The $T1$ Theorem.

Next. We will solve the problem: For an operator T with a standard kernel. When will T bdd on L^2. (So it's C-Z operator)

1) <u>Preliminary</u>:

① <u>Cotlar's Lemma</u>:

For H Hilbert space. (T_i) is seq of BLOs on H with adjoint (T_i^*). If $(a(j))_{\mathbb{Z}}$ is seq of nonnegative numbers. s.t.

$\| T_i T_j^* \|_{L(H)} + \| T_i^* T_j \|_{L(H)} \le a(i-j)$.

Then: $\forall\, n \le m$. $\| \sum_n^m T_j \|_{L(H)} \le \sum_{\mathbb{Z}} a(j)^{\frac{1}{2}}$.

<u>Pf</u>: Set $S = \sum_n^m T_j$.

i) $(SS^*)^k = \sum_{j_i \ge n}^{m} T_{j_1} T_{j_2}^* \cdots T_{j_{2k-1}} T_{j_{2k}}^*$

$\| T_{j_1} \cdots T_{j_{2k}}^* \| \le \| T_{j_1} T_{j_2}^* \| \| \cdots \| \cdots \| T_{j_{2k-1}} T_{j_{2k}}^* \|$

$\le a(j_1 - j_2) \cdots a(j_{2k-1} - j_{2k})$.

$\| T_{j_1} \cdots T_{j_{2k}}^* \| \le \| T_{j_1} \| \| T_{j_2}^* T_{j_3} \| \cdots \| T_{j_{2k}}^* \|$

$\le a^{\frac{1}{2}}(0)\, a(j_2 - j_3) \cdots a^{\frac{1}{2}}(0)$.

2°) $\|(SS^*)^k\| \le a(0)^{\frac{1}{2}} \sum a(j_1-j_2)^{\frac{1}{2}} a(j_2-j_3)^{\frac{1}{2}} \cdots a(j_{2k-1}-j_{2k})^{\frac{1}{2}}$

follow from geometric mean of estimate in 1°).

Actually, RHS $= a(0)^{\frac{1}{2}} (m-n+1) \left(\sum_{\mathbb{Z}} a(i)^{\frac{1}{2}} \right)^{2k-1}$

by fix the index and sum over from j_{2k} to j_2.

$m-n+1$ is number of index j_1.

3°) Note $\|S\| = \|(SS^*)^k\|^{\frac{1}{2k}} \le (m-n+1)^{\frac{1}{2k}} \sum_{\mathbb{Z}} a(i)^{\frac{1}{2}}$

Set $k \to \infty$. Obtain the result.

<u>Cor.</u> For (T_j) in Thm. above. $\sum_{\mathbb{Z}} T_j$ is a BLO

for $\sum_{\mathbb{Z}} a^{\frac{1}{2}}(j) < \infty$ on H.

<u>Rmk:</u> Note we assume: $\|T_i^* T_j\| = \|T_j T_i^*\|$

$\le a(i-j) \to 0$ as $|i-j| \to \infty$. It's

a weaker orthogonal property. (a.s. orthno.)

<u>e.g.</u> Set $H = L^2(\mathbb{R})$. $f \in L^2(\mathbb{R})$. $T_j f(x) = \int_{2^j < |t| \le 2^{j+1}} f(x-t)/t \, dt$.

Note $|T_j f| \lesssim |Mf|$. bdd.

We claim: $\sum_{\mathbb{Z}} T_j$ is bdd. (i.e. H. Hilbert Trans).

It's easy to check: $\|T_i T_j\| \le \|K_i * K_j\|_1 \lesssim 2^{-|i-j|}$.

Apply Cotlar's Lemma.

② <u>Carleson Measures:</u>

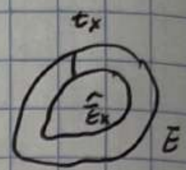
<u>Def:</u> Positive measure ν on $\mathbb{R}^n \times \mathbb{R}^+$ is carleson measure

if $\forall Q$ cube in $\mathbb{R}^n$. $\nu(Q \times (0, l(Q))) \le C|Q|$.

Rmk: Set $\|\nu\|_c$ is inf of possible C.

e.g. $dr\,d\theta$ on $\mathbb{R}\times\mathbb{R}^+$ is a Carleson measure.

$$\left(\int_\square dr\,d\theta = \int_\square \frac{dx\,dy}{\sqrt{x^2+y^2}} \right)$$



Lemma. Set $\hat{E} = \{(x,t)\in \mathbb{R}^n\times\mathbb{R}^+ \mid B(x,t)\subset E\}$.

If ν is a Carleson measure in $\mathbb{R}^n\times\mathbb{R}^+$.

Then: $\nu(\hat{E}) \lesssim_n \|\nu\|_c\,|E|$. for E open.

Pf: 1°) Apply C-Z decompose on I_E at $\lambda = \frac{1}{2}$.

$\Rightarrow E \subset \bigcup Q_j$. $|E| \le \sum |Q_j| \le C_n |E|$.

2°) For $(x,t)\in\hat{E}$. $\exists Q_j$ s.t. $x\in Q_j$.

Set $\tilde{Q}_j = 2Q_j \cap E^c \ne \emptyset$ (2nd method)

$\Rightarrow t < \sqrt{n}\, l(\tilde{Q}_j) = 2\sqrt{n}\, l(Q_j)$

So: $\hat{E} \subset \bigcup Q_j \times (0, 2\sqrt{n}\, l(Q_j))$

Directly apply def of ν.

Rmk: Converse is true: set $r = C_n l(Q)$. s.t. $\widehat{B(c_Q, r)} \supset Q\times(0, l(Q))$. Let $E = B$.

Thm. (Characterization of Carleson Measure)

$\phi \ge 0$. bdd. L^1. radial. & $\phi_t(x) = t^{-n}\phi(x/t)$.

Then ν is Carleson measure $\Leftrightarrow$ $\forall p\in(1,\infty)$.

$$\int |\phi_t * f(x)|^p\, d\nu(x,t) \le C \int |f|^p\, dx. \quad C \sim \|\nu\|_c.$$

Rmk: Poisson integral is bdd from $L^p(\mathbb{R}^n, dx)$ to

$L^p(\mathbb{R}^n \times \mathbb{R}^+, \nu) \iff \nu$ is Carleson measure.

<u>Pf</u>: 1°) Set $M_\phi f(x) = \sup\{|\phi_t * f(y)| \mid |x-y|<t\}$.

Recall: $M_\phi f(x) \lesssim Mf(x)$.

2°) Note $LHS = \int_0^\infty p\lambda^{p-1} \nu\{|\phi_t * f(x)| > \lambda\} d\lambda$

$\leq \int_0^\infty p\lambda^{p-1} \nu(\widehat{E_\lambda})$. $E_\lambda = \{M_\phi f > \lambda\}$.

$\lesssim_p \int_0^\infty \lambda^{p-1} \|\nu\|_c |E_\lambda|$.

$\lesssim_p \|\nu\|_c \int_0^\infty |M_\phi f|^p \lesssim_p \|\nu\|_c \|f\|_p^p$

This proved $(\Rightarrow)$.

3°) Conversely. Set $B = B(x_0, r)$. any ball.

$\forall (x,t) \in \widehat{B}$: $|\phi_t * \chi_B(x)| = \int_B \phi_t(x-y) dy$

$\overset{(B(x,t) \subset B)}{\geq} \int_{B(0,t)} \phi_t(y) dy = \int_{B(0,1)} \phi =: A$

$\Rightarrow \nu(\widehat{B}) \leq \frac{1}{A^p} \int_{\mathbb{R}^n \times \mathbb{R}^+} |\phi_t * \chi_B(x)|^p d\nu$

$\lesssim \int |\chi_B|^p = |B|$.

<u>Thm.</u> If $b \in BMO$. $\psi \in \mathcal{S}$. s.t. $\int \psi = 0$. Then measure

ν: $d\nu = |b * \psi_t|^2 dx dt / t$ is a Carleson measure

s.t. $\|\nu\|_c \lesssim \|b\|_*^2$.

<u>Pf</u>: Fix Q. cube. WLOG. Q is center at 0.

Q^* has the same center with $2\sqrt{n}$ l_Q.

$$\nu(Q\times(0,l(Q))) \overset{\int\psi=0}{\le} \int_{\mathbb{R}^n}\int_0^{l_Q} |(b-b_{Q^*})\chi_{Q^*} * \psi_t(x)|^2 \frac{dt\,dx}{t}$$

$$+\int_{\mathbb{R}^n}\int_0^{l_Q} |(b-b_{Q^*})\chi_{Q^{*c}} * \psi_t|^2 \,dt\,dx/t$$

$$=: I_1 + I_2$$

1°) $I_1 \le \int_{\mathbb{R}^n}\int_0^\infty |((b-b_{Q^*})\chi_{Q^*})^\wedge(\xi)|^2 |\hat{\psi}(t\xi)|^2 \frac{d\xi\,dt}{t}$

$\int_0^\infty |\hat{\psi}(t\xi)|^2 dt \le C$. follow from :

$|\hat{\psi}(t\xi)| \lesssim \min\{|t\xi|, |t\xi|^{-1}\}$. since $\hat{\psi}(0)=0$.

$\Rightarrow I_1 \lesssim \int_{Q^*} |b-b_{Q^*}|^2 \underset{\sim}{\overset{\text{(equi norm)}}{\le}} |Q|\,\|b\|_*^2$

2°) To estimate I_2. we separate $\mathbb{R}^n/Q^*$.

Denote Q_k^* has center at origin with length is $2^k l_{Q^*}$. $|x-\eta| \ge 2^{k-1} l_Q$. on $\mathbb{R}^n/Q^*$

$$I_2 \le \int_Q\int_0^{l_Q} \left|\sum_{k=0}^\infty \int_{Q^*_{k+1}/Q^*_k} (b(\eta)-b_{Q^*})\psi_t(x-\eta)\,d\eta\right|^2 \frac{dx\,dt}{t}$$

Note: $|\psi_t(x-\eta)| \lesssim t^{-n}(t^{-1}|x-\eta|)^{-n-1} \quad (\psi\in S)$

$\overset{(\text{on } \mathbb{R}^n/Q_k^*)}{\lesssim} t^{-n}(t^{-1}2^k l_Q)^{-n-1}$

So $I_2 \lesssim \int_Q\int_0^{l_Q} \left|\sum \frac{t}{2^k l_Q}\|b\|_*\right|^2 \frac{dx\,dt}{t}$

$\lesssim |Q|\,\|b\|_*^2$.

<u>Rmk</u>: Converse is true : For ψ. ν as above

if ν is Carleson measure, then $b\in BMO$.

<u>Cor</u>. As ϕ. ψ. in Thm above. $b\in BMO$. Then :

$$\int_{\mathbb{R}^n}\int_0^\infty |\phi_t * f|^2 |\psi_t * b|^2 \frac{dx\,dt}{t} \lesssim \|b\|_*^2 \int |f|^2 dx$$

(2) $T1$ Thm:

① **Statement:**

Consider $T: S(\mathbb{R}^n) \to S^*(\mathbb{R}^n)$

Def. T is associated with a standard kernel K.

if $\forall f, g \in S(\mathbb{R}^n)$. s.t. $supp(f)$, $supp(g)$ cpt

disjoint. $\langle Tf, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y) f(y) g(x)$

Rmk: Define $\langle Tf, g\rangle = \langle f, T^* g\rangle$. So:

$$T^* \sim K^*(x,y) = K(y,x).$$

Next. we extend the def of T on $C^\infty \cap L^\infty$:

Fix $g \in C_{c,0}^\infty = \{ f \in C_c^\infty \mid \int f = 0\}$. $f \in L^\infty \cap C^\infty$.

i) Set $\psi_1 \in C_c^\infty(\mathbb{R}^n)$. $supp(\psi_1) \subset B(0,3R)$.

$\psi_1 = 1$ on $B(0,2R)$. and $\psi_1 + \psi_2 = 1$.

where $supp\, g \subset B(0,R)$

ii) Note $f\psi_1 \in S(\mathbb{R}^n)$. $\Rightarrow \langle T(f\psi_1), g\rangle$ exists.

iii) Def: $\langle T(f\psi_2), g\rangle = \int_{\mathbb{R}^n\times\mathbb{R}^n} [K(x,y) - K(0,y)] f(y)\psi_2(y) g(x)$

Rmk: Since $\int g = 0$. it coincides the def

before when f has cpt support.

It makes sense by: $|K(x,y) - K(0,y)| \lesssim \frac{|x|^\delta}{|y|^{n+\delta}}$

$\Rightarrow$ Def: $\langle Tf, g\rangle =: \langle T(f\psi_1), g\rangle + \langle T(f\psi_2), g\rangle$.

(Next, we will use this def through this sect)

Rmk: The def is indept with choice of ψ_1, ψ_2 since $\int g = 0$.

Def: For $f \in L^\infty \cap C^\infty$, we say $Tf \in BMO$ if $\exists b \in BMO$. s.t. $\langle Tf, g\rangle = \langle b, g\rangle$. $\forall g \in C^\infty_{c,0}$.

Rmk: Note $C^\infty_{c,0} \underset{\text{dense}}{\subseteq} H^1$. It's equi: $\exists c > 0$. $|\langle Tf, g\rangle| \leq c\|g\|_{H^1}$. $\forall g \in H^1$. since $BMO = (H^1)^*$.

Def: i) φ is normalized bump function if $\varphi \in C_c^\infty(\mathbb{R}^n)$ supports in $B(0,10)$. s.t. $|D^\alpha \varphi| \leq 1$. for $\forall \alpha$. multindex. $|\alpha| \leq 2[n/2]+2$.

ii) $T: S(\mathbb{R}^n) \to S^*(\mathbb{R}^n)$ satisfies weak boundness property (WBP) if $\exists C > 0$. st. for $\forall x_0$. $\forall R > 0$. $\forall f, g$. normalized bump:

$\langle Tf^{x,R}, g^{x,R}\rangle \leq CR^n$. $(f^{x,R}(z) \triangleq f(\frac{z-x}{R}))$

Rmk: i) Denote inf of C by $\|T\|_{WB}$

ii) T is bdd on L^p. for some $p > 1$ $\underset{\text{(kiss)}}{\Rightarrow}$ T is WBP.

e.g. i) For standard kernel K. anti-symmetric (i.e. $K(x,y) = -K(y,x)$)

$$\langle Tf, g\rangle =: \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(x,y) f(y) g(x)\, dx\, dy$$

$$= \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} K(x,y)\,(f(y)g(x) - f(x)g(y))$$

satisfies WBP.

ii) $\partial/\partial x_1 : S \to S^*$. one-dim differentiation operator doesn't satisfy WBP. (It's associated with $k=0$. but not bdd on L^2)

<u>Thm.</u> ($T\mathbb{1}$ Thm)

For $T: S(\mathbb{R}^n) \to S^*(\mathbb{R}^n)$ associated with standard kernel K. It can be extended to operator on $L^2(\mathbb{R}^n)$. $\Leftrightarrow$ All of following conditions hold:

i) $T\mathbb{1} \in BMO$ ii) $T^*\mathbb{1} \in BMO$. iii) T has WBP.

<u>Rmk.</u> i). ii). iii) are all necessary conditions is easy to see: $\mathbb{1} \in L^\infty$. and T satisfies the cond. in VI. $\Rightarrow$ bdd from L^∞ to BMO. WBP is follows from bdd on $L^2(\mathbb{R}^n)$

<u>Cor.</u> For e.g. ii). It's bdd on $L^2(\mathbb{R}^n) \Leftrightarrow$ $T\mathbb{1} \in BMO$.

<u>pf:</u> $T^*\mathbb{1} = -T\mathbb{1}$. T satisfies WBP.

<u>e.g.</u> For $T_k \sim K_k(x,y) = \dfrac{(A(x)-A(y))^k}{(x-y)^{k+1}}$ on $\mathbb{R}^2$. s.t. $\|A'\|_\infty < \infty$. $T_k f = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \left(\dfrac{A(x)-A(y)}{x-y}\right)^k \dfrac{f(y)}{x-y}\,dy$

Calderón commutators.

T_k is bdd on L^2. and $\exists\, C>0$. s.t. $\|T_k\| \le C^k \|A'\|_\infty^k$.

③ <u>Pf of $T\mathbb{1}$ Thm:</u>

i) Case 1: $T\mathbb{1} = T^*\mathbb{1} = 0$. T has WBP.

1°) Fix $\phi \in S(\mathbb{R}^n)$. support on $B(0,1)$. $\int \phi = 1$.

Set $\phi_j(x) = 2^{-jn}\phi(x/2^j)$.

<u>Def</u>: $S_j f = \phi_j * f$. $\Delta_j = S_j - S_{j-1}$

<u>Lemma</u> (Decompose)

$$R_N = \sum_{-N}^{N} (S_j T \Delta_j + \Delta_j T S_j - \Delta_j T \Delta_j)$$

Then. $\lim_{N\to\infty} \langle R_N f, g\rangle = \langle f, g\rangle$. $\forall f, g \in C_c^\infty$.

<u>Pf</u>: Expand R_N: $R_N = S_{-N} T S_{-N} - S_{N+1} T S_{N+1}$.

1°) T is BLO: $S \to S^*$.

$$\Rightarrow \lim_{N\to\infty} \langle S_{-N} T S_{-N} f, g\rangle = \langle Tf, g\rangle.$$

2°) Prove: $\lim_{N\to\infty} \langle S_N T S_N f, g\rangle = 0$. $\forall f, g \in C_c^\infty$

Set $f_N(x) = 2^{nN}(\phi * f(2^N\cdot))(x) \in C_c^\infty$.

belongs to normalized bump for n large enough. Diff a const.

$\|D^\alpha f_N\|_\infty \leq \|f\|_1 \|D^\alpha \phi\|_\infty$

Similarly define for g_N. By WBP:

$$|\langle S_N T S_N f, g\rangle| = |\langle T S_N f, S_N g\rangle|$$
$$= |\langle T f_N, g_N\rangle| / 2^{2nN}$$

$$\lesssim \|f\|_1 \|g\|_1 \sup_{[\frac{n}{2}]+1} \|D_\alpha \phi\|^2\, 2^{nN} / 2^{2nN}$$

$$\to 0 \quad (\text{as } N \to \infty).$$

$\Rightarrow$ We can decompose $T = \sum_{\mathbb{Z}} (S_j T \Delta_j + \Delta_j T S_j - \Delta_j T \Delta_j)$

in sense of distribution.

2) Next, we show R_N's uniform bdd on $L^2(\mathbb{R}^n)$.

To apply Cotlar Lemma:

Set $T_j = S_j T \Delta_j$. (Similar for $\Delta_j T \Delta_j$, $\Delta_j T S_j$).

Def: $\psi = \phi - \phi_{-1}$, $\phi_j^x(u) = 2^{-jn} \phi\left(\frac{u-x}{2^j}\right)$

$\psi_j^x(u) = 2^{-jn} \psi\left(\frac{u-x}{2^j}\right)$.

$\Rightarrow \forall f, g \in C_c^\infty(\mathbb{R}^n)$, $\langle T_j f, g\rangle = \langle T \Delta_j f, S_j g\rangle$

$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle T \psi_j^\eta, \phi_j^x\rangle f(\eta) g(x)\, dx\, d\eta$.

follows from $\Delta_j f(x) = \int \psi_j^\eta(x) f(\eta)\, d\eta$. and

$S_j g(x) = \int \phi_j^\eta(x) g(\eta)\, d\eta$.

$\Rightarrow$ kernel of T_j is $k_j(x,\eta) = \langle T\psi_j^\eta, \phi_j^x\rangle$.

Lemma. Define $p(x) = (1+|x|)^{-n-\delta}$, $\delta = \delta(k) > 0$. depend on standard kernel k. Set $p_j(x) = 2^{-jn} p(x/2^j)$.

Then: i) $|k_j(x,\eta)| \le C\, p_j(x-\eta)$.

ii) $|k_j(x,\eta) - k_j(w,\eta)| \le C \min\{1, 2^{-j}|x-w|\}$

$[p_j(x-\eta) + p_j(w-\eta)]$. ($2^{nd}$ variable also ✓)

iii) $\int k(x,\eta)\, d\eta = \int k(x,\eta)\, dx = 0$. $\forall x, \eta$.

Pf: i) For $|x-\eta| \le 10\cdot 2^j$:

Set $\tilde{\Phi}(u) = \phi(u - 2^{-j}(x-\eta))$. $\Rightarrow \phi_j^x = \tilde{\phi}_j^\eta$.

By WBP. $LHS = |\langle T\psi_j^\eta, \tilde{\Phi}_j^\eta\rangle| \lesssim 2^{-nj}$

$\lesssim P_j(x-\eta)$

For $|x-\eta| > 10\cdot 2^j$:

$supp(\phi_j^x) \cap supp(\psi_j^\eta) = \emptyset$. So:

$|K_j(x,\eta)| = |\iint \phi_j(x-u)[K(u,v)-K(u,\eta)]\psi_j(v-\eta)\,du\,dv|$

$\lesssim \frac{2^{j\delta}}{|x-\eta|^{n+\delta}} \lesssim P_j(x-\eta)$. By std. kernel.

ii) For $|w-x| \ge 2^j$. it's from i)

We assume $|w-x| < 2^j$.

$|K_j(x,\eta) - K_j(w,\eta)| \le |x-w|\,|\nabla_x K_j(\xi,\eta)|$

$= |x-w|\,|\langle T\psi_j^\eta, \nabla\phi_j^x\rangle|$

$= |x-w|\cdot 2^{-j}|\langle T\psi_j^\eta, (\nabla\phi)_j^x\rangle|$

$\overset{(As\ i))}{\lesssim} 2^{-j}|x-w|\,P_j(\xi-\eta)$

With $P_j(\xi-\eta) \lesssim P_j(x-\eta) + P_j(w-\eta)$.

Since $x \overset{\xi}{\longrightarrow} w$. ξ is on line $\overline{xw}$.

iii) 1) $\int_{|\eta|\le R} K_j(x,\eta)\,d\eta = \langle Th, \phi_j^x\rangle$.

where $h(u) = 2^{-jn}\int_{|\eta|\le R}\psi(\frac{u-\eta}{2^j})\,d\eta$.

having support $\subset \{R-2^{j+1} < |u| < R+2^{j+1}\}$.

If R large enough. $supp(h)\cap\phi_j^x = \emptyset$.

Then, by standard estimate and $\int\psi = 0$:

$|LHS| \lesssim 2^{-j}R^{n-1}R^{-n} \to 0\ (R\to\infty)$

2') $\int_{|x|\le R} k_j(x,y)\,dx \overset{(T^*\mathbb{1}=0)}{=} \int_{|x|\le R} k_j\,dx - \langle T\psi_j^y, 1\rangle$

$= \langle \psi_j^y, T^* h\rangle$. where $h(u) = 2^{-jn}\int_{|u|\le R} \phi(\frac{u-x}{2^j})\,dx - 1$

Again by standard estimate of k, $\int\psi = 0$

$|LHS| \lesssim 2^{j\delta}\int_{|u|>\frac{R}{2}} du/|u|^{n+\delta} \to 0 \ (R\to\infty)$

Rmk: iii) i) $\Leftrightarrow T\mathbb{1} = 0$. iii) 2') $\Leftrightarrow T^*\mathbb{1} = 0$.

3) Estimate $\|T_j T_k^*\|_{L^2}$:

$T_j T_k^* \sim$ kernel $A_{j,k}(x,y) = \int_{\mathbb{R}^n} k_j(x,z) k_k(y,z)\,dz$

Lemma. $\forall j,k$: $\int |A_{j,k}|\,dy,\ \int |A_{j,k}|\,dx \lesssim 2^{-\delta|j-k|}$

for $\forall x, y$.

Pf: It follows from Lemma ii), iii) above.

Prop. $\|T_j T_k^*\|_{L^2\to L^2} \lesssim 2^{-\delta|j-k|}$. $\forall j,k$.

Pf: Directly test with $f\in S \underset{\text{dense}}{\subseteq} L^2$.

4) Apply Cotlar's Lemma. We're done.

ii) Case 2: Arbitrary operators

Lemma. $\forall b\in BMO$. $\exists$ C-Z operator L. st. $L\mathbb{1} = b$. $L^*\mathbb{1} = 0$

Pf: Fix $\phi, \psi \in S(\mathbb{R}^n)$, radial support on $B(0,1)$.

$\phi \ge 0$. $\int\phi = 1$. $\int\psi = 0$.

Let $Lf = c\int_0^\infty ((\psi_t * b)(\phi_t * f)) * \psi_t / t\,dt$

(This is called paraproduct)

Next, we show L is desired operator.

1°) $L \sim K = c\int_0^\infty \int_{\mathbb{R}^n} \psi_t(x-z)(\psi_t * b)(z)\, \phi_t(z-\eta)\, dz\, dt/t$

$= c\int k_t(x,\eta)\, dt/t$ is standard kernel:

Note: $|\psi_t * b| = |\int_Q \psi(z-\eta)(b(\eta)-b_Q)| \le 2^n \|\psi\|_\infty \|b\|_*$.

where $Q = Q(z, 2t)$. follow from $\int \psi = 0$.

$\Rightarrow |k_t| \le 2^n \|\psi\|_\infty \|b\|_* \|\psi_t\|_\infty \|\phi_t\|_1 \lesssim \|b\|_*/t^n$

With $\mathrm{supp}(k_t) \subset \{|x-\eta| \le 2t\}$.

So: $|k_t| \lesssim \|b\|_* t^{-n}(1+\frac{|x-\eta|}{t})^{-n-2} \Rightarrow |k| \lesssim \|b\|_* / |x-\eta|^n$

Similarly: $|\nabla_x k_t| + |\nabla_\eta k_t| \lesssim \|b\|_* t^{-n-1}(1+\frac{|x-\eta|}{t})^{-n-2}$

2°) L is bdd on L^2:

Test with $\eta \in S$. $\|\eta\|_2 = 1$.

Use Hölder inequi and Carleson measure from b.

3°) Choose c. st. $L\mathbb{1} = b$. $L^*\mathbb{1} = 0$.

Note $(\psi_t * ((\psi_t * b)\phi_t * \cdot))^* = \phi_t * ((\psi_t * b) * \cdot)$

First. $\psi * 1 = 0$ by $\int \psi = 0$. $\Rightarrow L^*\mathbb{1} = 0$.

And, $\langle L\mathbb{1}, \eta\rangle = c\int_0^\infty \int \psi_t * b(x)\, \phi_t * \mathbb{1}(x)\, \psi_t * \eta(x)/t$

$= c\int\int_0^\infty b(x)\, \psi_t * \psi_t * \eta /t\, dy\, dt$

Choose c. st. $c\int_0^\infty \psi_t * \psi_t * \eta\, dt/t = \eta$.

$\Leftrightarrow c\int_0^\infty |\hat{\psi}_t(\xi)|^2\, dt/t = 1$. $\Leftrightarrow c\int_0^\infty |\psi(t\xi)|^2/t = 1$

c doesn't depend on ξ since ψ is radial.

$\Rightarrow \exists L_1, L_2$ C-Z operator. St. $L_1\mathbb{1} = b_1 = T\mathbb{1}$. $L_1^*\mathbb{1} = 0$.

$L_2\mathbb{1} = 0$. $L_2^*\mathbb{1} = b_2 = T^*\mathbb{1}$. Let $\widetilde{T} = T - L_1 - L_2$.

Then reduce to Case 1.