Multinormal Dist.

(1) Definitions:

<u>Def</u>: X has r dimensional multinormal dist if

$\exists A \in M^{r\times n}$. $b \in \mathbb{R}^r$. $X = AZ + b$. where

$Z = (Z_1 \cdots Z_n)^T$. $Z_i \sim N(0,1)$. i.i.d.

We denote : $X \sim N_r(b.\ AA^T)$

<u>Rmk</u>: If $r(AA^T) < r$. Then density of X

doesn't exist (It should restrict on the

nonsingular subspace for density).

<u>Thm</u>. $X \sim N_n(\mu, \Sigma) \iff \xi = \Sigma l_i X_i \sim N(\mu^l, \Sigma^l)$.

for $\forall\ l = (l_1 \cdots l_n)^T \in \mathbb{R}^n$.

<u>Pf</u>: By ch.f. ($\Leftarrow$) First recover : $E(X)$. $Cov(X)$.

<u>Rmk</u>: If any marginal dist $\sim N(\tilde{\mu}, \tilde{\Sigma})$.

It's not neccessary $X \sim N_n(\mu, \Sigma)$.

<u>e.g.</u> $f(x_1, x_2) = \frac{1}{2\pi} e^{-\frac{x_1^2+x_2^2}{2}} (1 + \sin x_1 \sin x_2)$

$\Rightarrow X_1 \sim N(0,1)$. $X_2 \sim N(0,1)$. (Consider : f

$= \frac{1}{2\pi} e^{-\frac{x_1^2+x_2^2}{2}} g(x,y) \geq 0$. $\int f dx = \int f dy = \square$

Set $g = 1 + \sin x \sin y$. $\sin x \sin y$ odd w.r.t. x or y)

(2) Properties:

① Thm. (Linear Transformation)

$X \sim N_p(\mu, \Sigma)$. $A \in M^{q \times p}$. $c \in \mathbb{R}^q$. Then

$AX + c \sim N_q(A\mu + c, A\Sigma A^T)$.

Pf: write in origin form: $X = \mu + BZ$. $BB^T = \Sigma$.

Cor. $Y = \Sigma^{-\frac{1}{2}}(X - \mu) \sim N_p(0, I_p)$. $Y^T Y \sim \chi_p^2$.

② Indept:

Thm. $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_p(\mu, \Sigma)$, $X_1 \in \mathbb{R}^r$. $X_2 \in \mathbb{R}^{p-r}$

Define $X_{2.1} = X_2 - \Sigma_{21}\Sigma_{11}^{-1}X_1$. $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$

Then X_1 indept with $X_{2.1} \sim N_{p-r}(\mu_{2.1}, \Sigma_{22.1})$

$\mu_{2.1} = \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1$. $\Sigma_{22.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$

Pf: $X_1 = (I_r \ 0)X$. $X_{2.1} = (\Sigma_{21}\Sigma_{11}^{-1} \ I_{p-r})X$.

check $cov(X_1, X_{2.1}) = 0$.

Rmk: It's like some matrix transformation

to eliminate Σ_{12}, Σ_{21}.

cor. $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_p(\mu, \Sigma)$. $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$

X_1 indept with $X_2 \iff \Sigma_{12} = 0$.

Cor. AX indept with $BX \Leftrightarrow A\Sigma B = 0$.

Thm. (Condition)

$$X_2 \mid X_1 = x_1 \sim N_{p-r}(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1), \Sigma_{22.1})$$

Pf: Write $X_2 = X_{2.1} + \Sigma_{21}\Sigma_{11}^{-1}X_1$, $X_{2.1}$ indept with X_1.

Thm. If $X_1 \sim N_r(\mu_1, \Sigma_{11})$, $X_2 \mid X_1 = x_1 \sim N(Ax_1 + b, \Lambda)$, Λ is indept with x_1. Then $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N(\mu, \Sigma)$.

$$\mu = \begin{pmatrix} \mu_1 \\ A\mu_1 + b \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{11}A^T \\ A\Sigma_{11} & \Lambda + A\Sigma_{11}A^T \end{pmatrix}$$

Pf: Find $E(e^{it^TX_2}) = E(E(e^{it^TX_2} \mid X_1))$

③ Conditional Approxi.

Note $E(X_2 \mid X_1)$ is Predict of X_2. w.r.t $f(X_1)$

Write $X_2 = E(X_2 \mid X_1) + u$ $\therefore u \sim N(0, \Sigma_{22.1})$

$\Rightarrow X_2 = \beta_0 + BX_1 + u$. $\beta_0 = \mu_2 - B\mu_1$. $B = \Sigma_{21}\Sigma_{11}^{-1}$.

Denote: B is regressional coefficient of X_2 to X_1

(3) Normal Matrix:

Def: $X^i = (X_{i1} \cdots X_{ip})^T \sim N_p(\mu, \Sigma)$. $X = (X_{ij})_{n\times p}$ if

$$\mathrm{Vec}(X^T) \sim N_{np}(J_n \otimes \mu, I_n \otimes \Sigma).$$

Rmk: $AXB^T + D \sim N_{kq}(AMB^T + D, (AA^T) \otimes (B\Sigma B^T))$

$D \in M^{k\times q}$. $M = J_n \otimes \mu$. $A \in M^{k\times n}$. $B \in M^{q\times p}$