Regression Analysis.

(1) Common Case:

Consider $Y = C\beta + \varepsilon$. $C = (J_n\ X)$. $\beta = (\beta_0\ \beta)$

$X \in M^{n\times m}$. $r(C) = m+1$. $\varepsilon \sim N(0, \sigma^2 I)$

① Centralization:

Rewrite the equation into:

$$y_i - \bar{y} = \beta_0^* + \sum_{j=1}^{m} \beta_j (x_{ij} - \bar{x}_j) . \quad \beta_0^* = \beta_0 + \sum_i^m \beta_i \bar{x}_i - \bar{y}$$

i.e. $\tilde{Y} = \tilde{C}\tilde{\beta} + \varepsilon$. $\tilde{C} = \begin{pmatrix} J_n & \begin{matrix} x_{11}-\bar{x}_1 & \cdots & x_{1m}-\bar{x}_m \\ \vdots & & \vdots \\ x_{n1}-\bar{x}_1 & \cdots & x_{nm}-\bar{x}_m \end{matrix} \end{pmatrix}$

$$\tilde{\beta} = \begin{pmatrix} \beta_0^* \\ B \end{pmatrix}$$

For estimation: $\tilde{C}^T\tilde{C}\hat{\tilde{\beta}} = \tilde{C}^T\tilde{Y}$

$\tilde{C}^T\tilde{C} = \begin{pmatrix} n & 0 \\ 0 & \tilde{X}^T\tilde{X} \end{pmatrix}$, $L \triangleq \tilde{X}^T\tilde{X} = (l_{ij})$.

$\tilde{C}^T\tilde{Y} = (J_n\ \tilde{X})^T\tilde{Y} = \begin{pmatrix} 0 \\ \tilde{X}^T\tilde{Y} \end{pmatrix} \triangleq \begin{pmatrix} 0 \\ l \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} n & 0 \\ 0 & L \end{pmatrix}\begin{pmatrix} \hat{\beta}_0^* \\ \hat{B} \end{pmatrix} = \begin{pmatrix} n\hat{\beta}_0^* \\ L\hat{B} \end{pmatrix} = \begin{pmatrix} 0 \\ l \end{pmatrix}$

$\Leftrightarrow \hat{\beta}_0^* = 0$. $L\hat{B} = l$.

② Estimate of η:

<u>Thm.</u> i) $\hat{\eta} = x^T\hat{\beta}$ is BLUE of η.

$$\hat{\eta} \sim N(x^T\beta, \sigma^2 x^T (C^T C)^{-1} x)$$

ii) $y_0 - \hat{y}_0 \sim N(0, \sigma^2 (x^T (C^T C)^{-1} x + 1))$. So:

$$t = \frac{y_0 - \hat{y}_0}{\sqrt{MSE(1 + x^T (C^T C)^{-1} x)}} \sim t(n-m-1).\ MSE =$$

$Y^T (I - M) Y / (n-m-1)$.

<u>Rmk:</u> We can construct confidence interval of y_0.

(2) <u>Multilinear Regression:</u>

Consider: $X = (x_{ij})_{n \times m}$. $Y = (y_{ij})_{n \times m}$ $\beta = (\beta_{ij})_{m \times p} = (\beta_1 \cdots \beta_p)$

$E = (\varepsilon_{ij})_{n \times p}$. $Y = (J_n\ X) \begin{pmatrix} \beta_0 \\ \beta \end{pmatrix} + E$. where.

$E = (\varepsilon_1 \cdots \varepsilon_p)$. $\varepsilon_i \sim N(0, \Sigma)$. $C = (J_n\ X)$

① <u>For LSE:</u>

Note that $Vec(Y) = (I_p \otimes C) Vec(\beta) + Vec(E)$.

<u>Denote:</u> $D = I_p \otimes C$. $\therefore Vec(\hat{\beta}) = (D^T D)^{-1} D^T Vec(Y)$

$$\Rightarrow \begin{pmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_p \end{pmatrix} = \begin{pmatrix} (C^T C)^{-1} C^T Y_1 \\ \vdots \\ (C^T C)^{-1} C^T Y_p \end{pmatrix}$$

For $\hat{\beta} = \begin{pmatrix} \hat{b}_0 \\ \hat{B} \end{pmatrix} = (C^T C)^{-1} C^T Y$. Since $(C^T C)^{-1} =$

$\begin{pmatrix} \frac{1}{n} + \bar{X}^T L_{XX}^{-1} \bar{X} & -\bar{X}^T L_{XX}^{-1} \\ L_{XX}^{-1} \bar{X} & L_{XX}^{-1} \end{pmatrix}$. $L_{XX} = X^T (I - P_n) X$. $\bar{X} =$

$\frac{1}{n} X^T J_n$. $L_{XY} = X^T (I - P_n) Y$.

$$\Rightarrow \begin{cases} \hat{b}_0 = \bar{Y}^T - \bar{X}^T \hat{B} \\ L_{XX} \hat{B} = L_{XY}. \end{cases}$$

② For SSE $\hat{\Sigma}$:

$$\hat{Y} = (J_n \ X) \begin{pmatrix} \hat{b}_0 \\ \tilde{B} \end{pmatrix} = J_n \hat{b}_0 + X \tilde{B}$$

$$= J_n \bar{Y}^T + (I_n - P_n) X \hat{B}.$$

$$\Rightarrow Y - \hat{Y} = (I - P_n)(Y - X L_{xx}^{-1} L_{xY})$$

$$= (I - C(C^TC)^{-1}C^T) Y. \text{ alternatively.}$$

We obtain: $SSE = L_{YY} - L_{Yx} L_{xx}^{-1} L_{xY}$

$$= Y^T (I - M) Y.$$

Set $\hat{\Sigma} = SSE/(n-m-1)$

③ Properties of $\hat{\beta}$. $\hat{\Sigma}$:

Thm. $E(\hat{\beta}) = \beta$. $E(\hat{\Sigma}) = \Sigma$

Lemma. Denote $\hat{\beta} = (b_{ij})_{(m+1)\times p}$. Then:

$$Cov(b_{ik}, b_{jl}) = \sigma_{kl} \, e_{i+1}^T (C^TC)^{-1} e_{j+1}$$

Pf: Note that $\begin{cases} b_{ik} = e_{i+1}^T \hat{\beta} e_k \\ b_{jl} = e_{j+1}^T \hat{\beta} e_l \end{cases}$

It's direct.

Lemma. Denote $\hat{\beta} = \begin{pmatrix} \hat{b}_0 \\ \hat{B} \end{pmatrix}$. $L_{xx}^{-1} = (L_{ij})$

$\hat{B} = (\hat{b}_1 \cdots \hat{b}_p) = \begin{pmatrix} \hat{b}_{(1)} \\ \vdots \\ \hat{b}_{(m)} \end{pmatrix}$. Then:

i) $\sigma^2(\hat{b}_0^T) = (\frac{1}{n} + \bar{x}^T L_{xx}^{-1} \bar{x}) \Sigma$

ii) $Cov(\hat{b}_{(i)}^T, \hat{b}_{(j)}^T) = l^{ij} \Sigma$

iii) $Cov(\hat{b}_i, \hat{b}_j) = \sigma_{ij} L_{xx}^{-1}$.

Pf: i) is from ii, iii). $\hat{b}_0 = \bar{Y}^T - \bar{X}^T \hat{B}$.

Thm. i) $\hat{\Sigma} \sim W_p(n-m-1, \Sigma)$

ii) $\hat{\beta}$ is indept with $\hat{\Sigma}$.

iii) $\hat{\beta}$ is normal dist. st. $\hat{b}_0^T \sim N(b_0^T, (\frac{1}{n} + \bar{x}^T L_{xx}^{-1} \bar{x})\Sigma)$

$\hat{b}_{(i)}^T \sim N_p(b_{(i)}^T, l_{ii}\Sigma)$, $\hat{b}_j \sim N_m(b_j, \sigma_{jj} L_{xx}^{-1})$

Pf: i) $P = I - C(C^TC)^{-1}C^T$ is orthogonal proj.

$Y \sim N_{n\times p}(C\beta, I_n \otimes \Sigma)$. $\therefore A = (C\beta)P(C\beta)^T = 0$

ii) $C\hat{\beta} = HY$. indept with $\hat{\Sigma} = (I-H)Y$.

iii) By Lemma. It's direct.

④ Significance Testing:

Consider $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \begin{matrix} m_1 \\ m_2 \end{matrix}$. $Y = (J_n \ X_1)\begin{pmatrix} b_0 \\ B_1 \end{pmatrix} + X_2 B_2 + E$.

$B = \begin{pmatrix} b_0 \\ b_{(1)} \\ \vdots \\ b_{(m)} \end{pmatrix} = (b_0^T \ b_1 \cdots b_p)$

i) Test $H_0^{(i)}: \beta_{(i)} = 0_{(p)}$:

$\hat{\beta}_{(i)} = \hat{b}_{(i)} \sim N(b_{(i)}, l_{ii}\Sigma)$. Set $E_i = I_p / l_{ii}$

$T^2 = (n-m-1)(E_i \hat{\beta}_{(i)})^T \hat{\Sigma}^{-1}(E_i \beta_{(i)}) \overset{H_0^{(i)}}{\sim} T^2(p, n-m-1)$

$F = \dfrac{(n-m-1)-p+1}{(n-m-1)p} T^2 \sim F(p, n-m-p)$.

ii) Test $H_0: B_2 = 0$:

Denote: $\hat{\Sigma}_1 = Y^T(I - C_1(C_1^T C_1)^{-1} C_1^T)Y$.

$C_1 = (J_n \ X_1)$. $M_1 = C_1(C_1^T C_1)^{-1} C_1^T$

$\Rightarrow \hat{\Sigma}_1 - \hat{\Sigma} = Y^T(I - M_1)X_2 D^{-1} X_2^T (I - M_1) Y$.

where $D = X_2^T(I - M_1)X_2$.

Set $\hat{B}_2 = D^{-1} X_2^T (I - M_1) Y$. $\Rightarrow \hat{\Sigma}_1 - \hat{\Sigma} = \hat{B}_2^T D \hat{B}_2$

Thm. i) $\hat{\Sigma}_1 - \hat{\Sigma} \overset{H_0}{\sim} W_p(m_2, \Sigma)$

ii) $\hat{\Sigma}$ is indept with $\hat{\Sigma}_1 - \hat{\Sigma}$.

Pf. i) Denote $R = (I - M_1)X_2$. $B = R D^{-1} R^T$

B is orthogonal proj. $tr(B) = m_2$

ii) $P = (I - M_1 - B)$. $PB = 0$.

Thm. $$\lambda(Y) = \frac{\max_{H_0} L(\beta, \Sigma)}{\max L(\beta, \Sigma)} = \frac{L(\hat{\beta}(1), \hat{\Sigma}_1)}{L(\hat{\beta}, \hat{\Sigma})} = \frac{|\hat{\Sigma}|^{\frac{n}{2}}}{|\hat{\Sigma} + (\hat{\Sigma}_1 - \hat{\Sigma})|^{\frac{n}{2}}}$$

where $\beta(1) = (C_1^T C_1)^{-1} C_1^T Y$.

Rmk: $$U = \frac{|\hat{\Sigma}|}{|\hat{\Sigma} + (\hat{\Sigma}_1 - \hat{\Sigma})|} \sim \Lambda(p, n-m-1, m_2)$$