Testing

(1) Quadratic Forms:

① Chi square:

Def: $X_i \sim N(\mu_i, 1)$. indept. $\sum_1^n X_i^2 \sim \chi_n^2(\delta)$.

δ is noncentral parameter. $\delta = \sum \mu_i^2$.

Rmk: i) $T = \dfrac{X}{\sqrt{\frac{Y}{n}}} \sim t(n, \delta)$. where $X \sim N(\delta, 1)$

$Y \sim \chi_n^2$. indept.

ii) $F = \dfrac{X/n}{Y/m} \sim F(n, m, \delta, \tilde{\delta})$ where $X \sim \chi_n^2(\delta)$.

$Y \sim \chi_m^2(\tilde{\delta})$. indept.

prop. i) $E(\chi_n^2(\delta)) = n + \delta$

ii) $Var(\chi_n^2(\delta)) = 2n + 4\delta$.

properties: i) Y_i indept. $1 \le i \le k$. $Y_i \sim \chi^2(n_i, \delta_i)$

$\Rightarrow \sum_1^k Y_i \sim \chi^2(\sum_1^k n_i, \sum_1^k \delta_i)$ (By ch.f)

ii) $X \sim N_p(\mu, I_p)$. $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{matrix} r \\ p-r \end{matrix}$ $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$

$\Rightarrow X^T X \sim \chi^2(p, \mu^T \mu)$

$X_1^T X_1 \sim \chi^2(r, \mu_1^T \mu_1)$

$X_2^T X_2 \sim \chi^2(p-r, \mu_2^T \mu_2)$

Lemma. $\{A_i\}_1^k$ symmetric $r(A_i)=r_i$. $A_i \in M^{p\times p}$.

Set $A=\sum_1^k A_i$, $r(A)=r\le p$. Then

i) A_i are proj. matrix

ii) $A_iA_j=0$, $\forall i\neq j$.

iii) A is proj.

iv) $r=\sum_1^k r_i$. Any two of i), ii), iii) concludes others. iii), iv) $\Rightarrow$ i), ii)

Thm. i) $X\sim N_n(0,I_n)$. $A^T=A$. $r(A)=r$. Then
$X^TAX\sim\chi_r^2 \iff A^2=A$.

ii) $X\sim N_n(\mu,I_n)$. $A\in M^n$. $B\in M^{m\times n}$. $A^T=A$

Then $BA=0 \iff BX$ indept with X^TAX.

Pf. i) ($\Rightarrow$): $\exists\, \Gamma$. s.t. $\Gamma^TA\Gamma=\mathrm{diag}\{\lambda_1\cdots\lambda_n\}$

$X^TAX=Y^T\Gamma^TA\Gamma Y=\sum\lambda_iY_i^2$. $Y=\Gamma^TX$.

$$\varphi_{\sum\lambda_iY_i^2}=\prod_{i=1}^n(1-2i\lambda_it)^{-\frac{1}{2}}=(1-2it)^{-\frac{r}{2}}$$

$\therefore \lambda_i=1$. $1\le i\le r$. $\lambda_k=0$. $k>r$.

($\Leftarrow$) Similarly. $\Gamma^TA\Gamma=\begin{pmatrix}1&&&&\\&\ddots&&&\\&&1&&\\&&&0&\\&&&&\ddots\end{pmatrix}$.

Set $Y=\Gamma^TX$. $X^TAX=\sum Y_i^2\sim\chi_r^2$.

ii) ($\Rightarrow$) $\exists\,\Gamma$. orthonormal $\Gamma^TA\Gamma=\begin{pmatrix}\lambda_1&&&\\&\ddots&&\\&&\lambda_r&\\&&&0\end{pmatrix}$

$BA=B\Gamma\begin{pmatrix}D_r&0\\0&0\end{pmatrix}\Gamma^T$. Denote $B\Gamma=\begin{pmatrix}C_1\\C_2\end{pmatrix}$

$\Rightarrow C_1D_r=0 \quad\therefore C_1=0$

$\because X^T A X = Y^T \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} Y = \sum_1^r \lambda_i Y_i^2.$

$BX = BIY = C_2 \begin{pmatrix} Y_{r+1} \\ \vdots \\ Y_n \end{pmatrix}$. $Y = IX$. $\{Y_i\}$ indept.

$\therefore$ BX indept with $X^T A X$.

Cor. $X \sim N_n(\mu, I_n)$. $A, B \in M^{n\times n}$. symmetric.

Then $AB = 0 \Leftrightarrow X^T A X$ indept with $X^T B X$.

Thm. (Cochran)

$Y \sim N_p(\mu, \Sigma)$. $\Sigma > 0$. $\{A_i\}_1^k$ sym. $r(A_i) = r_i$. $A = \sum_1^k A_i$

$r(A) = r$. Then:

i) $\Sigma^{\frac{1}{2}} A_i \Sigma^{\frac{1}{2}}$ is idempotent $\Leftrightarrow Y^T A_i Y \sim \chi^2(r_i, \mu^T A_i \mu)$

ii) $A_i \Sigma A_j = 0 \Leftrightarrow Y^T A_i Y$ indept. with $Y^T A_j Y$. $i \neq j$.

iii) $\Sigma^{\frac{1}{2}} A \Sigma^{\frac{1}{2}}$ is idempotent $\Leftrightarrow Y^T A Y \sim \chi^2(r, \mu^T A \mu)$

iv) $r = \sum_1^k r_i$. Any two of i). ii). iii) $\Rightarrow$ others.

Pf. Set $X = \Sigma^{-\frac{1}{2}} Y$. As above.

Thm. (general)

i) $Y \sim N(\mu, V)$. If $VAVAV = VAV$. $\mu^T AVA\mu = \mu^T A \mu$. $VAVA\mu = VA\mu$. Then $Y^T A Y \sim \chi^2(\mathrm{tr}(AV), \mu^T A \mu)$

ii) $Y \sim N(\mu, V)$. A, B nonnegative definite. $VAVBV = 0$

Then $Y^T A Y$ indept with $Y^T B Y$.

Cor. $Y \sim N(\mu, V)$. $\mu \in C(V) \Rightarrow Y^T V^- Y \sim \chi^2(r(V), \mu^T V^- \mu)$

Rmk: If $\mu = 0$ in ii) Then only require: $A^T = A$. $B^T = B$

② Wishart Dist:

Def: $X_k \overset{\text{indept}}{\sim} N_p(\mu_k, \Sigma)$. $W = \sum_1^n X_k X_k^T \sim W_p(n, \Sigma, \Delta)$

where $\Delta = \sum_1^n \mu_k \mu_k^T$ noncentral para.

properties:

i) $X_k \sim N_p(\mu, \Sigma)$. $A = \sum_1^n (X_k - \bar{X})(X_k - \bar{X})^T \sim W_p(n-1, \Sigma)$

pf: $A = \sum_1^{n-1} Y_k Y_k^T$. $Y_k \sim N(0, \Sigma)$. i.i.d.

ii) $W_i \sim W_p(n_i, \Sigma)$. indept. $\Rightarrow \sum_1^n W_i \sim W_p(\sum_1^n n_i, \Sigma)$

iii) $W \sim W_p(n, \Sigma)$. $C \in M^{m \times p} \Rightarrow CWC^T \sim W_m(n, C\Sigma C^T)$

iv) For $X = (X_1 \cdots X_n)$. $X_k \sim N_p(0, \Sigma)$. i.i.d.

$W = \sum X_k X_k^T = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}_{p-r}^{r} \sim W_p(n, \Sigma)$

Then $W_{11} \sim W_r(n, \Sigma_{11})$. $W_{22} \sim W_{p-r}(n, \Sigma_{22})$

$\Sigma_{21} = 0 \Rightarrow W_{11}$ indept with W_{22}

pf: The first claim is from iii). Besides.

$\forall k$: $f(X_k^1 \cdots X_k^r)$ indept with $g(X_k^{r+1} \cdots X_k^p)$

v) $W_{22.1} = W_{22} - W_{21} W_{11}^{-1} W_{12} \sim W_{p-r}(n-r, \Sigma_{22.1})$

indept with W_{11}. $\Sigma_{22.1} = \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}$

vi) $W \sim W_p(n, \Sigma) \Rightarrow E(W) = n\Sigma$

Pf. Directly check $E(\sum_1^n X_k X_k^T) = n(\sigma_{ij})_{p\times p}$

vii) $X \sim N_{n\times p}(M, I_n \otimes \Sigma)$. $A^T = A$. Then :

$X^T A X \sim W_p(r, \Sigma, M^T A M) \Leftrightarrow A^2 = A$. $r(A) = r$

Cor. $X \sim N_{n\times p}(M, I_n \otimes \Sigma)$, A, B orthonormal proj. Then $X^T A X$ indept with $X^T B X$

$\Leftrightarrow \quad AB = 0$.

③ Hotelling Dist:

Def: For $X \sim N_p(0, \Sigma)$, $W \sim W_p(n, \Sigma)$ indept.

We call $T^2 = n X^T W^{-1} X$ Hotelling Statistics.

Denote $T^2 = T(p, n)$

Rmk: i) For $X \sim N_p(M, \Sigma)$: $T^2 = T(p, n, M)$

ii) It generilizes t-dist: $\frac{X}{\sqrt{S/n}}$ in dim = 1.

properties:

i) $X_k \sim N_p(M, \Sigma)$. $T^2 = (n-1)(\sqrt{n}(\bar{X}-M))^T A^{-1}(\sqrt{n}(\bar{X}-M))$

$= n(n-1)(\bar{X}-M)^T A^{-1}(\bar{X}-M) \sim T^2(p, n-1)$

ii) $T^2 \sim T^2(p, n) \Rightarrow \frac{n-p+1}{np} T^2 \sim F(p, n-p+1)$

Rmk: When $p=1 \Rightarrow T^2 = \left(\frac{X}{\sqrt{S/n}}\right)^2 \sim F(1, n)$

Cor. $X_k \sim N_p(\mu, \Sigma)$. $T^2 = n(n-1)\bar{x}^T A^{-1} \bar{x}$

$$\Rightarrow \frac{n-p}{p}\frac{T^2}{n-1} \sim F(p, n-p, n\mu^T \Sigma^{-1} \mu)$$

iii) $T^2 = T^2(p, n)$ is irrevelant with Σ

Pf: For $u \sim N_p(0, I_p)$. $W_0 \sim W_p(n, I_p)$

$\therefore n u^T W_0^{-1} u \sim n X^T W^{-1} X$. where $X \sim N_p(0, \Sigma)$

iv) $X \sim N_p(0, \Sigma)$. $Y = CX + d$. $|C| \neq 0$. $C \in M^{p \times p}$.

Then: $T_X^2(p, n) = T_Y^2(p, n)$.

④ Wilks Dist:

Def: i) $\left|\frac{A}{n-1}\right|$ is called generalized sample variance

ii) $A_1 \sim W_p(n_1, \Sigma)$. $A_2 \sim W_p(n_2, \Sigma)$. $\Sigma > 0$, $n_1 \geq p$.

$$\Lambda = \frac{|A_1|}{|A_1 + A_2|} \sim \Lambda(p, n_1, n_2) \text{ is Wilks}$$

Statistics.

Rmk: It generalizes F-dist: $F = \frac{S/m}{h/n}$

since $S_x^2 \sim \chi^2_{m_1}$. $S_y^2 \sim \chi^2_{n_1}$

$F = \frac{S_x^2}{S_y^2} \sim F(m_1, n_1)$ can be used

to testing.

Properties:

i) When $n > p$. $\Lambda(p, n, 1) \sim \dfrac{1}{1 + \frac{1}{n} T^2(p, n)}$

<u>pf</u>: $X_k \sim N_p(0, \Sigma)$. i.i.d.

Set $W_1 = \sum_1^n X_k X_k^T \sim W_p(n, \Sigma)$

$W = \sum_1^{n+1} X_k X_k^T \sim W_p(n+1, \Sigma)$

$\therefore \Lambda(p, n, 1) \sim \dfrac{|W_1|}{|W|} = \dfrac{1}{1 + X_{n+1}^T W_1^{-1} X_{n+1}}$. by:

$$|W| = |W_1 + X_{n+1} X_{n+1}^T| = \begin{vmatrix} W_1 & -X_{n+1} \\ X_{n+1}^T & 1 \end{vmatrix} = |W_1|(1 + X_{n+1}^T W_1^{-1} X_{n+1})$$

ii) When $n_2 < p$. $\Lambda(p, n_1, n_2) \sim \Lambda(n_2, p, n_1 + n_2 - p)$

<u>Rmk</u>: It generalizes $F(m, n) \sim \dfrac{1}{F(n, m)}$.

(2) Testing on Mean:

① Single Population:

For $X \sim N_p(\mu, \Sigma)$. $\Sigma > 0$. μ_0 known.

Consider $H_0: \mu = \mu_0$ v.s. $H_1: \mu \neq \mu_0$

i) <u>$\Sigma = \Sigma_0$ Known</u>:

Choose statistics: $T_0^2 = n(\bar{X} - \mu_0)^T \Sigma_0^{-1} (\bar{X} - \mu_0)$

$\sim \chi_p^2$. under H_0.

Rejection Region is $\{T_0^2 \geq \chi_{p,\alpha}^2\}$.

ii) Σ is unkown:

Choose $T^2 = n(\bar{X}-\mu_0)^T \left(\frac{A}{n-1}\right)^{-1}(\bar{X}-\mu_0) \sim T^2(p, n-1)$

Testing statistics: $\frac{n-p}{(n-1)p} T^2 \sim F(p, n-p)$

under H_0. Rejection Region is:

$R = \{ \frac{n-p}{(n-1)p} T^2 \geq F_\alpha(p, n-p) \}$.

② Double Population:

For $X \sim N_p(\mu_1, \Sigma_1)$, $Y \sim N_p(\mu_2, \Sigma_2)$ indept.

Consider $H_0: \mu_1 = \mu_2$ v.s. $H_1: \mu_1 \neq \mu_2$

i) $\Sigma_1 = \Sigma_2$ but unknown:

Note that $\bar{X} - \bar{Y} \sim N_p(0, (\frac{1}{n}+\frac{1}{m})\Sigma)$ under H_0.

from sample $\{X_k\}_1^n$, $\{Y_k\}_1^m$. Then we obtain:

$A_1 + A_2 \sim W_p(n+m-2, \Sigma)$

$\therefore T^2 = (n+m-2)\frac{nm}{n+m}(\bar{X}-\bar{Y})^T(A_1+A_2)^{-1}(\bar{X}-\bar{Y})$

$\sim T^2(p, n+m-2)$ under H_0.

Testing Statistics: $\frac{n+m-p-1}{(n+m-2)p} T^2 \sim F(p, n+m-p-1)$

Rejection Region is:

$\{ \frac{n+m-p-1}{(n+m-2)p} T^2 \geq F_\alpha(p, n+m-p-1) \}$

ii) Σ_1, Σ_2 are known:

$$T^2 = (n+m)(\bar{X}-\bar{Y})^T\left(\frac{\Sigma_1}{n}+\frac{\Sigma_2}{m}\right)^{-1}(\bar{X}-\bar{Y}) \sim \chi^2_{n+m}$$

$$R = \{T^2 \geq \chi^2_{n+m}(\alpha)\}.$$

iii) $n=m$:

Consider $Z_k = X_k - Y_k \sim N_p(0, 2\Sigma)$. reduce to ①.

③ Multi-population:

For $X^i \sim N_p(\mu^i, \Sigma)$. $1 \leq i \leq k$ from k populations.

$H_0: \mu^1 = \mu^2 = \cdots = \mu^k$ v.s. $H_1: \exists i \neq j.\ \mu^i \neq \mu^j$.

If we have samples $\{X_t^i\}_{\substack{1\leq i\leq k\\ 1\leq t\leq n_i}}$. $X_t^i \sim N_p(\mu^i, \Sigma)$.

Denote: $T = \sum_{i=1}^{k}\sum_{t=1}^{n_i}(X_t^i - \bar{X})(X_t^i - \bar{X})^T$. total

$$A = \sum_{i=1}^{k}\sum_{t=1}^{n_i}(X_t^i - \bar{X}^i)(X_t^i - \bar{X}^i)^T.\ \text{intra-class}$$

$$B = \sum_{i=1}^{k} n_i(\bar{X}^i - \bar{X})(\bar{X}^i - \bar{X})^T.\ \text{inter-class}$$

where $\bar{X} = \sum_{i=1}^{k}\sum_{t=1}^{n_i} X_t^i / n$. $n = \sum_1^k n_i$

$$\bar{X}^i = \sum_1^{n_i} X_t^i / n_i$$

$\Rightarrow$ If $|T|$ is invariant. Under H_0. $|B|$ will be small and $|A|$ will be relatively larger. $(T = A+B)$

Choose $\tilde{T} = \dfrac{|A|}{|A+B|}$. since $A = \Sigma A_i$. $A_i \sim W_p(n_i-1, \Sigma)$

$B \sim W_p(k-1, \Sigma)$. $\therefore\ \tilde{T} \sim \Lambda_p(n-k, k-1)$ under H_0.

Then $R = \{ T \le \lambda_\alpha \}$. reject. region.

<u>Rmk</u>: When $p=1$. Note that $\dfrac{B/k-1}{A/n-k}$

$\sim F(k-1, n-k)$. If we consider

$H_0: a^T\mu_1 = \cdots a^T\mu_k$ v.s. $H_1: \exists i \neq j . \mu_i \neq \mu_j$.

Then consider $a^T X^i \sim N_1(a^T\mu^i, a^T\Sigma a)$

$$\therefore F = \frac{a^T B a / k-1}{a^T A a / n-k} \sim F(k-1, n-k)$$

Choose $\lambda = \max\limits_a \dfrac{a^T B a}{a^T A a} \cdot \dfrac{n-k}{k-1}$. compare to λ_α.

(3) <u>Test on Covariance</u>:

① <u>Single population</u>:

Consider $X_i \sim N_p(\mu, \Sigma)$. $\Sigma > 0$.

$H_0: \Sigma = \Sigma_0$ v.s. $H_1: \Sigma \neq \Sigma_0$

i) <u>$\Sigma_0 = I_p$</u>:

By MLE: $\hat{\lambda} = \max\limits_{\mu} L(\mu, I_p) / \max\limits_{\mu, \Sigma} L(\mu, \Sigma)$

$$= \left(\frac{e}{n}\right)^{np/2} |A|^{\frac{n}{2}} e^{-\frac{1}{2}\operatorname{tr}(A)}$$

$\xi = -2\ln\hat{\lambda} \sim \chi^2\left(\frac{p(p+1)}{2}\right)$ when n is large

ii) <u>$\Sigma_0 \neq I_p$</u>:

Set $Y = QX$. $Y \sim N_p(Q\mu, I_p)$ under H_0.

i.e. $Q\Sigma_0 Q^T = I_p$. reduce to i).

iii) For $H_0: \Sigma = \sigma^2 \Sigma_0$. σ^2 unknown:

$$MLE: \hat{\lambda} = |\Sigma_0^{-1} S|^{\frac{n}{2}} / (tr(\Sigma_0^{-1} S)/p)^{\frac{pn}{2}}$$

② Multi-population:

Consider k populations $N_p(\mu^t, \Sigma_t)$. $1 \le t \le k$.

We have samples $\{X_i^t\}_{1 \le i \le n_t}$. Hypothesis:

$H_0: \Sigma_1 = \Sigma_2 = \cdots = \Sigma_k$ v.s. $H_1: \exists i \ne j.\ \Sigma_i \ne \Sigma_j$.

$$\hat{\lambda} = \left| \frac{\Sigma A_t}{\Sigma n_t} \right|^{-\frac{\Sigma n_t}{2}} / \prod_1^k \left| \frac{A_t}{n_t} \right|^{-\frac{n_t}{2}}$$

Rmk: If consider $H_0: \Sigma_1 = \cdots = \Sigma_k.\ \mu_1 = \cdots \mu_k$

$$\hat{\lambda} = \frac{n^{np/2}}{\prod_1^k n_t^{\frac{n_t p}{2}}} \cdot \frac{\prod_1^k |A_t|^{\frac{n_t}{2}}}{|T|^{\frac{n}{2}}},\quad T = \Sigma\Sigma(X_i^t - \bar{X}^t)(X_i^t - \bar{X}^t)^T.$$

(4) Test on independence:

For $X \sim N_p(\mu, \Sigma)$. $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$. Test:

$H_0: \Sigma_{12} = 0$ v.s. $H_1: \Sigma_{12} \ne 0$. Denote $\Sigma_0 = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}$

$MLE: \hat{\lambda} = (|S|/|S_1||S_2|)^{\frac{n}{2}}$. $S_1 = S^2(X_1)$. $S_2 = S^2(X_2)$.

Rmk: For $X = \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix}$, $H_0 = \Sigma_{ij} = 0.\ i \ne j$.

$$\hat{\lambda} = (|S| / \prod_1^k |S_i|)^{\frac{n}{2}}$$

(5) Likelihood Ratio Test:

Find $\hat{\lambda} = \dfrac{L^*(\Theta_0)}{L^*(\Theta)}$. $-2\ln\hat{\lambda} \sim \chi_p^2$. $p = dim(\Theta) - dim(\Theta_0)$

(6) Linear Hypothesis:

For $X^i \sim N_p(\mu, \Sigma)$. $1 \le i \le n$. populations.

consider H_0: $A\mu = a$ v.s. H_1: $A\mu \ne a$.

Then set: $Y^i = AX^i \sim N_p(A\mu, A\Sigma A^T)$

Construct $T = n(\bar{Y} - a)^T (A\Sigma A^T)^{-1} (\bar{Y} - a)$

$\sim \chi_q^2$. $\bar{Y} = A\bar{X}$.

(7) Confidence Intervals:

Def: If $X \sim N_p(\mu, \Sigma)$. $\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}$. $\prod_1^p A_i$

is simultaneous confidence interval of

μ if $A_j = [\hat{a}_j(X_1 \cdots X_n), \hat{b}_j(X_1 \cdots X_n)]$. s.t.

$P(\mu_j \in A_j) \ge 1-\alpha$. $\forall j$ (with level $1-\alpha$)

Rmk: It's convenient to check a test.

① First Method:

consider $a^T X \sim N_1(a^T\mu, a^T\Sigma a)$

$\therefore T_a = \dfrac{a^T\bar{X} - a^T\mu}{\sqrt{a^T S a / n}} \sim t(n-1)$

set $P(T_{a_i} \ge t_{\frac{\alpha_i}{2}}(n-1)) = \alpha_i$. $\sum_1^n \alpha_i = \alpha$.

$\therefore P(a_i\mu \in [a_i^T\bar{X} \pm \sqrt{\frac{a_i^T S a_i}{n}}\, t_{\frac{\alpha_i}{2}}]) = 1-\alpha_i$

$\Rightarrow P(a_i\mu \in [a_i^T\bar{X} \pm t_{\frac{\alpha_i}{2}}\sqrt{\frac{a_i^T S a_i}{n}}], \forall i) \ge 1 - \sum_1^n \alpha_i = 1-\alpha$

Let $a_i = e_i$. We can obtain interval for each variable.

② Second Method:

$$p\left(\frac{n[a^T(\bar{X}-\bar{\mu})]^T[a^T(\bar{X}-\bar{\mu})]}{a^T S a} \le t^2_{\frac{\alpha}{2}} \right) = 1-\alpha$$

Find a. st. $\max_a \frac{n(a^T(\bar{X}-\mu))^2}{a^T S a} = n(\bar{X}-\mu)^T S^{-1}(\bar{X}-\mu) =: T^2$

If $T^2 \le c^2$. Then $\forall a$ holds.

Set $c = \sqrt{\frac{(n-1)p}{(n-p)} F_\alpha}$. $p(a^T\mu \in [a^T\bar{X} \pm \sqrt{c^2 a^T S a / n}]) = 1-\alpha$.

Let $a^T = e_i$. Then we obtain each interval.

$p(\mu_i \in [\bar{X}_i \pm \sqrt{c^2 \frac{s_{ii}}{n}}], \forall i) \ge 1-\alpha$.