High Order Equations

(1) Depression of order:

① $F(x, y^{(k)}, \dots, y^{(n)}) = 0 \Rightarrow$ let $p = y^{(k)}$

② $F(y, y', \dots, y^{(n)}) = 0$. (Autonomous)

let $\frac{dy}{dx} = p. \Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} \dots$

③ $L(y) = y^{(n)} + a_1(x) y^{(n-1)}(x) + \dots + a_n(x) y(x) = 0$. where $a_i(x)$ conti. on $[a,b]$. If we know k linearly indept solutions. $\{y_i\}_1^k$. Then we can reduce the order to $n-k$.

Pf: Lemma. $\forall I \subseteq [a,b]$ interval. $\{y_i(x)\}_1^k$ are linearly indept.

Pf: $\exists \{c_i\}_1^k \neq \{0\}$. s.t. $y = \sum_1^k c_i y_i = 0$ on I

Then $\forall x_0 \in I$, $y^{(k)}(x_0) = 0$. $1 \le k \le n-1$

$\therefore y \equiv 0$ on $[a,b]$. by Uniqueness

Contradict with $\{y_i\}$ l.i on $[a,b]$.

$\Rightarrow$

$\exists I_1$. s.t. $y_1(x) \neq 0$ on I_1. $\exists I_2 \subseteq I_1$. s.t. $y_2(x) \neq 0 \dots$

$\exists I^*$. s.t. $\forall k \in [1, n]$. $y_k(x) \neq 0$.

by conti. and the same argument above. If $\exists x_0$. s.t. $y(x_0) \neq 0 \Rightarrow \exists I_{x_0}$. $y(x_0) \neq 0$

Let $\eta = \eta_k(x) Z(x)$. Then we obtain:

$$L(\eta_k Z(x)) = [\eta_k^{(n)}(x) + a_1 \eta_k^{(n-1)}(x) + \cdots + a_n(x)] Z(x)$$
$$+ \eta_k(x) \cdot L_k(Z(x)) = \eta_k(x) \cdot L_k(Z(x))$$

$$L_k(Z(x)) = Z^{(n)} + b_1(x) Z^{(n-1)}(x) + \cdots + b_n(x) Z(x)$$

Since $L(\eta_k(x) \cdot 1) = 0$ $\therefore L_k(1) = 0 = b_n(x)$.

$\Rightarrow$ $\eta = \eta_k(x) Z(x)$ is solution of equation $\Leftrightarrow$
$Z = Z(x)$ is solution of $Z^{(n)} + b_1(x) Z^{(n-1)} + \cdots b_{n-1}(x) Z'(x) = 0$.

Let $p(x) = Z'(x)$. Since $\{\eta_i\}_1^k$ is the original solution

$\therefore p_i(x) = \left(\frac{\eta_i(x)}{\eta_k(x)}\right)'$. $1 \le i \le k-1$. linearly indept. solution of
$p^{(n-1)} + b_1(x) p^{(n-2)} + \cdots b_{n-1}(x) p(x) = 0$. Repeat the procedure!

(Pf: If $\sum_1^{k-1} C_i \left(\frac{\eta_i(x)}{\eta_k(x)}\right)' = 0$. Then
$\left(\sum_1^{k-1} C_i \left(\frac{\eta_i(x)}{\eta_k(x)}\right)\right)' = 0$ $\therefore \sum_1^{k-1} C_i \left(\frac{\eta_i(x)}{\eta_k(x)}\right) = C_k$)

$\Rightarrow$ In sum: Let $\eta = \eta_k Z$. $p_i(x) = \left(\frac{\eta_i(x)}{\eta_k(x)}\right)'$
Solve for Z $\to$ generate!

(2) <u>Equations in dimension n linear space</u>:

<u>Def</u>: $\vec{\eta}$ in $\mathbb{R}^n$. with norm $\|\cdot\|$. where $\|\vec{\eta}\| = \max_{1 \le i \le n} \{|\eta_i|\}$

$$\frac{d\vec{\eta}}{dx} = \begin{pmatrix} \frac{d\eta_1}{dx} \\ \vdots \\ \frac{d\eta_n}{dx} \end{pmatrix}, \quad \vec{f}(x, \vec{\eta}) = \begin{pmatrix} f_1(x, \eta_1 \cdots \eta_n) \\ \vdots \\ f_n(x, \eta_1 \cdots \eta_n) \end{pmatrix}$$

It's easy to see Peano and Cauchy Thm. hold!

(3) The dependence of solution on initial value and parameter:

· In reality, there're measuring errors in some parameter. We hope it will not make much disturbance!

$\Rightarrow$ For $\frac{d\vec{\eta}}{dx} = \vec{f}(x, \vec{\eta}, \vec{\lambda})$. $\vec{\eta}(x_0) = \vec{\eta}_0$.

Let $x - x_0 = t$. $\vec{u} = \vec{\eta} - \vec{\eta}_0$. We only need to consider:

$$\frac{d\vec{u}}{dt} = \vec{g}(t, \vec{u}, \vec{\lambda}), \quad \vec{u}(0) = 0.$$

We claim: (About conti of parameter)

Thm $\vec{f}(x, \vec{\eta}, \vec{\lambda})$ conti. in G: $|x| \le a$. $|\vec{\eta}| \le b$. $|\vec{\lambda} - \vec{\lambda}_0| \le c$

satisfies Lipschitz condition on $\vec{\eta}$. Then:

If $M = \sup_G |\vec{f}(x, \vec{\eta}, \vec{\lambda})|$. $h = \min\{a, \frac{b}{M}\}$.

Then solution $\vec{\eta} = \vec{\varphi}(x, \vec{\lambda})$ conti on D: $|x| \le h$. $|\vec{\lambda} - \vec{\lambda}_0| \le c$

Pf: Construct Picard Sequence.

Prove $\varphi_k(x, \vec{\lambda})$ conti on D by induction. $\forall k$.

Cor. Let the parameter be in the initial value:

$\vec{f}(x, \vec{\eta})$ conti. on R: $|x - x_0| \le a$. $|\vec{\eta} - \vec{\eta}_0| \le b$.

satisfies Lipschitz condition on $\vec{\eta}$.

For $\frac{d\vec{\eta}}{dx} = \vec{f}(x, \vec{\eta})$ $\vec{\eta}(x_0) = \vec{\eta}$. $h = \min\{a, \frac{b}{M}\}$.

The solution $\vec{\eta} = \varphi(x, \vec{\eta})$ conti. on $|x - x_0| \le \frac{h}{2}$. $|\vec{\eta} - \vec{\eta}_0| \le \frac{b}{2}$

Pf: Let $t = x - x_0$. $\vec{u} = \vec{\eta} - \vec{\eta}$. $\Rightarrow$ $\frac{d\vec{u}}{dt} = \vec{g}(t, \vec{u}, \vec{\eta})$. $u(0) = 0$.

$|t| \le a$. $|\vec{u}| \le |\vec{\eta} - \vec{\eta}_0| - |\vec{\eta} - \vec{\eta}_0| = \frac{b}{2}$. $|\vec{\eta} - \vec{\eta}_0| \le \frac{b}{2}$, by Thm.

Remark: Local Straightening:

$$G: \begin{cases} |x-x_0| \le \frac{a}{2} \\ |\eta-\eta_0| \le \frac{b}{2} \end{cases} \longrightarrow T(G) \qquad T: \begin{cases} x = x \\ \vec{\eta} = \vec{\varphi}(x, \vec{\eta}) \end{cases}$$

T is one-to-one. by local uniqueness of equation.

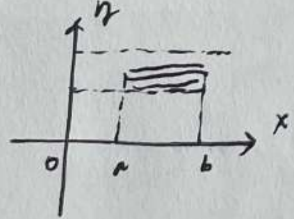
$\Rightarrow$ For fixed $\bar{\eta}$, $|\bar{\eta}-\eta_0| \le \frac{b}{2}$. $T\left(\begin{matrix} \eta = \bar{\eta} \\ |x-x_0| \le \frac{a}{2} \end{matrix}\right) = T_{\bar{\eta}} = I_{\bar{\eta}}$.

$\therefore T^{-1}(I_{\bar{\eta}}) = \left\{\begin{matrix} \eta = \bar{\eta} \\ |x-x_0| \le \frac{a}{2} \end{matrix}\right\}$. a stright line!

Thm. $\vec{f}(x,\vec{\eta})$ conti. at G. satisfies Lipschitz condition on $\vec{\eta}$. $\vec{\eta} = \vec{\xi}(x)$ is one of solutions of $\frac{d\vec{\eta}}{dx} = \vec{f}(x,\vec{\eta})$ on zone J. $\exists \delta > 0$. $\forall (x_0, \eta_0) \in \left\{\begin{matrix} a \le x_0 \le b \\ |\eta_0 - \vec{\xi}(x_0)| \le \delta \end{matrix}\right. \subseteq J$.

$\frac{d\vec{\eta}}{dx} = \vec{f}(x,\vec{\eta})$, $\vec{\eta}(x_0) = \vec{\eta}_0$. has conti. solution $\varphi(x, x_0, \eta_0)$ on $a \le x \le b$, $a \le x_0 \le b$, $|\vec{\eta}_0 - \vec{\xi}(x_0)| \le \delta$.

Remark:



locally, in conti. zone of x_0, η_0, full of the conti solutions!

Pf: By the optness. we consider a sufficient small open interval in it:

Construct Picard Sequence. $\varphi_0(x, x_0, \eta_0) = \eta_0 + \xi(x) - \xi(x_0)$

$$\Rightarrow |\varphi_{k+1} - \varphi_k| \le \frac{(L|x-x_0|)^{k+1}}{(k+1)!} |\eta_0 - \xi(x_0)|.$$

Need: $|\varphi_k(x, x_0, \eta_0) - \xi(x)| < \sigma \Rightarrow$ chose $\delta = \frac{1}{2} e^{-L(b-a)} \sigma$

Thm. $\vec{f}(x,\vec{y})$ conts. on R. If for $\frac{d\vec{y}}{dx}=\vec{f}(x,\vec{y})$, $\forall (x_1,\vec{y}_1)\in R$.

$\exists$ solution curve cross $(x_1,\vec{y}_1)$. and it's unique.

Then the solutions of the equation is continuously dependent on $(x_1,\vec{y}_1)$, the initial values.

Pf: By contradiction: suppose $\varphi(x,x_0,y_0)$ is unique solution.

If $\exists (x_0,\vec{y}_0)$, $(\bar{x}_0,\vec{\bar{y}}_0)$. $d((x_0,\vec{y}_0),(\bar{x}_0,\vec{\bar{y}}_0))<\delta$.

$\exists \varepsilon_0$. $|\varphi(x,x_0,y_0)-\varphi(x,\bar{x}_0,\bar{y}_0)|\geq\varepsilon_0$. for $\forall\delta>0$.

Let $x=x_0,\bar{x}_0$ $\begin{cases}|\varphi(\bar{x}_0,x_0,y_0)-\bar{y}_0|\geq\varepsilon_0\\|\varphi(x_0,\bar{x}_0,\bar{y}_0)-y_0|\geq\varepsilon_0\end{cases}$

Choose $\delta=\min\{\frac{\varepsilon_0}{2},\delta_0\}$. $\therefore |x_0-\bar{x}_0|<\frac{\varepsilon_0}{2}$. $|\vec{y}_0-\vec{\bar{y}}_0|<\frac{\varepsilon_0}{2}$

$$|\vec{y}_0-\vec{\bar{y}}_0| = |\vec{y}_0-\varphi(x_0,\bar{x}_0,\bar{y}_0)+\varphi(x_0,\bar{x}_0,\bar{y}_0)-\vec{\bar{y}}_0|$$

$$\geq |\vec{y}_0-\varphi(x_0,\bar{x}_0,\bar{y}_0)|-\underline{|\varphi(x_0,\bar{x}_0,\bar{y}_0)-\vec{\bar{y}}_0|} \leftarrow \text{From conts.}$$

$$> \varepsilon_0-\frac{\varepsilon_0}{2}=\frac{\varepsilon_0}{2}.$$ Contradict!

(4) Differentiability of solution on parameters and initial value:

Thm. For $\frac{d\vec{y}}{dx}=\vec{f}(x,\vec{y},\vec{\lambda})$. $\vec{y}(0)=\vec{0}$. $\vec{f}$ is differentiable at $\vec{y}$ and $\vec{\lambda}$ on $G: |x|\leq a$. $|\vec{y}|\leq b$. $|\vec{\lambda}-\vec{\lambda}_0|\leq c$.

Then the solution $\vec{y}=\varphi(x,\vec{\lambda})\in C^1(D)$. $D: \begin{array}{l}|x|\leq h=\min\{\cdots\}.\\|\lambda-\lambda_0|\leq c.\end{array}$

Pf: Construct Picard Sequence. Induction!

(5) Sumary:

· $\frac{d\vec{\eta}}{dx} = \vec{f}(x, \vec{\eta}, \vec{\lambda})$. The property of solutions depend on $\vec{f}(x, \vec{\eta}, \vec{\lambda})$:

i) $f(x, \vec{\eta}, \vec{\lambda})$. conti. Lipschitz (For uniqueness)

$\Rightarrow \varphi(x, x_0, \vec{\eta}_0, \vec{\lambda})$ conti. on $(x_0, \vec{\eta}_0, \vec{\lambda})$

ii) $f(x, \vec{\eta}, \vec{\lambda})$ differentiable

$\Rightarrow \varphi(x, \vec{\lambda})$ differentiable.

Recall that the property is transitted by constructing picard sequence!