ANOVA

(1) One-Way ANOVA:

① Projection Decomposition:

Consider: $Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$, $i = 1, \cdots t$, $j = 1, \cdots n_i$

Set $n = \sum_1^t n_i$, $\varepsilon \sim N(0, \sigma^2 I)$. $Y = X\beta + \varepsilon$.

Design matrix $X = (J \; X_1 \cdots X_t)_n$, where X_k:

$X_k = (t_{ij})$, $t_{ij} = \delta_{ik}$, has n_k "1", $n - n_k$ "0".

Rmk: i) $J = \sum_1^t X_k$. ii) X_k's are orthogonal.

$\Rightarrow C(X) = C(Z)$, $Z = (X_1 \cdots X_t)$, $r(Z) = t$.

We obtain:

$M_X = Z(Z^T Z)^{-1} Z^T$, $Z^T Z = \mathrm{diag}(n_1 \cdots n_t)$, by $\begin{cases} X_i^T X_j = 0 \\ X_i^T X_i = n_i \end{cases}$

$\Rightarrow M_X = \sum n_i^{-1} X_i X_i^T = \mathrm{blk\,diag}\{ n_1^{-1} J_{n_1}^{n_1} \cdots n_t^{-1} J_{n_t}^{n_t} \}$.

Denote $M_\mu = \frac{1}{n} J_n^n$, projects on $C(J)$.

$\therefore M_X = M_\mu + M_\alpha = M_\mu + (M_X - M_\mu)$, $C(M_\alpha) \perp C(M_\mu)$

Besides $r(M_\mu) = 1$, $r(M_\alpha) = t-1$.

Rmk: $\mathbb{R}^n = C(M_\mu) + C(M_\alpha) + C(I - M_X)$

② Estimation:

- Note that $\mu + \alpha_i$ is estimable, $\forall \, 1 \le i \le t$.

But μ, α_i are not estimable, $\forall \, 1 \le i \le t$.

By $X\hat\beta = MY = \begin{pmatrix} J_{n_1} \bar{Y}_{1\cdot} \\ \vdots \\ J_{n_t} \bar{Y}_{t\cdot} \end{pmatrix}$. $\therefore \hat\mu + \hat\alpha_i = \bar{Y}_{i\cdot} = \frac{1}{n_i} \sum_k^{n_i} Y_{ik}$

Rmk: If add linear restriction: $\sum_1^t n_i \alpha_i = 0$

$\exists \{n_i\} \neq \{0\}$. $\Rightarrow \mu = \sum_1^t n_i(\mu + \alpha_i) / n$.

$\Rightarrow \hat{\mu} = \bar{Y}$, $\hat{\alpha}_i = \bar{Y}_i - \bar{Y}$.

Def: A contrast in a One-Way ANOVA is a function $\sum_1^t \lambda_i \alpha_i$, s.t. $\sum_1^t \lambda_i = 0$.

Rmk: Write $\lambda = (0, \lambda_1 \cdots \lambda_t)^T$. $\sum \lambda_i = 0$. Then contrast is $\lambda^T \beta$. $\lambda^T = \ell^T X$. One possible choice is $\ell^{*T} = (\frac{\lambda_1}{n_1} J_{n_1}^T, \cdots, \frac{\lambda_t}{n_t} J_{n_t}^T) \in C(X)$. Since: $M\ell = M\ell^* = \ell^*$ is unique. It fixs the proj.

Thm. $\ell^T X \beta$ is a contrast $\Leftrightarrow \ell^T J_n = 0$

Pf: $(\Rightarrow)$ $\sum \ell_i(\mu + \tau_i) = \sum \ell_i \tau_i$ $\therefore \ell^T J_n = 0$.

$(\Leftarrow)$ $\ell^T (J_n \; Z) \beta = (0 \;\; \ell^T Z) \beta$ doesn't involve μ.

Check $\lambda^T J_{t+1} = \ell^T X J_{t+1} = 0$. (i.e. $\sum \lambda_i = 0$)

From $X J_{t+1} = J_n$. We're done.

prop. $\ell^T X \beta$ is a contrast $\Leftrightarrow M\ell \in C(M_\alpha)$

Pf: $\Leftrightarrow$ $\ell^T J_n = 0$ $\therefore$ $\ell^T M J_n = 0$. $M\ell \in C(M_\alpha)$

prop. $C(M_\alpha) = \{ \ell \mid \ell = [t_{ij}],\ t_{ij} = \lambda_i / n_i,\ \sum_1^t \lambda_i = 0 \}$

Pf: "$\supseteq$" Note $\ell^T J_n = 0$ $\therefore \ell^T$ determines a contrast.

$\Rightarrow M\ell = \ell \in C(M_\alpha)$

"$\subseteq$" $\ell \in C(M_\alpha) \Rightarrow \ell^T J = 0$ $\therefore \ell^T X \beta$ is contrast.

$\therefore M\ell = \ell^*$ (in Rmk). $\overset{\ell \in C(M)}{\Longrightarrow} \ell = \ell^*$.

Rmk: To test contrast: $H_0: \lambda^T\beta = 0$. $\lambda^T = (0, \lambda_1 \cdots \lambda_t)$. $\Sigma\lambda_i = 0$.

$$\Rightarrow \lambda^T\hat{\beta} = \ell^T M Y = \sum_1^t \lambda_i \bar{Y}_i \text{. } \ell^T M \ell = (M\ell)^T(M\ell) = \sum_1^t \frac{\lambda_i^2}{n_i}$$

$$F = \frac{(\ell^T MY)^T(\ell^T M\ell)^{-1}(\ell^T MY)}{MSE} = \frac{(\Sigma\lambda_i \bar{Y}_i)^2}{MSE(\sum_1^t \lambda_i^2/n_i)} \sim F(1, n-t, \gamma)$$

$$\gamma = \frac{(\Sigma\lambda_i(\mu+\alpha_i))^2}{\sigma^2 \Sigma \lambda_i^2/n_i} \text{. } \quad \gamma = 0 \text{ under } H_0.$$

(1) Orthogonal contrasts:

Def: Contrasts $\lambda_1^T\beta$, $\lambda_2^T\beta$. $\lambda_i^T = (0, \lambda_{i1} \cdots \lambda_{it})$ orthogonal if $\sum_{k=1}^t \frac{\lambda_{1k}\lambda_{2k}}{n_k} = 0$.

Rmk: Since $r(C(M_\alpha)) = t-1$. We can break it into $t-1$ orthogonal subspace. $M_\alpha = \sum_1^{t-1} M_i$

From: $\lambda_i^T = \ell_i^T X$. $\Rightarrow \ell_1^T M \ell_2 = (M\ell_1)^T(M\ell_2)$

$$= \sum_{k=1}^t \sum_{l=1}^{n_k} \frac{\lambda_{1k}\lambda_{2k}}{n_k^2} = \sum_1^t \frac{\lambda_{1k}\lambda_{2k}}{n_k} = 0.$$

So. $\lambda_i^T\beta \perp \lambda_j^T\beta \iff \ell_i^T M \ell_j = 0 \iff \ell_i^T M_\alpha \ell_j = 0$

(since $\ell_i^T M_\mu = 0$) $\iff \ell_i^T \ell_j = 0$. ($\ell_i, \ell_j \in C(X)$)

For test one of contrast $\ell_i^T X\beta$. The proj. $M_i = \frac{(M\ell_i)^T(M\ell_i)}{(\ell_i^T M\ell_i)}$

$= \frac{\ell_i\ell_i^T}{\ell_i^T\ell_i}$ if $\ell_i \in C(M)$

Thm. To test $H_0: \ell_i^T X\beta = 0$. $F = \frac{\|M_i Y\|^2}{MSE} = \frac{(\ell_i^T Y)^2}{(\ell_i^T\ell_i)MSE}$

$\sim F(1, n-t, \gamma)$. if $\ell_i \in C(X)$.

(2) Multifactor Analysis of Variance:

(1) Decomposition:

Consider two-way balanced ANOVA without the interaction: $Y_{ijk} = \mu + \alpha_i + \eta_j + \varepsilon_{ijk}$. for $1 \le i \le a$ $1 \le j \le b$. $k = 1, 2, \cdots N$. Denote $n = abN$. total number.

Write: $Y = X\beta + \varepsilon$. $X = (J\ X_1\ X_2 \cdots X_a\ X_{a+1} \cdots X_{a+b})_{n \times (a+b+1)}$

$$X_r = (t_{ijk}) \quad t_{ijk} = \delta_{ir} \quad 1 \le r \le a.$$

$$X_s = (t_{ijk}) \quad t_{ijk} = \delta_{j(s-a)}. \quad a+1 \le s \le a+b$$

$$C(M) = C(M_\mu) + C(M_\alpha) + C(M_\eta)$$

Define: $Z = (J\ Z_1 \cdots Z_a\ Z_{a+1} \cdots Z_{a+b})$. $Z_r = X_r - \frac{X_r^T J}{J^T J} J$

so that $Z_i \perp J$. for $1 \le i \le a+b$.

since $J^T J = n = abN$. $X_r^T J = \begin{cases} bN. & 1 \le r \le a. \\ aN. & a+1 \le r \le a+b \end{cases}$

$$\therefore Z_r = \begin{cases} X_r - \frac{1}{a} J. & 1 \le r \le a \\ X_r - \frac{1}{b} J. & a+1 \le r \le a+b \end{cases}$$

Rmk: $C(J\ X_1 \cdots X_a) = C(J\ Z_1 \cdots Z_a)$

$C(J\ X_{a+1} \cdots X_{a+b}) = C(J\ Z_{a+1} \cdots Z_{a+b})$

Besides. $C(Z_1 \cdots Z_a) \perp C(Z_{a+1} \cdots Z_{a+b})$

By $Z_s^T Z_r = \sum_{ijk} (\delta_{j(s-a)} - \frac{1}{b})(\delta_{ir} - \frac{1}{a})$

$$= N - \frac{aN}{a} - \frac{bN}{b} + N = 0$$

Denote: $C(M_\mu) = C(J)$. $C(M_\alpha) = C(Z_1 \cdots Z_a)$

$C(M_\eta) = C(Z_{a+1} \cdots Z_{a+b})$.

$$\Rightarrow \sum_{i=1}^{a} Z_i = \sum_{a+1}^{b} Z_i = J - J = 0. \quad \begin{cases} r(C(M_\alpha)) = a-1 \\ r(C(M_\eta)) = b-1 \end{cases}$$

Table: $M_\alpha Y = (t_{ijk})$ $t_{ijk} = \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot}$

$SS(\alpha) = Y^T M_\alpha Y = bN \sum_i^a (\bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot})^2$. $df = a-1$.

$M_\eta Y = (t_{ijk})$. $t_{ijk} = \bar{Y}_{\cdot j \cdot} - \bar{Y}_{\cdots}$ $\quad df = b-1$

$SS(\eta) = Y^T M_\eta Y = aN \sum_1^b (\bar{Y}_{\cdot j \cdot} - \bar{Y}_{\cdots})^2$.

$(I-M) Y = (I - \frac{1}{n} J_n - M_\tau - M_\eta) Y = (t_{ijk})$. where

$t_{ijk} = y_{ijk} - \bar{Y}_{\cdots} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j \cdot} + 2\bar{Y}_{\cdots} = Y_{ijk} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\cdots}$

$SSE = \Sigma (Y_{ijk} - \bar{Y}_i - \bar{Y}_j + \bar{Y})^2$.

② Contrast:

. Estimation and testing in balanced two-way ANOVA is done exactly as one-way ANOVA by ignoring other group of parameters.

Thm. $\lambda^T \beta = \ell^T X \beta$ is contrast in α_i's $\Leftrightarrow \ell^T M = \ell^T M_\tau$

Rmk: Then $\lambda^T \hat{\beta} = \ell^T M Y = \ell^T M_\tau Y$. which is estimation ignores η_j's.

Pf: $\lambda^T \beta = \sum_1^a c_i \alpha_i$ with $\Sigma c_i = 0 \quad \Leftrightarrow \lambda^T J_{a+b+1} = 0$.

$\lambda^T = (0, c_1 \cdots c_a, 0 \cdots 0)$. $\Leftrightarrow$ (by $\lambda^T = \ell^T X = (\ell^T J, \cdots \ell^T X_{a+b+1})$)

$\ell \perp C(J, Z_{a+1} \cdots Z_{a+b}) = C(M - M_\tau)$. $\ell^T X_k = 0$. $a+1 \le k$.

$\Leftrightarrow \ell^T (M - M_\tau) = \ell^T M - \ell^T M_\tau = 0$

prop. If we have a contrast in $\vec{\tau}$. $\ell^T = (c_1 J_{bN}^T, \cdots c_a J_{bN}^T)/bN$

$\sum_1^a c_i = 0$ Then $\ell^T \in C(M_\tau)$. $\ell^T M Y = \ell^T Y = \sum_1^a c_i \bar{Y}_{i\cdots}$

$Var(\ell^T M Y) = \sigma^2 \ell^T M \ell = \frac{\sigma^2}{bN} \sum_1^a c_i^2$.

For test H_0: $\ell^T X \beta = \lambda^T \beta = 0$. $F = \frac{(\ell^T Y)^2}{(\ell^T \ell) MSE} \sim$

$F(1, n-a-b+1, \gamma)$

Rmk: $\lambda_1^T \beta = \ell_1^T X \beta \perp \lambda_2^T \beta = \ell_2^T X \beta \Leftrightarrow \ell_1^T M_\tau \ell_2 = 0 \Leftrightarrow \Sigma c_{1i} c_{2i} = 0$

(3) Balanced Two-Way ANOVA

With Interactions:

Consider $Y_{ijk} = \mu + \alpha_i + \eta_j + \gamma_{ij} + \varepsilon_{ijk}$. $\varepsilon_{ijk} \overset{iid}{\sim} N(0, \sigma^2)$

$i = 1, \cdots a$. $j = 1, \cdots b$. $k = 1, \cdots N$. γ_{ij} is interaction terms.

Rmk: It means $Y_{ijk} = \mu_{ij} + \varepsilon_{ijk}$. This model has no interaction if:

i) $\mu_{ij} = a_i + b_j$ ii) $\mu_{ij} - \mu_{ij'}$ indept with i. $\forall j, j'$

iii) $\mu_{ij} - \mu_{i'j}$ indept with j. $\forall i, i'$.

iv) $\mu_{ij} - \mu_{ij'} - \mu_{i'j} + \mu_{i'j'}$ is const. $\forall (i, j, i', j')$.

Actually, we can write:

$$\mu_{ij} = \bar{\mu}_{..} + (\bar{\mu}_{i.} - \bar{\mu}_{..}) + (\bar{\mu}_{.j} - \bar{\mu}_{..}) + (\mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..})$$

$$= \mu + \alpha_i + \eta_j + \gamma_{ij}.$$

① Projection:

Write in $Y = X\beta$. $X = (J, X_1 \cdots X_a, X_{a+1} \cdots X_{a+b}, X_{a+b+1} \cdots X_{a+b+ab}) \in M^{n \times (a+b+ab+1)}$. Reindex: X_{a+b+k}'s into $X_{(1,1)} \cdots X_{(1,b)} \cdots X_{(a,b)}$

Then $X_1 \sim X_{a+b}$ is same as before. For $X_{(i,j)}$:

$X_{(r,s)} = (t_{ijk})$. $t_{ijk} = \delta_{(i,j)(r,s)}$

$$\Rightarrow J = \sum_{i=1}^{a} \sum_{j=1}^{b} X_{(i,j)}. \quad X_r = \sum_{i=1}^{b} X_{(r,i)}, \ 1 \le r \le a.$$

$$X_s = \sum_{1}^{a} X_{(i, s-a)}, \ a+1 \le s \le a+b.$$

$$\Rightarrow C(X) = C(X_{a+b+1} \cdots X_{a+b+ab}).$$

Break into orthogonal spaces:

$C(M) = C(M_\mu) + C(M_\alpha) + C(M_\eta) + C(M_\gamma)$, where M_α. M_η are obtained in one-way case.

$\Rightarrow M_\gamma = M - M_\mu - M_\alpha - M_\eta$ with $df = (a-1)(b-1)$

$M = X_\gamma (X_\gamma^T X_\gamma)^{-1} X_\gamma^T = \text{Blk diag}\{\frac{1}{N} J_N^N \cdots \frac{1}{N} J_N^N\}$. $r(M) = ab$

There're ab such blocks.

$\Rightarrow M_\gamma Y = (\bar{Y}_{ij\cdot} - \bar{Y}_{\cdots}) - (\bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdots}) - (\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdots})$

$= \bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\cdots}$.

$E(Y^T M_\gamma Y) = \sigma^2 (a-1)(b-1) + \beta^T X^T M_\gamma X \beta$

$= \sigma^2 (a-1)(b-1) + N \sum_i \sum_j (\gamma_{ij} - \bar{\gamma}_{i\cdot} - \bar{\gamma}_{\cdot j} + \bar{\gamma}_{\cdot\cdot})^2$

Rmk: Expected Values of $Y^T M_\alpha Y$. $Y^T M_\eta Y$ are different in no interaction case:

$E(Y^T M_\alpha Y) = \sigma^2 (a-1) + bN \sum^a (\alpha_i + \bar{\gamma}_{i\cdot} - \bar{\alpha} - \bar{\gamma}_{\cdot\cdot})^2$

$E(Y^T M_\eta Y) = \sigma^2 (b-1) + aN \sum^b (\eta_j + \bar{\gamma}_{\cdot j} - \bar{\eta} - \bar{\gamma}_{\cdot\cdot})^2$

So. to test "No α treatment" is:

$H_0: \alpha_1 + \bar{\gamma}_{1} = \cdots = \alpha_a + \bar{\gamma}_{a} \iff H_0: M_\alpha X\beta = 0$

② Contrast:

Note that: Any estimable functions must involve γ_{ij}'s. since $\ell^T X = \lambda^T = (\lambda_1 \cdots \lambda_{a+b+ab})$. if: $\lambda_{a+b+1} \sim \lambda_{a+b+ab} = 0 \Rightarrow \ell^T X_{a+b+i} = 0$. $1 \le i \le ab$.

$\Rightarrow \ell^T X = 0$ since $C(X) = C(X_\gamma)$.

For contrasts in M_α. M_η spaces:

$\lambda^T\beta = \ell^T X\beta = \ell^T M_\alpha X\beta$, or $\ell^T M_\eta X\beta$. For M_α case:

$M_\alpha X\beta = (\alpha_i + \bar{\gamma}_{i\cdot} - \bar{\alpha} - \bar{\gamma}_{\cdot\cdot} \cdots) \in M^{n\times 1}$. Then:

Contrast is in form: $\sum \lambda_i (\alpha_i + \bar{\gamma}_{i\cdot})$, $\sum \lambda_i = 0$

$\sum \mu_i (\eta_j + \bar{\gamma}_{\cdot j})$, $\sum \mu_i = 0$

For contrasts in interaction space:

$H_0: \ell^T X\beta = 0$ constriant on interaction space iff

$M\ell = M_\gamma \ell$. i.e. $\ell \in C(M_\mu + M_\alpha + M_\eta)^\perp \Leftrightarrow \ell X_i = 0$.

for $0 \le i \le a+b$ and $M\ell = \ell$. $(X_0 = J)$

Thm. $\lambda^T = (\lambda_1 \cdots \lambda_a) \in M^{1\times a}$. $c^T = (c_1 \cdots c_b) \in M^{1\times b}$.

where $\sum_1^a \lambda_i = \sum_1^b c_i = 0$. (i.e. coefficient of contrasts in α. η space)

Then: $\lambda^T \otimes c^T$ is coefficient of contrasts in interaction space.

Thus, $\ell^T = \frac{1}{N}(\lambda^T \otimes c^T) \otimes J_N^T$. correspond vector

Pf. check: 1°) $M\ell = \ell$. 2°) $\ell^T X_i = 0$. $0 \le i \le a+b$.

Rmk. It doesn't characterize all contracts in the interaction space. Such vectors don't form LS.

Thm. $Q \in M^{a\times b}$. $J_a^T Q = 0$. $Q J_b = 0$. Then: We have

$\ell^T = \frac{1}{N}(q_{11} \otimes J_N^T, \cdots\ q_{ab} \otimes J_N^T)$ satisfies that

$M\ell = \ell$. $\ell^T X_i = 0$. $0 \le i \le a+b$. It precisely express all forms of vector in contrast of interaction

Rmk: i) Not every a can be written in the form: $(\lambda^T \otimes c^T) \otimes J_N^T$. since $r(\lambda^T \otimes c^T) = 1$

ii) For $c^T = (\lambda^T \otimes c^T) \otimes J_N^T$. Then:

$c^T M Y = \sum^a \sum^b c_j \lambda_i \bar{Y}_{ij\cdot}$

$c^T M c = \sum \sum c_j^2 \lambda_i^2 / N$

iii) $r(V = \{ a \mid a J_b = 0.\ J_a^T a = 0 \}) = (a-1)(b-1)$

iv) To break down interaction span:

consider the form: $(\lambda^T \otimes c^T) \otimes J_N^T$.

$c^T M_Y c_* = 0 \iff \sum \lambda_i \lambda_i^* \sum c_j c_j^* = 0$.

There're $(a-1)(b-1)$ ways. So forms an orthogonal basis.

③ Three or Higher way:

Consider $Y_{ijkl} = \mu + \alpha_i + \eta_j + \gamma_k + (\alpha\eta)_{ij} + (\alpha\gamma)_{ik}$
$+ (\eta\gamma)_{jk} + (\alpha\eta\gamma)_{ijk} + \varepsilon_{ijkl}$. where

$1 \le i \le a.\ 1 \le j \le b.\ 1 \le k \le c.\ 1 \le l \le N.\ n = abcN.$

It's similar with ② since we can ignore other para's.

e.g. $SS(\alpha) = bcN \sum (\bar{y}_{i\cdots} - \bar{y}_{\cdots})^2$

$SS(\alpha\gamma) = bN \sum (\bar{y}_{i\cdot k\cdot} - \bar{y}_{i\cdots} - \bar{y}_{\cdot\cdot k\cdot} + \bar{y}_{\cdots})^2$

$C(M_\alpha) = \{ V \mid V = [V_{ijkl}].\ V_{ijkl} = a_i.\ \sum a_i = 0 \}$

$C(M_{\alpha\gamma}) = \{ U \mid U = [V_{ijkl}].\ V_{ijkl} = r_{ik}.\ \sum_i r_{ik} = \sum_k r_{ik} = 0 \}$

(4) Unified Approach for Balanced ANOVA:

· We can develop a unified way to obtain orthogonal proj. in arbitrary k-way ANOVA.

Consider: $Y_{ijk} = \mu + \alpha_i + \eta_j + \gamma_{ij} + \varepsilon_{ijk}$. (two-way case)

Denote: $P_s = \frac{1}{s} J_s J_s^T$. $Q_s = I_s - P_s$.

i) Computing M_M:

Note that $J_n = J_a \otimes J_b \otimes J_N$. Since $M_M = J_n (J_n^T J_n)^- J_n^T$

$$\Rightarrow M_M = (J_a \otimes J_b \otimes J_N)(abN)^-(J_a \otimes J_b \otimes J_N)^T$$
$$= J_a J_a^T/a \otimes J_b J_b^T/b \otimes J_N J_N^T/N$$
$$= P_a \otimes P_b \otimes P_N$$

ii) Computing M_α:

Note that α space is $C(Q_a \otimes J_b \otimes J_N)$

By $(A \otimes B)^- = (A^- \otimes B^-)$:

$$M_\alpha = Q_a Q_a^- Q_a \otimes J_b J_b^T/b \otimes J_N J_N^T/N$$
$$= Q_a \otimes P_b \otimes P_N$$

iii) Computing M_η:

Similarly. $M_\eta = P_a \otimes Q_b \otimes P_N$

iv) Computing M_γ:

First note γ space: $C(Q_a \otimes Q_b \otimes J_N)$

$\therefore M_\gamma = Q_a \otimes Q_b \otimes P_N$.

Rmk: $M = M_M + M_\alpha + M_\gamma + M_\eta = I_a \otimes I_b \otimes P_N$.

Rmk: In three way: e.g. $C(M_\alpha) = C(Q_a \otimes J_b \otimes J_k \otimes J_N)$

$\therefore M_\alpha = Q_a \otimes P_b \otimes P_k \otimes P_N$.

(5) Unbalanced Two-Way ANOVA:

Consider $Y_{ijk} = \mu + \tau_i + \eta_i + \varepsilon_{ijk}$. $i = 1, \cdots a$ $j = 1 \cdots b$. $k = 1, \cdots n_{ij}$.

When the number of replicates is unequal. We can't use the method before to decompose $L(X)$ into ortho. subspaces.

① Proportion Case:

Def: The model is proportional if $\dfrac{n_{ij}}{n_{i'j}} = \dfrac{n_{ij'}}{n_{i'j'}}$

Rmk: In this model. Orthogonal subspace can be attained as. before:

prop. (n_{rs}) is proportional in the sense above.

Then: $n_{rs} = \dfrac{n_{r\cdot} n_{\cdot s}}{n_{\cdot\cdot}}$

Pf: $n_{rs} \sum_i \sum_j n_{ij} = \sum_i \sum_j n_{rj} n_{is} = n_{r\cdot} n_{\cdot s}$

Denote: $X = (J\ X_1 \cdots X_a\ X_{a+1} \cdots X_{a+b})$, where

$X_r = (t_{ijk})$ $t_{ijk} = \delta_{ir}$. when $1 \le r \le a$.

$X_{s+a} = (t_{ijk})$ $t_{ijk} = \delta_{js}$. when $1 \le s \le b$

Similarly: $Z_r = X_r - \dfrac{n_{r\cdot}}{n_{\cdot\cdot}} J$. $1 \le r \le a$.

$Z_{s+a} = X_{s+a} - \dfrac{n_{\cdot s}}{n_{\cdot\cdot}} J$. $1 \le s \le b$

$$\Rightarrow Z_{s+a}^T Z_r = n_{rs} - n_{\cdot s}\frac{n_{r\cdot}}{n_{\cdot\cdot}} - n_{r\cdot}\frac{n_{\cdot s}}{n_{\cdot\cdot}} + \frac{n_{r\cdot} n_{\cdot s}}{n_{\cdot\cdot}} = 0$$

② General Case:

If the model isn't proportional. Then $Z_{s+a}^T Z_r \neq 0$

i.e. orthogonality doesn't hold. In this case. Sum of squares depend on the order of inclusion of the effects (para's). generally

$R(\alpha|\mu,\eta) \neq R(\alpha|\mu)$. $R(\eta|\alpha,\mu) \neq R(\eta|\mu)$

(6) Experimental Design Model:

① Completely Randomized Design (CRD):

In this design, we have homogeneous experimental units. If we have t treatments. Then we can divides the experiment units into t groups randomly. Apply a treatment to each unit in one group.

e.g. Standard Model: $Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$. $i = 1 \sim t$. $j = 1 \sim n_i$

Rmk: Our goal is to compare t treatments

② Blocking:

Block is for reduce variability. So the difference between treatments can be accessed.

It consists of grouping homogeneous experimental units into blocks. Then apply treatment to the units in each block.

③ Randomized Complete Block Design (RCB):

Standard model: $Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$ where α_i's are treatments effect. β_j's are block effect.

Rmk: Our goal is to compare a treatments after adjusting the blocks.

Proceed: Arrange experiment units into blocks.

$\Rightarrow$ Assign treatments $\Rightarrow$ remove extraneous source of variability

Rmk: It's a variance reduction design.

④ Latin Square Designs:

· It allows two different blocking factors.

e.g.

H \ M	1	2	3	4
1	A	B	C	D
2	B	C	D	A
3	C	D	A	B
4	D	A	B	C

A. B. C. D are treatments on two factors: Machine 1.2.3.4 and Hospital 1.2.3.4.

Standard Model: $Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \varepsilon_{ijk}$.

where $1 \le i, j \le a$. $k = f(i,j)$. s.t. one-to-one on $\{1, 2, \cdots a\}$ when fix i or j. $\varepsilon_{ijk} \sim N(0, \sigma^2)$.

Rmk: α_i's represents i^{th} row factor effect.

β_j's represents j^{th} column effect.

γ_k's represent k^{th} treatment effect.

Writen in: $Y = X\beta + \varepsilon$. $X = (J \; X_1 \cdots X_a \; X_{a+1} \cdots X_{2a} \cdots X_{3a})$

$X_r = (t_{ijk})$. $t_{ijk} = \delta_{ir}$. $1 \le r \le a$.

$X_{a+s} = (t_{ijk})$ $t_{ijk} = \delta_{js}$. $1 \le s \le a$.

$X_{2a+t} = (t_{ijk})$ $t_{ijk} = \delta_{tk}$. $1 \le t \le a$.

To decompose into orthogonal space:

$$Z_0 = J. \quad Z_i = X_i - \frac{(X_i^T J)}{J^T J} J = X_i - \frac{1}{a} J.$$

Then $C(J) \perp C(Z_1 \cdots Z_a) \perp C(Z_{a+1} \cdots Z_{2a}) \perp C(Z_{2a+1} \cdots Z_{3a})$

Table:

$$SS(\alpha) = a \sum_{1}^{a} (\bar{\eta}_{i\cdot\cdot} - \bar{\eta}_{\cdots})^2 . \qquad \xrightarrow[\text{by para.}]{\text{replace } \eta} \qquad \text{obtain: } E(MS)$$

$$SS(\beta) = a \sum_{1}^{a} (\bar{\eta}_{\cdot j \cdot} - \bar{\eta}_{\cdots})^2 .$$

$$SS(\gamma) = a \sum_{1}^{a} (\bar{\eta}_{\cdot\cdot k} - \bar{\eta}_{\cdots})^2$$

⑤ Factorial Treatment Structure:

It arises when we need to treat two or more factors or treatments and wish to construct all possible treatment combinations.

e.g. Two factors A, B. A has a level. B has b level. Then there're ab combinations.