\ & \ddots & \\ & & \lambda_i\end{pmatrix} + x\begin{pmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & & 0\end{pmatrix}}$$

$$= e^{\lambda_i x} e^{x\begin{pmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & & 0\end{pmatrix}} = e^{\lambda_i x}(I + x\begin{pmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & & 0\end{pmatrix} + \frac{x^2}{2!}\begin{pmatrix}0 & 0 & 1 & \\ & \ddots & \ddots & \ddots \\ & & & 0\end{pmatrix}$$

$$+ \frac{x^{n_i-1}}{(n_i-1)!}\begin{pmatrix}0 & \cdots & 0 & 1 \\ & \ddots & & 0 \\ 0 & & \ddots & \\ & & & 0\end{pmatrix}) = e^{\lambda_i x}\begin{pmatrix}1 & x & \frac{x^2}{2!} & \cdots \\ & \ddots & \ddots & \\ & & \ddots & x \\ 0 & & & 1\end{pmatrix})$$

Thm. λ_i is n-multiple-eigenvalues of A. Denote $\vec{\varphi}(x) =$

$e^{\lambda_i x}(\vec{r_0} + x\vec{r_1} + \cdots \frac{x^{n_i-1}}{(n_i-1)!}\vec{r}_{n_i-1})$ is solution in FSM

$\Leftrightarrow (A-\lambda_i E)^{n_i}\vec{r_0} = 0$. $\vec{r_k} = (A-\lambda_i E)\vec{r}_{k-1}$. $k \geq 1$.

Pf: $\vec{\varphi}(x)' = A\vec{\varphi}(x) \Leftrightarrow e^{\lambda_i x}(A-\lambda_i E)(\vec{r}_0 + \frac{x}{1!}\vec{r}_1 + \cdots \frac{x^{n_i-1}}{(n_i-1)!}\vec{r}_{n_i-1})$

$$= e^{\lambda_i x}(\vec{r}_1 + \frac{x}{1!}\vec{r}_2 \cdots + \frac{x^{n_i-2}}{(n_i-2)!}\vec{r}_{n_i-2})$$

$$\Leftrightarrow (A-\lambda_i E)\vec{r}_1 = 0, \quad \vec{r}_k = (A-\lambda_i E)\vec{r}_{k-1}.$$

Remark: Since $V_i = \{r \in \mathbb{R}^n \mid (A-\lambda_i E)^{n_i}\vec{r} = 0\}$

$\dim V_i = n_i$. $\sum_1^s n_i = n$. We can obtain $\phi(x)$

Thm. The FSM of $\frac{d\vec{\eta}}{dx} = A\vec{\eta}$ can be expressed:

$$(e^{\lambda_1 x}P_1^{(1)}(x), \cdots e^{\lambda_1 x}P_{n_1}^{(1)}(x), e^{\lambda_2 x}P_1^{(2)}(x) \cdots e^{\lambda_s x}P_{n_s}^{(s)}(x))$$

$$P_j^{(i)} = r_{j0}^{(i)} + \frac{x}{1!}r_{j1}^{(i)} + \cdots \frac{x^{n_i-1}}{(n_i-1)!}r_{j\,n_i-1}^{(i)}.$$

(3) High order linear Differential Equations:

- $y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = f(x)$. $a_i(x)$ conti on (a,b). $f \not\equiv 0$.

Let $\begin{cases} y_1 = y \\ y_2 = y' \\ \vdots \\ y_n = y^{(n-1)} \end{cases} \Rightarrow \frac{d\vec{y}}{dx} = A(x)\vec{y} + \vec{f}(x)$

where $A(x) = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ & 0 & & 1 \\ -a_n(x) & \cdots & & -a_1(x) \end{pmatrix}$. $\vec{f}(x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f(x) \end{pmatrix}$

Then convert to case (2)!

① General case: $A(x) \not\equiv A$

If $\varphi_k(x)$ is solution of $\frac{d\vec{\eta}}{dx} = A(x)\vec{\eta}$. $1 \le k \le n$.

$$W(x) = \begin{pmatrix} \varphi_1(x) & \cdots & \varphi_n(x) \\ \varphi_1'(x) & \cdots & \varphi_n'(x) \\ & \vdots & \\ \varphi_1^{(n-1)}(x) & \cdots & \varphi_n^{(n-1)}(x) \end{pmatrix}$$. Then: easy to prove

$|W(x)| \neq 0 \iff [\varphi_k(x)]_1^n$ l.i. on (a,b)

Note that $tr(A(x)) = -a_1(x)$. $\therefore W(x) = W(x_0) e^{-\int_{x_0}^{x} a_1(s) ds}$.

<u>A special example</u>: $(n=2)$

$\eta''(x) + p(x)\eta' + q(x) = 0. \Rightarrow$ <u>properties</u> $\begin{cases} \text{i) } \varphi(x) \text{ is a solution, } \varphi \not\equiv 0. \text{ Then the general solution: } \\ \eta = \varphi(x)[C_1 + C_2 \int_{x_0}^{x} \frac{1}{\varphi^2(s)} e^{-\int_{x_0}^{s} p(t)dt} ds] \\ \text{ii) For } u(x), v(x) \text{ are fundamental solutions. Then it determines } p(x), q(x), \text{ has no common zeros.} \end{cases}$

<u>pf</u>: i) 1°) Special case: $\varphi \neq 0$.

Then by Liouville Thm

$$\begin{vmatrix} \varphi & \eta \\ \varphi' & \eta' \end{vmatrix} = C e^{-\int p(x)dx}$$

$$\therefore \varphi\eta' - \eta\varphi' = C e^{-\int p(x)dx} \quad \therefore d\left(\frac{\eta}{\varphi}\right) = C \frac{e^{-\int p(x)dx}}{\varphi^2}$$

We obtain $\eta = \varphi[C_1 + C_2 \int_{x_0}^{x} \frac{1}{\varphi^2(s)} e^{-\int_{x_0}^{s} p(t)dt} ds]$

2°) General case:

<u>Lemma</u>. If $\varphi \not\equiv 0$. Then: if $x_0 \in I$. $\varphi(x_0) = 0$.

then $\varphi'(x_0) \neq 0$. (So, there're at most finite zeros on $[a,b]$.)

<u>pf</u>: If $\varphi'(x_0) = \varphi(x_0) = 0 \Rightarrow \varphi(x) \equiv 0$

is the unique solution. Contradict!

Then $\exists (x_0-\varepsilon, x_0+\varepsilon)$. $\varphi'(x_0) > 0$ (or < 0)

$\therefore$ In $(x_0-\varepsilon, x_0+\varepsilon)$. $\varphi(x)$ has only one zero

By cpt of $[a,b]$. There're finite zeros!

$\Rightarrow$

suppose $\{z_k\}_1^n \subseteq (a,b)$ are zeros of $\varphi(x)$.

In each (a, z_1). or (z_k, z_{k+1}) or (z_n, b).

$y = \varphi [c_1 + c_2 \int_{x_1}^{x} \frac{1}{\varphi^2(s)} e^{-\int_{x_1}^{s} p(t) dt} ds]$ holds

Define: $y(z_k) = a_k, 1 \le k \le n$. Since By Lospital Thm. $\lim_{x \to z_k} \frac{\int_{x_0}^{x} \frac{1}{\varphi^2(s)}}{\frac{1}{\varphi(x)}}$ exist! $(= a_k)$ $\square$

$\therefore y$ is conti.

(2) $\begin{pmatrix} u & v \\ u' & v' \end{pmatrix} \triangleq \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{21} \end{pmatrix}$ determines $A(x)$!

If $u(x_0) = v(x_0) = 0 \Rightarrow W(x_0) = 0$. contradict!

Thm. $\{\varphi_k\}_1^n$ is FS of homogenous equations $y^{(n)} + \cdots = 0$.

Then general solution of $y^{(n)} + a_1 y^{(n-1)} + \cdots a_n(x) = f(x)$ is

$y = \sum c_i \varphi_i + \varphi^*$. $\varphi^* = \sum_1^n \varphi_k(x) \int \frac{W_k(s)}{W(s)} f(s) ds$.

where $W_k(s) = A_{nk}(W(s))$

$$\vec{y}^* = \begin{pmatrix} y^* \\ y'^* \\ \vdots \\ y^{(n-1)*} \end{pmatrix} = \int_{x_0}^{x} \phi(x) \phi^{-1}(s) \vec{f}(s) ds = \int_{x_0}^{x} \frac{\phi(x)}{W(s)} \begin{pmatrix} * & \begin{matrix} W_1(s) \\ \vdots \\ W_n(s) \end{matrix} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ f(s) \end{pmatrix} ds$$

$$= \int_{x_0}^{x} \frac{f(s)}{W(s)} \begin{pmatrix} \varphi_1 \cdots \varphi_n \\ * \end{pmatrix} \begin{pmatrix} * & \begin{matrix} W_1(s) \\ \vdots \\ W_n(s) \end{matrix} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} ds$$

Remark: By method of variation on const:

suppose $y = \sum_1^n c_i(x) \varphi_i(x)$ is particular solution

$$\therefore \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \varphi_1 \cdots \varphi_n \\ \\ \varphi_1^{(n-1)} \cdots \varphi_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1(x) \\ \vdots \\ c_n(x) \end{pmatrix} = \phi(x) \vec{C}(x)$$

Since $\frac{d\vec{\eta}}{dx} = \frac{d\phi(x)}{dx}\vec{C}(x) + \phi(x)\vec{C}'(x)$

$= A(x)\phi(x)\vec{C}(x) + \phi(x)\vec{C}'(x)$

$= A(x)\vec{\eta} + \vec{f}(x) \quad \therefore \phi(x)\vec{C}'(x) = \vec{f}(x) = \begin{pmatrix} 0 \\ \vdots \\ f(x) \end{pmatrix}$

$\vec{C}'(x) = \phi^{-1}(x)\begin{pmatrix} 0 \\ \vdots \\ f(x) \end{pmatrix}$. Since $AA^* = |A|I \therefore A^{-1} = \frac{A^*}{|A|}$.

② $\underline{A(x) \in M^{n\times n}(\mathbb{R})}$:

· $\eta^{(n)}(x) + \cdots a_n\eta = f(x)$

Similarly. substitute $\eta^{(k)}$. $A = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ -a_n & \cdots & & -a_1 \end{pmatrix}$ Froberius Form!

$\therefore |\lambda I - A| = \lambda^n + a_1\lambda^{n-1} + \cdots a_n = 0 = f(\lambda)$. the eigenpoly-!

Thm $\sigma(A) = EV(A) = \{\lambda_i\}_1^s$. Then the fundamental

solutions : $\begin{cases} e^{\lambda_1 x}, xe^{\lambda_1 x} \cdots, x^{n_1-1}e^{\lambda_1 x} \\ \vdots \\ e^{\lambda_s x}, \cdots\cdots x^{n_s-1}e^{\lambda_s x} \end{cases}$ $\sum_1^s n_i = n$.

Pf: For $\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ -a_n & \cdots & & -a_1 \end{pmatrix} - \lambda I = M$. $|M(2:n, 2:n)| = 1$

$\therefore D_{n-1}(M) = 1 \quad \therefore$ There exists only one block for each λ_i

$AP = P\begin{pmatrix} J_{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & J_{\lambda_s} \end{pmatrix}$. where P is transpose of Jordan

For $m_i = \sum_1^{i-1} n_i + 1$. $P_{1m_i} \neq 0$. $0 < i < s$ (check)

Only consider the first row :

$e^{xA}P = Pe^{J} \Rightarrow P_{1m_1}e^{\lambda_1 x} \cdots, *e^{\lambda_1 x} + \cdots \frac{P_{1m_1}x^{n_1-1}}{(n_1-1)!}e^{\lambda_1 x} \cdots$

By cancelling ... Done!

③ Finding the particular solution by undetermine variation:

· It's complicated to use the Wronsky determinant to find particular solution $\varphi^*(x)$.

i) For $f(x) = P_m(x) e^{\mu x}$. where $P_m(x)$ is m degree polynomial.

Then conject:

$$\varphi^*(x) = \begin{cases} a_m(x) e^{\mu x}. & a_m \text{ is m-degree-poly. when } \mu \text{ isn't eigenroot.} \\ x^k a_m(x) e^{\mu x}. & a_m \text{ is m-degree-poly. } \mu \text{ is k-repeat-eigenroot.} \end{cases}$$

ii) For $f(x) = [A_m(x) \cos\beta x + B_l(x) \sin\beta x] e^{\alpha x}$. $\deg A_m = m$. $\deg B_l = l$.

Then conject:

$$\varphi^*(x) = \begin{cases} [C_n(x) \cos\beta x + D_n(x) \sin\beta x] e^{\alpha x}. & C_n . D_n \text{ is poly of } \max\{m, l\} \stackrel{\Delta}{=} n \text{ degree. } \alpha \pm i\beta \notin EV(A) \\ x^k [C_n(x) \cos\beta x + D_n(x) \sin\beta x] e^{\alpha x}. & C_n . D_n \text{ is n-degree poly. } n = \max\{m, l\}. \ \alpha \pm i\beta \text{ is k-repeat-eigen roots.} \end{cases}$$

Remark: For $\mu \in EV(A)$. That's because the fundamental solutions: $\{e^{\mu x} x^i\}_0^{n_s - 1}$

So we need multiply x^k. $(k = n_s)$ to test!

since $\varphi^*(x) = \sum_1^n \varphi_i(x) + T^*(x)$. φ_i is FS. T^* is another particular solution. $1 \le i \le n$

④ Euler's equations:

$$x^n y^{(n)}(x) + a_1 x^{n-1} y^{(n-1)}(x) \cdots + a_{n-1} x y' + a_n x = 0 .$$

where $a_k \in$ const. $x>0$. Then we can apply

a transformation to simplify it:

Note that: $x^{-(n+1)}(x^n y^{(n)}+\cdots+a_n y)=0$

$\therefore (\ln x)' y^{(n)} + (\ln x)^{(2)} y^{(n-1)} a_1 \cdots + a_n y (\ln x)^{(n+1)} = 0$

It may be from $\mathrm{span}\{(y\ln x)^{(k)}\}_1^{n+1}$? Let $\ln x = t$.

i.e. $x = e^t \quad \therefore \frac{dy}{dx} = e^{-t}\frac{dy}{dt}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\frac{dt}{dx}\left(e^{-t}\frac{dy}{dt}\right) = -e^{-2t}\frac{dy}{dt} - e^{-2t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}$$

$\cdots \Rightarrow$ we obtain $y^{(n)} + b_1 y^{(n-1)} + \cdots b_n y = 0$. $b_k \in \mathbb{R}$.