Solve by Elementary Integration

(1) Exact Equation:

· For $P(x,y)\,dx + Q(x,y)\,dy = 0$

If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $LHS = d\phi(x,y)$

(2) Separation of variables:

· For $P(x,y)dx + Q(x,y)dy = 0$. If $\begin{cases} P = X_1(x)\,Y_1(y) \\ Q = X_2(x)\,Y_2(y) \end{cases}$

$$\Rightarrow \frac{X_1(x)}{X_2(x)}\,dx + \frac{Y_2(y)}{Y_1(y)}\,dy = 0.$$

Caution that $Y_1(y) \equiv 0$ is a particular solution!

The integrate x, y separately!

(3) Linear Equation with one order:

① Homogeneous Case:

$$\frac{dy}{dx} + p(x)y = 0 \Rightarrow \text{From (2)}:$$

$$y = C e^{-\int p(x)dx}$$

② Nonhomogeneous Case:

$$\frac{dy}{dx} + p(x)y = q(x).$$

Multiply $\mu(x) = e^{\int p(x)dx}$

$$\Rightarrow d(ye^{\int p(x)dx}) = d(\int q(x)e^{\int p(x)dx})$$

$$\therefore y = e^{-\int p(x)dx}(C + \int q(x)e^{\int p(x)dx})$$

(4) Elementary Transformation:

① When $Pdx + Qdy = 0$, where $\frac{P(tx,ty)}{Q(tx,ty)} = \frac{P}{Q}$:

Let $y = ux$ $\therefore P(x,y) = x^m P(1,u)$. $Q(x,y) = x^m Q(1,u)$

We can reduce to the case of separation of Variables.

· A special Case:

$$\frac{dy}{dx} = f\left(\frac{ax+by+c}{mx+ny+l}\right)$$

i) $\Delta = an - bm = 0$, then $mx+ny = \lambda(ax+by)$. Let $v = ax+by$

ii) $\Delta = an - bm \neq 0$, Solve α, β from:

$$\begin{cases} a\alpha + b\beta + c = 0 \\ m\alpha + n\beta + l = 0 \end{cases} \quad \text{Let } \begin{cases} u = x + \alpha \\ v = y + \beta \end{cases}$$

$$\Rightarrow \begin{cases} \text{i) } \dfrac{dv}{dx} = a + b\dfrac{dy}{dx} = a + b f\left(\dfrac{v+c}{\lambda v + l}\right) \\ \text{ii) } \dfrac{dv}{du} = f\left(\dfrac{au+bv}{mu+nv}\right) \end{cases}$$

② Bernoulli Equation:

$$\frac{dy}{dx} + p(x)y = y^n, \quad n \neq 0, 1. \Rightarrow \text{Divide } y^n \ (y \neq 0)$$

Let $u = y^{1-n}$, reduce to (3)

③ Riccati Equation:

$$\frac{dy}{dx} = p(x)y^2 + q(x)y + r(x)$$

Approach to find solutions:

1°) Find a particular solution: $\varphi_1(x)$.

2°) Suppose $\varphi(x) + \varphi_1(x)$ is a solution

$\Rightarrow$ offset $r(x)$. Then become a Bernoulli equation

Thm. $\frac{dy}{dx} + ay^2 = bx^m$. $a \neq 0$. When $m = 0, -2,$ $\frac{-4k}{2k+1}, \frac{-4k}{2k-1}$. We can solve it by elementary transformation.

Remark: Actually: Riccati Equation
$\Updownarrow$
Homogeneous Linear Equation with order 2

: $(\Rightarrow)$ Let $y = -\frac{u'(x)}{r(x)u(x)}$

$(\Leftarrow)$ From $y'' + p(x)y' + q(x)y = 0$

Let $z = \frac{y'}{y} \Rightarrow \frac{dz}{dx} = \frac{y''}{y} + z^2$.

We have: $\frac{dz}{dx} - z^2 + p(x)z + q(x) = 0$

So $y = Ce^{\int z(x)}$.

(5) Integrating Factor:

① · Find $\mu(x,y)$. s.t. Let $P dx + Q dy = 0$

be an exact equation, i.e.

$$\frac{\partial(\mu P)}{\partial y} = \frac{\partial(\mu Q)}{\partial x} \Leftrightarrow P\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial x} = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mu.$$

For simplification:

Suppose μ is univariate to reduce calculation:

i) $\mu = \mu(x) \Rightarrow \frac{d(\ln \mu(x))}{dx} = \frac{1}{Q(x,y)}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) \rightarrow$ only depend on x

ii) $\mu = \mu(y) \Rightarrow \frac{d(\ln \mu(y))}{dy} = \frac{1}{P(x,y)}(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) \rightarrow$ only depend on y.

$\Rightarrow$ Check: $\frac{1}{Q}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) = F(x)$ Or $\frac{1}{P}(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) = G(y)$

② Thm. If $\mu(x,y)$ is an intergrating factor. s.t.

$\mu P dx + \mu Q dy = d\phi(x,y)$. Then so $\mu g(\phi(x,y))$ is!

where g is differentiable.

Pf: $\Rightarrow$ RHS $= d\int g(\phi(x,y))$.

$\Rightarrow$ Separation: $P dx + Q dy$

$$= (P_1 dx + Q_1 dy) + (P_2 dx + Q_2 dy) = 0.$$

Find $\mu_1 . \mu_2$. s.t. $\begin{cases} \mu_1 P_1 dx + \mu_1 Q_1 dy = d\phi_1(x,y) \\ \mu_2 P_2 dx + \mu_2 Q_2 dy = d\phi_2(x,y) \end{cases}$

$\Rightarrow$ Find g_1, g_2 differentiable. s.t.

$$\mu_1 g_1(\phi_1(x,y)) = \mu_2 g_2(\phi_2(x,y)) = \mu.$$

Then μ is the common factor!

③ Characterization of

Integrating Factors:

Thm. 1: If $P\,dx + Q\,dy = 0$, $\frac{P}{Q} = g(\frac{y}{x})$

Then $\frac{1}{xP + yQ}$ is its factor.

Pf: $P = Q\,g(\frac{y}{x})$ $\therefore$ $Q(g(\frac{y}{x})dx + dy) = 0$. Let $u = \frac{y}{x}$.

Thm. 2: If $P\,dx + Q\,dy = 0$, μ is a factor. s.t.

$\mu(P\,dx + Q\,dy) = d\phi(x,y) = 0$. Then, for μ_1

$\mu_1 = \mu\, g(\phi)$, where μ_1 is another factor.

Pf: From: $\begin{cases} \mu(P\,dx + Q\,dy) = d\phi \\ \mu_1(P\,dx + Q\,dy) = d\varphi \end{cases}$ we obtain:

$$\frac{D(\phi,\varphi)}{D(x,y)} = \begin{vmatrix} \phi_x & \phi_y \\ \varphi_x & \varphi_y \end{vmatrix} = 0.$$

$\therefore$ $\phi(x,y)$ are dependent with φ. $\therefore$ $\phi = F(\varphi)$

$$\therefore \frac{\mu}{\mu_1} = \frac{d\phi}{d\varphi} = F'(\varphi)$$

Cor. μ_1, μ_2 are two factors. then $\frac{\mu_1(x,y)}{\mu_2(x,y)} = C$

is one solution in form of integration.

Pf: $\frac{M_1}{M_2} = f(\phi) = C$. Since $\phi = C$ is

One of solutions. $\therefore \phi = f^{-1}(C)$ is one solution!

(6) Practical Case:

① Equiangular Family:

· For $\phi(x, y, C_1) = 0$. find $\varphi(x, y, C_2)$. st. $\measuredangle(\varphi, \phi) = \alpha$.

1°) Solve $\frac{dy}{dx} = y'$ of ϕ:

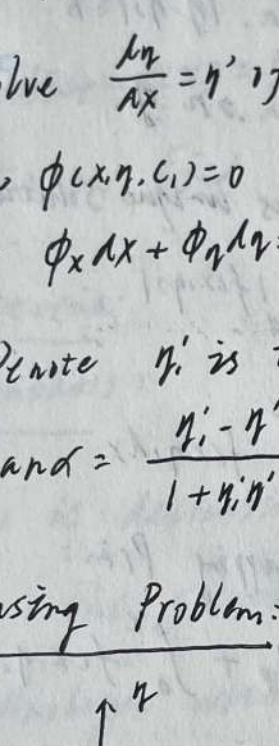
$$\begin{cases} \phi(x, y, C_1) = 0 \\ \phi_x dx + \phi_y dy = 0 \end{cases} \Rightarrow \text{Cancel "}C_1\text{"}: \frac{dy}{dx} = H(x, y)$$

2°) Denote y_1' is tanangent of $\varphi(x, y, C_2)$, then

$\tan\alpha = \frac{y_1' - y'}{1 + y_1' y'}$. solve y_1'!

② Chasing Problem:

e.g.



P starts at origin walking on Axis-x with speed a. Q starts at (0,1) walking toward P. with speed b. Find the track of Q.

Pf. $Q(x, y)$. suppose after time t: $P(at, 0)$

$$\begin{cases} \frac{dy}{dx} = \frac{y}{at - x} \cdots ① \\ (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = b^2 \cdots ② \end{cases}$$ We want to cancel t: $at - x = \frac{y\,dx}{dy}$.

Differentiate for y: $a\frac{dt}{dy} - \frac{dx}{dy} = \frac{dx}{dy} + \frac{d^2x}{dy^2}$

Solve $\frac{dt}{dy}$ from ② $\Rightarrow \frac{dy}{dx} = \frac{1}{2}(y^r - y^{-r}),\ r = \frac{b}{a}$.