Singular Solution

- An equation may have general solution and particular solution which isn't contained in the former. It's phenomenon of degeneration on general solution —— The uniqueness is broken geometrically.

$\Rightarrow$ Next we introduce a special kind of particular solution —— Singular solution

(1) Implicit Equation with one order:

- $F(x, y, \frac{dy}{dx}) = 0$

① Methods of Solving:

i) By factorization, obtain explicit differential equation.

ii) Solve for $\begin{cases} y = f(x, p) \\ \text{or} \\ x = f(y, p) \end{cases}$ where $p = \frac{dy}{dx}$.

Then by d/dx or d/dy $\Rightarrow$ $\begin{cases} p = \frac{d}{dx} f(x, p) \\ \frac{1}{p} = \frac{d}{dy} f(y, p) \end{cases}$

iii) By parametrization. for $F(y, p) = 0$ or $F(x, p) = 0$. Let $y = g(t)$, $p = h(t)$ or $x = g(t)$, $p = h(t)$ satisfies equation above.

$$\Rightarrow \begin{cases} h(t)dx = dy = g'(t)dt \\ \frac{1}{h(t)}dy = dx = g'(t)dt \end{cases}$$ solve x or y!

Or for $F(x, y, p) = 0$. Let $\begin{cases} x = f(u,v) \\ y = g(u,v) \\ p = h(u,v) \end{cases}$

$\Rightarrow \quad dy = g_1' du + g_2' dv = h(u,v)\, dx = h(u,v)\,(f_1' du + f_2' dv)$

Then we can solve for u, v!

② Example:

Clairaut equation

$$y = xp + f(p), \quad p = \frac{dy}{dx}, \quad f''(p) \neq 0$$

→ It can construct an equation, whose singular solution is the given $y = g(x)$.

1') Tangent line of $g(x)$: $y = g'(t)(x-t) + g(t)$

$\therefore c = g'(t)$, $t = k(c)$

$\Rightarrow f(c) = g(k(c)) - c\,k(c)$

2') Since we obtain f

$\Rightarrow y = px + f(p)$ ✓

Pf: By $\frac{d}{dx} \Rightarrow \frac{dp}{dx}(x + f'(p)) = 0$

$$\Rightarrow \begin{cases} y = Cx + f(C), \text{ general solution} \\ x = -f'(p),\ y = -f'(p)p + f(p). \text{ particular solution.} \end{cases}$$

Since $f''(p) \neq 0$, solve $p = w(x) = y'$

$y = x\,w(x) + f(w(x))$. for any (x, y) on it.

$\exists\, C_0$, s.t. $y = C_0 x + f(C_0)$ tangent to it!

(2) Singular solutions:

Def: For $F(x, y, \frac{dy}{dx}) = 0$. has a particular solution Γ:

$y = \varphi(x)$, s.t. $\forall\, Q \in \Gamma$, $\forall\, U_Q$, $\exists$ solution $\neq \Gamma$.

s.t. it tangent to Γ at Q.

① Necessarity:

$F(x, y, p)$ conti. in G for (x, y, p). F_y', F_p' conti.

If $\varphi(x) = y$ is a singular solution, i.e.

$F(x, \varphi(x), \varphi'(x)) = 0$. $(x, \varphi(x), \varphi'(x)) \in G$. Then it satisfies:

p-criterion: $\begin{cases} F(x, y, p) = 0 \\ F_p'(x, y, p) = 0. \end{cases}$

Pf: By Implicit Func. Thm: If $F_p \neq 0$. $\Rightarrow \frac{dy}{dx} = f(x, y)$.

$f_y' = -\frac{F_y'}{F_p'}$ conti. by Picard Thm.

It has unique solution locally. Contradict!

② Sufficience:

$F(x, y, p) \in C^2[\mathbb{R}^3 \cap G]$. satisfies p-criterion.

If $y = \varphi(x)$ is the solution from p-criterion

by cancelling p. satisfies $\begin{cases} F_y'(x, \varphi(x), \varphi'(x)) \neq 0 \\ F_{pp}''(x, \varphi(x), \varphi'(x)) \neq 0 \\ F_p'(x, \varphi(x), \varphi'(x)) = 0 \end{cases}$

$\Rightarrow$ Then $y = \varphi(x)$ is singular

Pf. The ideal is find the point-tangent solution which is different from $y = \varphi(x)$.

The differentials are for applying Implicit Function Thm to reduce multivariation!

(3) Envelop:

Def. An envelop I for family of curves:

$k(c)$: $V(x, y, c) = 0$. $V \in C[\mathbb{R}^3 \cap D]$.

which $\in C^1[J]$. If $\forall q$ on I. $\exists k(c_0) \in k(c)$, $\exists U_q$ of q. tangent to I at q in U_q. Besides. $k(c_0) \neq I$ in U_q.

Remark: The envelop won't exist always!

① Equivalence:

Thm. For $F(x,y,\frac{dy}{dx})=0 \Rightarrow$ general solution $u(c)$: $u(x,y,c)=0$. Then the envelop of $u(c)$ is the singular solution for $F(x,y,\frac{dy}{dx})=0$.

Pf: Check the envelop $y=\varphi(x)$ is a solution!

② Dual Propositions:

Thm. 1 (Neccessarity)

If I is an envelop for $V(x,y,c)=0$, Then it satisfies C-Criterion $\begin{cases} V(x,y,c)=0 \\ V_c'(x,y,c)=0 \end{cases}$ → Gaurantee it's not unique!

(we can solve for $\Lambda(x,y)=0$)

Pf. Only Consider the parametrized Case:

$I: \begin{cases} x=f(c) \\ y=g(c) \end{cases}$ Then $V(f(c),g(c),c)=0$

$\Rightarrow d/dc$, we obtain: $(V_x', V_y', V_c')(f', g', 1)=0$

1°) $(V_x', V_y')=\vec{0}$ or $(f', g')=\vec{0}$.

2°) By tangent: $(-V_y', V_x') \parallel (f', g')$, since it's an envelope

Remark: The slope of $V(x,y,c)$ is from:

by Implicit Function Thm. $y=\varphi(x)$

$\therefore \varphi'(x) = -\frac{V_x'}{V_y'}$ is the slope!

Thm 2: (Sufficient)

For $V(x,\eta,c)=0$. by C-Criterion, we determine

a curve $\Delta = \begin{cases} x=\varphi(c) \\ \eta=\psi(c) \end{cases} \in C^1$. (Parametrized expression) → If exists a $\vec{0}$. Then the slope will be ∞. It may be a "pole"! e.g. $\frac{d\eta}{\eta-1} = \frac{dx}{x-1}$. $x=\eta=1$! So they won't be tangent!

If Δ satisfies $\begin{cases} (\varphi'(c),\psi'(c)) \neq \vec{0} \\ (V_x', V_\eta')\big|_{(\varphi(c),\psi(c),c)} \neq \vec{0} \end{cases}$ (Non-degenerated(*) Condition)

Then Δ is an envelop for $V(x,\eta,c)=0$.

Pf: The non-degenerated condition is for the existence of slope! Then $(-V_\eta', V_x') \parallel (\varphi'(c),\psi'(c))$

Remark: Envelop is for the solution family (Not primary form) Its property may be better than singular.

(4) Complement: Characterization

· If a solution of equation isn't unique anywhere. Then it's the singular solution.

Since two solutions tangent at the same point!

e.g. If $E(\eta)$ satisfies. $E(\eta)=0$ in $[0,1]$ only when $\eta=0$. Then

for $\frac{d\eta}{dx}=E(\eta)$. $\eta=0$ is singular $\Leftrightarrow \int_0^1 \frac{1}{E(\eta)}$ converges.

Pf: Consider the uniqueness

($\Leftarrow$) Construct $x = C + \int_0^\eta \frac{d\eta}{E(\eta)}$.

($\Rightarrow$) If $\int_0^1 \frac{d\eta}{E(\eta)} = \infty$. Then $\eta=0$ is locally unique.