Existence and Uniqueness for the Solution.

(1) Picard Thm:

① Main Thm:

For I.V.P: $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$.

f is conti in $R: |x-x_0| \le a$. $|y-y_0| \le b$.

satisfies Lipschitz condition on y.

Then in $[x_0-h, x_0+h]$. y has unique solution

$h = \min\{a, \frac{b}{M}\}$, $M = \max\limits_{x,y \in R} |f(x,y)|$.

Pf: 1°) $\Leftrightarrow$ $y(x) = y(x_0) + \int_{x_0}^{x} f(x,y)\,dx$

2°) By Contracting Mapping Prin:

Picard Seq: $y_{n+1} = y_0 + \int_0^x f(x, y_n(x))\,dx$

We obtain: $|y_n(x) - y_0(x_0)| \le M|x-x_0|$. by induction.

M guarantee $y_n(x)$ falls in $|y-y_0| \le b$!

3°) $y_n \rightrightarrows y$, which is solution of I.V.P.

By Induction: $|y_n(x) - y_{n-1}(x)| \le \frac{M}{L} \frac{(L|x-x_0|)^n}{n!}$

4°) Uniqueness by Disposition.

Since the convergence is unique.

It's obvious to obtain!

② Osgood Condition:

· For I.V.P. $\frac{dy}{dx} = f(x,y)$. f conti at R. for y.

$f(x,y)$ satisfies Osgood condition. i.e. $\forall y_1, y_2 \in R$.

$|f(x,y_1) - f(x,y_2)| \le F(|y_1 - y_2|)$. $\exists F$. $\int_0^{r_1} \frac{dr}{F(r)} = \infty$.

Then solution for I.V.P. is unique.

Pf: By contradiction:

Consider $r(x) = y_1(x) - y_2(x)$. $\bar{x} = \sup\{x \mid y_1(x) = y_2(x), x > x_0\}$.

$r(\bar{x}) = 0$. Find Neighbour of $\bar{x}$. s.t. $r(x) \ne 0$.

③ Other Criteria for Uniqueness:

i) $f(x,y)$ is decreasing on y. Then for I.V.P above. the solution is uniques on $[x_0, +\infty)$

Pf: Anologue with before. $\bar{x} = \sup\{x \mid y_1(x) = y_2(x)\}$.

WLOG, on $[\bar{x}, x_1]$. $r(x) = y_1 - y_2 \ge 0$. for some x_1

Then $r'(x) = f(x, y_1) - f(x, y_2) \le 0$. Since $y_1 \ge y_2$

when x is on $[\bar{x}, x_1]$

$\therefore r(x) \le r(\bar{x}) = 0$. on $[\bar{x}, x_1]$. $r(x) = 0$!

ii) Bellman - Grownwall Inequality:

$y(x) \in [a,b]$. $a \le x_0 \le b$. if $\exists\ \delta, k \ge 0$. s.t.

$y(x) \le \delta + k \left| \int_{x_0}^{x} y(t)\, dt \right|$, $x \in [a,b]$. $y \ge 0$.

Then $y(x) \le \delta e^{k|x - x_0|}$.

Remark: i) It's convenient to handle with the Iteration equation:

$$y(x) = f(x) + \int_{x_0}^{x} g(t) f(t)\, dt.$$

ii) If $\delta = 0$, Then $y \equiv 0$.

Pf: 1°) $x \in [x_0, b]$. Denote $\varphi(x) = \int_{x_0}^{x} y(t)\,dt$.

$$\therefore\ \varphi'(x) \le \delta + K\varphi(x)$$

$$\therefore\ \frac{d}{dx}\left(e^{-K(x-x_0)}\varphi(x)\right) \le \delta e^{-K(x-x_0)}$$

Intergrate from x_0 to x:

$$\delta + K\varphi(x) \le \delta e^{K(x-x_0)}.$$

Replace $\varphi(x)$ in the original equation

2°) $x \in [a, x_0]$. Similarly!

(2) Peano Existense Thm:

- We only consider the existence but not uniqueness to lose some condition:

$\Rightarrow$ Only need Conti. property. The Lipschitz condition can be dropped!

By Euler's broken lines:

- Separate $|x - x_0| \le h$ into $2n$ intervals:

$\{[x_{k-1}, x_k]\}$. Replace the curve by straight

line: $\eta_k = \eta_{k-1} + f(x_{k-1}, \eta_{k-1})(x - x_{k-1})$. $x \in [x_{k-1}, x_k]$.

$\Rightarrow$ Approximation of solution $\eta(x)$:

$$\varphi_n(x) = \begin{cases} \eta_0 + \sum_{0}^{s-1} f(x_k, \eta_k)(x_{k+1} - x_k) + f(x_s, \eta_s)(x - x_s). & x_s \le x \le x_{s+1} \\ \eta_0 + \sum_{0}^{-s+1} f(x_k, \eta_k)(x_{k+1} - x_k) + f(x_{-s}, \eta_{-s})(x - x_{-s}). & x_{-s-1} \le x \le x_{-s}. \end{cases}$$

$\Rightarrow$

1) Ascoli Thm: $\{\varphi_n(x)\}$ has convergent subseq.

2) On $|x - x_0| \le h$. $\varphi_n(x) = \eta_0 + \int_{x_0}^{x} f(x, \varphi_n(x))\,dx + \delta_n(x)$

$\delta_n(x) \to 0\ (n \to \infty)$.

Pf: $\int_{x_0}^{x} f(x, \varphi_n(x))\,dx = \sum_{1}^{s} \int_{x_{k-1}}^{x_k} f(x, \varphi_n(x))\,dx + \int_{x_s}^{x} f(x, \varphi_n(x))\,dx$

By matching to $\varphi_n(x)$. $\delta_n(x) = \sum_{0}^{s-1} d_n(s) + d_n^{*}(x)$

By conti. of $f(x, \eta)$: $d_n(s) \le \frac{\varepsilon C}{n}$.

$\Rightarrow$ Let $n_k \to \infty$. $\varphi_{n_k}(x)$ converges to one of the solution. If the solution is unique. Then every subseq converges to it!

(3) Extension of Solution:

① By Existence Thm:

From local to global:

$\frac{d\eta}{dx} = f(x, \eta)$. f conti at area G. For I.V.P:

$P(x_0, \eta_0) \in G$. $\frac{d\eta}{dx} = f(x, \eta)$. $\eta(x_0) = \eta_0$. For $\forall G_1 \underset{\text{closed bounded}}{\subseteq} G$

We can extend the solution curve I outside G_1!

<u>Remark</u>: i) The maximal existence interval can't be the form of close or semiclose

(Since we can extend the solution $y(x)$ to the Boundary. By conti of $f(x,y)$)

By Peano Thm!

ii) If $f(x,y)$ satisfies Lipschitz condition for y on G. Then exists unique extension.

② <u>Maximal Existence Interval</u>:

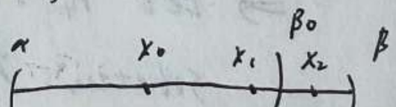
<u>Thm</u>. For $\frac{dy}{dx} = f(x,y)$. $S = \alpha < x < \beta$. $y \in \mathbb{R}$

f conti at S. Besides. $|f| \le A(x)|y| + B(x)$

where $A(x)$. $B(x)$ conti on $[\alpha,\beta]$. $A(x), B(x) \ge 0$.

Then (α,β) is the MEI of each solution.

<u>Pf</u>: For I.V.P: $y(x_0)=y_0$



If $[x_0,\beta_0)$ is the right MEI. $\beta_0 < \beta$

Find x_1. x_2. From $x = x_1$. i.e.

Consider I.V.P: $y(x_1) = y_1$.

Extend x_1 to x_2. which is contradiction!

i.e. $h > \beta_0 - x_1 \Leftarrow h > x_2 - x_1$ ∴ Δx should be small!

<u>Remark</u>: For $\alpha, \beta \in \overline{\mathbb{R}}_\infty$. It still holds by similar argument.

(3) Comparison Thm:

① The First Thm.

f, F conti. in G. $f(x,\eta) < F(x,\eta)$. And

$\varphi(x)$. $\phi(x)$ are solutions of I.V.P's:

$$\begin{cases} \frac{d\eta}{dx} = f(x,\eta), & \eta(x_0)=\eta_0 \\ \frac{d\eta}{dx} = F(x,\eta), & \eta(x_0)=\eta_0 \end{cases} \quad \text{separately. } (x_0,\eta_0) \in G$$

Then $\varphi(x) < \phi(x)$, when $x_0 < x < b$.

$\varphi(x) > \phi(x)$, when $a < x < x_0$.

② The second Thm.

i) For. I.V.P. $\frac{dx}{dx} = f(x,\eta)$, $\eta(x_0) = \eta_0$. on R:

$|x-x_0| \le a$. $|\eta-\eta_0| \le b$. $\exists\ \sigma < h$. where $|x-x_0| \le h$

is the existent interval of solution. s.t.

On $|x-x_0| \le \sigma$. there're maximal solution $Z(x)$

and minimal $W(x)$. i.e. $\forall \eta(x)$. solution. $Z(x) \ge \eta(x) \ge W(x)$.

Pf: Consider (E_m): $\frac{d\eta}{dx} = f(x,\eta) + \varepsilon_m$. $\varepsilon_m \downarrow 0$

$\therefore h_m \to h$. σ is choosen for existence of $\varphi_m(x)$

By Ascoli Thm. $\exists$ subseq of $\{\varphi_m(x)\}$. converges.

Besides $\varphi_m(x) > \eta(x)$, $x \in (x_0, x_0+\sigma)$. By 1st Thm.

Remk: Extend $Z(x)$. $W(x)$ to the boundary

Then $Z(x) = W(x) \Leftrightarrow$ The solution is unique.

ii) f, F conti on G. $f \le F$. φ is the right minimal and the left maximal solution of E_1. ψ is a solution of E_2. Then, we obtain:

$\psi \ge \varphi$, when $x \in [x_0, b]$. converse on $[a, x_0]$

Remark: "$=$" holds when $f = F$, the solution is unique!

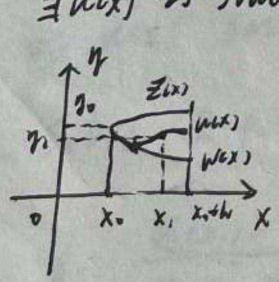
iii) On R: $|x - x_0| \le a$, $|\eta - \eta_0| \le b$. $[\,|x - x_0| \le h\,]$ is the existence interval of solution of I.V.P. $\frac{d\eta}{dx} = f(x, \eta)$, $\eta(x_0) = \eta_0$. Then

$\forall\ (x_1, \eta_1)$, st. $|x_1 - x_0| \le h$, $w(x_1) \le \eta_1 \le Z(x_1)$

$\exists\ u(x)$ is solution of I.V.P. (I), $u(x_1) = \eta_1$

Pf: Suppose $x_0 < x_1 < x_0 + h$. By Peano Thm.

$\exists u(x)$ is solution for I.V.P. $\frac{d\eta}{dx} = f$, $\eta(x_1) = \eta_1$



extend $u(x)$ in $[x_0, x_0 + h]$.

Then it will intersect with one of $Z(x)$, $w(x)$.

$\Rightarrow$ Joint them for the needed solution!

iv) Existence interval from Comparison:

e.g. $\frac{d\eta}{dx} = x^2 + (\eta + 1)^2$, $|x| \le 1$, $\eta(0) = 0$

$\Rightarrow (\eta + 1)^2 \le \frac{d\eta}{dx} \le 1 + (\eta + 1)^2$. By Integration:

we obtain: $\frac{\pi}{4} \le x \le 1$ (*)

(*) If a solution can't extend to x_0. That's mean: $\eta(x_0) = \infty$!