Littlewood Paley Theory.

(1) Littlewood Paley Theory:

① Vector-Valued Inequi:

Thm. T is convolution operator with kernel k satisfying Hörmander condition. and bdd on L^2. Then: $\forall r, p \in (1, \infty)$.

$$\| (\sum_j |Tf_j|^r)^{\frac{1}{r}} \|_p \lesssim_{p,r} \| (\sum_j |f_j|^r)^{\frac{1}{r}} \|_p$$

$$| \{ (\sum_j |Tf_j|^r)^{\frac{1}{r}} > \lambda \} | \le \frac{C_r}{\lambda} \| (\sum_j |f_j|^r)^{\frac{1}{r}} \|_1$$

Pf: 1°) For strong-(p,p):

$p = r$. trivial from T is bdd on L^p.

2°) Set $\tilde{T} : \ell^r \to \ell^r$ with kernel
$(f_i) \mapsto (Tf_i)$

$\tilde{k} = k(x) I$. (I is id. on ℓ^r).

Note $\| (k(x-y) - k(x)) I \|_{\ell^r} =$
$|k(x-y) - k(x)| \, \| I \|_{\ell^r}$.

$\Rightarrow \tilde{k}$ satisfies Hörmander condition

3°) With $\tilde{T}$ is bdd on $L^r(\ell^r)$.

Cor. For $T_j = k_j *$. bdd on $L^2(\mathbb{R}^n)$. $\forall j$.

and $\int_{|x|>2|y|} \sup_j |k_j(x-y) - k_j(x)| \, dx \le C$

Then: $\| (\sum_j |T_j f_j|^r)^{\frac{1}{r}} \|_p \underset{p,r}{\lesssim} \| (\sum_j |f_j|^r)^{\frac{1}{r}} \|_p$.

for $1 < p, r < \infty$.

Pf: Set $\tilde{T} : \ell^r \to \ell^r$ similarly.
$(f_i) \mapsto (T_i f_i)$

Cor. (I_j) intervals of $\mathbb{R}^1$. $(S_j f)^\wedge = \chi_{I_j} \hat{f}$.

Then: $\| (\sum_j |S_j f_j|^r)^{\frac{1}{r}} \|_p \underset{p,r}{\lesssim} \| (\sum_j |f_j|^r)^{\frac{1}{r}} \|_p$.

for $1 < p, r < \infty$.

Pf: $S_j f_j = \frac{i}{2} (M_{a_j} H M_{-a_j} - M_{b_j} H M_{-b_j}) f_j$.

satisfies cond. of Cor. above.

Thm. If (T_j) is seq of LOs st. bdd on $L^2(w)$ if $w \in A_2$, with const. uniform in j, depending only on A_2 const. of w. Then $\forall p \in (1, \infty)$.

$\| (\sum_j |T_j f_j|^2)^{\frac{1}{2}} \|_p \lesssim \| (\sum |f_j|^2)^{\frac{1}{2}} \|_p$.

Pf: 1) $p = 2$ is trivial.

2) $p > 2$: Note $\exists u \in L^{(\frac{p}{2})'}$. $\|u\|_{(\frac{p}{2})'} = 1$. st.

$$\| (\sum |T_j f_j|^2)^{\frac{1}{2}} \|_p^2 = \int \sum |T_j f_j|^2 u \, dx$$

$$\leq \int \sum |T_j f_j|^2 M(u^{\delta^{-1}})^{\delta}.$$

$$\underset{\delta}{\lesssim} \int \sum |f_j|^2 M(u^{\delta^{-1}})^{\delta}.$$

$$\underset{\delta}{\lesssim} \| (\sum_j |f_j|^2) \|_{\frac{p}{2}} \| M(u^{\delta^{-1}})^{\delta} \|_{(\frac{p}{2})'}$$

Fix δ. st. $\delta (\frac{p}{2})' < 1$.

3') $p<2$: Consider adjoint operator $\widetilde{T}_j^*$

bdd on $L^2(w)$ if $w \in A_2$ as well.

② Theory:

Observe : i) $f \in L^2$. $q\hat{f} \in L^2$ if $\|q\|_{L^\infty} \le 1$.

ii) $f \in L^2$. $f = \sum_k a_k e^{2\pi i k x}$. Then : $\tilde{f} = \sum_k a_k e^{2\pi i k x} \cdot \sigma_k$. $\sigma_k \in \{-1, 1\}$. $\tilde{f} \in L^2$.

Both follows from Plancherel Thm.

However. for $p \ne 2$. These don't hold for L^p.

Littlewood Paley Theory provides a partial substitute in L^p for result from Plancherel Thm.

Denote: $\Delta_j = (-2^{j+1}, -2^j] \cup [2^j, 2^{j+1})$. $j \in \mathbb{Z}$.

$(S_j f)^\wedge = \chi_{\Delta_j} \hat{f}$ on $\mathbb{R}^1$.

Rmk: For $f \in L^2$. $\|(\sum_j |S_j f|^2)^{\frac{1}{2}}\|_2 = \|f\|_2$

Thm.$^{(*)}$ For $f \in L^p(\mathbb{R}^1)$. $1<p<\infty$. Then $\exists\, C_p > 0$.

s.t. $\|(\sum_j |\tilde{S}_j f|^2)^{\frac{1}{2}}\|_p \le C_p \|f\|_p$. where

$(\tilde{S}_j f)^\wedge = \psi_j(\xi) \hat{f}(\xi)$. $\psi_j(\xi) = \psi(\xi/2^j)$.

$\psi \ge 0 \in S(\mathbb{R}^1)$. $\operatorname{supp} \psi = \{\frac{1}{2} < |\xi| < 4\}$. $\psi = 1$ on $1 \le |\xi| \le 2$.

Rmk: $S_j \tilde{S}_j = \tilde{S}_j S_j = S_j$.

Pf: 1°) $\hat{\varphi} = \psi$. $\varphi_j(x) = 2^j \varphi(2^j x)$. $\tilde{S}_j f = \varphi_j * f$.

At any value of ξ. at most 3 $\psi_j \neq 0$.

2°) $p=2$. direct by Plancherel Thm

3°) Check Hörmander condition:

Prove: $\| \varphi_j' \|_{\ell^2} \lesssim |x|^{-2}$. (Gradient)

$LHS \leq (\sum |\varphi_j'|^2)^{\frac{1}{2}} \leq \sum |\varphi_j'|$

$= \sum 2^{2j} |\varphi'(2^j x)|$

Note: $\varphi \in S \Rightarrow |\varphi'| \lesssim \min\{1, |x|^{-3}\}$.

$\forall x$. suppose $x \in (2^i, 2^{i+1}]$.

$\Rightarrow LHS \lesssim \sum_{j \leq i} 2^{2j} + |x|^{-3} \sum_{j > i} 2^{-j}$

$\lesssim |x|^{-2}$.

Thm (Littlewood - Paley)

For $f \in L^p(\mathbb{R}^1)$. $1 < p < \infty$. Then $\exists\, c_p, C_p > 0$. s.t.

$c_p \|f\|_p \leq \| (\sum |S_j f|^2)^{\frac{1}{2}} \|_p \leq C_p \|f\|_p$.

Rmk: i) It mean if $f \in L^p$. then the modification of f by multiplying ± 1 on $\hat{f}$ on each dyadic interval still belongs to L^p.

ii) General: (I_j) is disjoint intervals in $\mathbb{R}^1$. $(\pi_j f)^\wedge := \chi_{I_j} \hat{f}$. Then $\forall p \in [2, \infty)$.

We have: $\| (\sum_j |\pi_j f|^2)^{\frac{1}{2}} \|_p \lesssim_p \|f\|_p$.

<u>Pf</u>: i) $\|(\sum_j |S_j f|^2)^{\frac{1}{2}}\|_p = \|(\sum_j |\tilde{S}_j S_j f|^2)^{\frac{1}{2}}\|_p$

$$\overset{(\text{cor} ①)}{\leq} C_p \|(\sum |\tilde{S}_j f|^2)^{\frac{1}{2}}\|_p$$

$$\underset{p}{\lesssim} \|f\|_p$$

ii) Note: $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$ holds for $f \in L^1$, $g \in L^\infty$. (Approxi. by $\mathcal{S}$)

$$\Rightarrow \int \sum S_j f \cdot \overline{S_j g} = \int f \bar{g}.$$

$$\|f\|_p = \sup \{ |\int f\bar{g}| \mid \|g\|_{p'} \leq 1 \}.$$

$$\leq \sup \{ \|(\sum_j |S_j f|^2)^{\frac{1}{2}}\|_p \|(\sum_j |S_j g|^2)^{\frac{1}{2}}\|_{p'} \}$$

$$\underset{p}{\overset{(i)}{\lesssim}} \|(\sum_j |S_j f|^2)^{\frac{1}{2}}\|_p \qquad \square$$

To extend $\mathbb{R}^1$ to $\mathbb{R}^n$:

i) Cut $\hat{f}$ with C^∞ functions support on $\{2^{j-1} \leq |\xi| \leq 2^j\}$.

ii) Cut $\hat{f}$ with χ_A. A is product of dyadic intervals.

<u>Rmk</u>: We can't use the method i):

<u>Thm</u>. χ_B isn't a multiplier on L^p, $p \neq 2$.

for B is ball on $\mathbb{R}^n$, $n \geq 2$.

Thm (A) For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\varphi(0) = 0$, $(S_j f)^\wedge = \varphi(\xi/2^j)\hat{f}$, $j \in \mathbb{Z}$.

If $\sum_{\mathbb{Z}} |\varphi(\xi/2^j)|^2 \leq C < \infty$. Then, we have:

$$\|f\|_p \underset{p,n}{\lesssim} \|(\sum_j |S_j f|^2)^{\frac{1}{2}}\|_p \underset{p,n}{\lesssim} \|f\|_p, \quad 1 < p < \infty.$$

Pf: 1') $p=2$ is trivial. by condition.

2') Check Hörmander condition (as before in (*))

3') Use dual representation and 2').

Rmk: To construct $\sum |\psi(2^{-j}\xi)|^2 \le c < \infty$. $\forall \xi \neq 0$:

We can fix $\varphi \in S(\mathbb{R}^2)$. s.t. $\varphi \ge 0$. $\downarrow$.

radial. $\varphi = 1$ if $|\xi| \le \frac{1}{2}$. $\varphi = 0$. if $|\xi| > 1$.

Set $\psi^2(\xi) = \varphi(\xi/2) - \varphi(\xi)$.

Denote: $(S_j^1 f)^\wedge = \chi_{\Delta_j}(\xi_1)\hat{f}$. $(S_j^2 f)^\wedge = \chi_{\Delta_j}(\xi_2)\hat{f}$.

Lemma. Extend Thm (*): For $p \in (1,\infty)$. $f_k \in L^p(\mathbb{R}^1)$. $\forall k$.

$$\| (\sum_{j,k} |\tilde{S}_j f_k|^2)^{\frac{1}{2}} \|_p \underset{p}{\lesssim} \| (\sum_k |f_k|^2)^{\frac{1}{2}} \|_p.$$

Pf: 1') $p=2$. trivial.

2') Check Hörmander condition:

$$S: L^p(\ell^2) \longrightarrow L^p(\hat{\otimes}\ell^2)$$

$$(f_k) \longmapsto ((S_j f_k)_k)_j$$

with kernel $k = (k_j I_{\ell^2})_j$. $k_j = \psi_j$.

Similarly. $\|k'\|_{\otimes\ell^2}^2 \le \sum_j |\psi_j'|^2 \lesssim |x|^{-3}$

Thm. $p \in (1,\infty)$. $f \in L^p(\mathbb{R}^2)$. Then. we have:

$$\|f\|_p \underset{p}{\lesssim} \| (\sum_{j,k} |S_j^1 S_k^2 f|^2)^{\frac{1}{2}} \|_p \underset{p}{\lesssim} \|f\|_p.$$

Pf: 1') First $\lesssim$ is from dual representation

argue twice as in 2') of pf of

Littlewood - Paley Thm.

2) For second $\lesssim$:

consider : $\tilde{S}_j^1 S_j^1 = S_j^1 \tilde{S}_j^1 = S_j^1$.

$$\|(\sum_{j,k} |S_j^1 S_k^2 f|^2)^{\frac{1}{2}}\|_p \lesssim_p \|(\sum_{j,k} |\tilde{S}_j^1 S_k^2 f|^2)^{\frac{1}{2}}\|_p$$

$$\lesssim_p \|(\sum_k |S_k^2 f|^2)^{\frac{1}{2}}\|_p$$

$$\lesssim_p \|f\|_p .$$

Rmk: By induction: extend $n=2$ to $n=d$ on $\mathbb{R}^n$.

(2) Apply on Multipliers:

Next. we will characterize multipliers on L^p.

i.e. given m. when $(T_m f)^\wedge = m\hat{f}$ bdd on L^p for $1 \le p \le \infty$.

① Hörmander Multipliers:

Recall : $g \in W^{n,2} =: H^n \iff \langle \xi \rangle^n \hat{g} \in L^2$.

Prop. If $n > n/2$. $g \in H^n$. Then $\hat{g} \in L^1(\mathbb{R}^n)$

Pf: $\int |\hat{g}| \le (\int |\hat{g}\langle\xi\rangle^n|^2)^{\frac{1}{2}} (\int \langle\xi\rangle^{-2n})^{\frac{1}{2}}$

Rmk: $n > n/2$. $g \in H^n(\mathbb{R}^n) \Rightarrow g$ is a multiplier on L^p. $1 \le p \le \infty$.

<u>Lemma.</u> $m \in W^{2,\alpha}(\mathbb{R}^n)$, $\alpha > \frac{n}{2}$. For $\lambda > 0$. Set :

$(T_\lambda f)^\wedge = m(\lambda\xi)\hat{f}(\xi)$. Then: we have.

$$\| T_\lambda f \|_{L^2(u)} \lesssim \| f \|_{L^2(Mu)}$$

<u>Pf</u>: Set $\hat{k} = m$. $\Rightarrow \langle\xi\rangle^\alpha k(\xi) \triangleq k(\xi) \in L^2$.

$$\int |T_\lambda f|^2 u = \int \left| \int \lambda^{-n} k(\lambda^{-1}(x-y)) \langle \lambda^{-1}(x-y)\rangle^{-\alpha} f \right| u$$

$$\lesssim \|k\|_2 \iint \frac{\lambda^{-n}|f|^2}{\langle\lambda^{-1}(x-y)\rangle^{2\alpha}} u \, dy\, dx$$

$$= \|k\|_2 \int |f|^2 (u * \phi_\lambda)\, dy.$$

$$\lesssim_n \|k\|_2 \int |f|^2 Mu\, dy$$

Where $\phi_\lambda = 1/\lambda^n \langle \lambda^{-1}(x-y)\rangle^{2\alpha}$ is radial.

$\downarrow$. ≥ 0. $\phi_\lambda \in L^1$ since $\alpha > n/2$.

<u>Thm.</u> (Hörmander)

Fix $\psi \in C^\infty$. radial. support on $\{1/2 < |\xi| < 2\}$. and

$\sum_{j\in\mathbb{Z}} |\psi(\xi/2^j)|^2 = 1$. $\forall \xi \neq 0$.

If m satisfies : for some $\alpha > \frac{n}{2}$. $\sup_j \|m(2^j\cdot)\psi\|_{H^\alpha} < \infty$. Then: T_m is bdd on $L^p(\mathbb{R}^n)$. $1 < p < \infty$.

<u>Pf</u>: 1°) Def: $(S_j f)^\wedge = \psi(\xi/2^j)\hat{f}$.

$\tilde{\psi} \in C^\infty$. Supp on $\frac{1}{4} \leq |\xi| \leq 4$. and

$\tilde{\psi} = 1$ on $\{1/2 \leq |\xi| \leq 2\}$.

$(\tilde{S}_j f)^\wedge = \tilde{\psi}(\xi/2^j)\hat{f}$.

$\Rightarrow S_j \tilde{S}_j = \tilde{S}_j S_j = S_j$. $\tilde{S}_j$ and S_j both satisfies Thm (D).

2') So: $\|Tf\|_p \underset{p}{\sim} \|(\sum_j |S_j T f|^2)^{\frac{1}{2}}\|_p$

$= \|(\sum_j |S_j T \tilde{S}_j f|^2)^{\frac{1}{2}}\|_p$.

3') Note $S_j T$ associate $m(2^j x)\psi(x) \in H^n$

By Lemma: $\int |S_j T f|^2 u \le C \int |f|^2 Mu$.

$\exists u \in L^{(p/2)'}$. $\|u\|_{(p/2)'} = 1$. (for $p>2$). s.t.

$\|(\sum |S_j T g_j|^2)^{\frac{1}{2}}\|_p^2 = \int \sum |S_j T g_j|^2 u$

$\lesssim \int \sum |g_j|^2 Mu$

$\underset{p}{\lesssim} \|(\sum |g_j|^2)^{1/2}\|_p^2$.

4') So by 2'). 3'):

$\|Tf\|_p \underset{p}{\lesssim} \|(\sum |\tilde{S}_j f|^2)^{\frac{1}{2}}\|_p \underset{p}{\lesssim} \|f\|_p$. $p>2$.

5') $p<2$. by dual argument.

<u>Cor.</u> For $k = [\frac{n}{2}]+1$. $m \in C^k$ away from origin. If $\sup_R R^{|\beta|} (\frac{1}{R^n} \int_{R<|\xi|<2R} |D^\beta m(\xi)|^2 d\xi)^{\frac{1}{2}}$

$< \infty$ for $\forall |\beta| \le k$.

Then: m is multiplier on L^p. $1<p<\infty$.

<u>Rmk</u>: In particular. it holds when

$|D^\beta m(\xi)| \le C |\xi|^{-|\beta|}$. $\forall |\beta| \le k$.

<u>Pf</u>: It equi: $\sup_R \int_{1<|\xi|<2} |D^\beta m(R\xi)|^2 d\xi < \infty$.

$D^\beta (m(2^j \cdot)\psi) = \sum_{|\gamma| \le |\beta|} C_{\beta,\gamma} D^\gamma m(2^j \cdot) D^{\beta-\gamma}\psi$.

with $|D^r \psi| \le C$. $\forall |r| \le |\beta| \le k$.

$\Rightarrow \sup_j \|m(2^j \cdot)\psi\|_{H^k} < \infty$.

e.g. i) $|\xi|^{it}$ satisfies: $|D^\beta m(\xi)| \lesssim_t |\xi|^{-|\beta|}$.

ii) m has degree 0. $m \in C^k(S)$ for some $k > [\frac{n}{2}]$.

i), ii) both bdd in $L^p(\mathbb{R}^n)$, $1<p<\infty$.

② Marcinkiewicz Multiplier:

Thm. m is bdd func. having uniform BV on each dyadic interval in $\mathbb{R}^1$. Then m is multiplier on $L^p(\mathbb{R}^1)$. $1<p<\infty$.

Pf: 1°) WLOG. m is right-conti. Denote $I_j = (2^j, 2^{j+1})$

Set T_j associates multiplier $(m\chi_{I_j})(\xi)$

$= m(2^j) + \int_{2^j}^{\xi} dm(t)$. for $\xi \in I_j$.

2°) $T_j f(x) = m(2^j) S_j f(x) + \int_{2^j}^{2^{j+1}} S_{t,2^{j+1}} f(x) \, dm(t)$

where $S_{t,2^{j+1}} \sim \chi_{[t,2^{j+1}]}$. $S_j \sim \chi_{[2^j,2^{j+1}]}$.

are uniform bdd on $L^2(w)$. $w \in A_2$. by condition

3°) $\|T_j f\|_{L^2(w)} \overset{(\text{Minkowski})}{\leq} \|m\|_\infty \|S_j f\|_{L^2(w)} + \int_{2^j}^{2^{j+1}} \|S_\sigma f\|_{L^2(w)} dm$

$\lesssim \|f\|_{L^2(w)} + TV_m[2^j, 2^{j+1}] \|f\|_{L^2(w)}$

$\lesssim \|f\|_{L^2(w)}$.

4°) $\|Tf\|_p \overset{(\text{Little Paley})}{\lesssim_p} \|(\Sigma |S_j T f|^2)^{\frac{1}{2}}\|_p$

$= \|(\Sigma |S_j T_j f|^2)^{\frac{1}{2}}\|_p$

$\overset{(1)\ ①}{\lesssim_p} \|(\Sigma |S_j f|^2)^{\frac{1}{2}}\|_p \lesssim_p \|f\|_p$

Cor. m is bdd having uniform bdd variation on each dyadic intervals in $\mathbb{R}^1$. If $w \in A_p$. Then T_m is bdd on $L^p(\mathbb{R}^1, w)$. $(1<p<\infty)$.

Thm. $m \in C^2(M_i)$. $1 \le i \le 4$. M_i is one quadrant of $\mathbb{R}^2$. st. $\sup_i \int_{I_j} |\frac{\partial m}{\partial t_1}| dt_1$. $\sup_j \int_{I_j} |\frac{\partial m}{\partial t_2}| dt_2$

$\sup_{i,j} \int_{I_i \times I_j} |\frac{\partial^2 m}{\partial t_1 \partial t_2}| dt_1 dt_2 < \infty$. for (I_j) is dyadic interval in $\mathbb{R}^1$. If m is bdd.

Then m is multiplier on $L^p(\mathbb{R}^2)$. $p \in (1,\infty)$.

Pf: Fix $I_i = (2^i, 2^{i+1})$. $I_j = (2^j, 2^{j+1})$

$$m(\xi_1, \xi_2) = \int_{2^i}^{\xi_1} \int_{2^j}^{\xi_2} \frac{\partial^2 m}{\partial t_1 \partial t_2} dt_1 dt_2 + \int_{2^i}^{\xi_1} \frac{\partial m}{\partial t_1} dt_1$$

$$+ \int_{2^j}^{\xi_2} \frac{\partial m}{\partial t_2} dt_2 + m(2^i, 2^j)$$

for $(\xi_1, \xi_2) \in I_i \times I_j$.

Similarly. prove $T_{i,j} \sim (m \chi_{I_i \times I_j})(\xi)$ is bdd on $L^2(w)$. for $w \in A_2$.

Rmk: By induction: extend $n=2$ to $n=d$:

$$\sup_{j_1 \cdots j_k} \int_{I_{j_1} \times \cdots I_{j_k}} \left| \frac{\partial^k m}{\partial \xi_{i_1} \cdots \partial \xi_{i_k}} \right| d\xi_{i_1} \cdots d\xi_{i_k} < \infty.$$

$\{i_1 \cdots i_k\} \subset \{1, 2, \cdots d\}$. $k \le d$

③ Bochner - Riesz Multipliers:

Note that for $S_R f(x) = \int_{|\xi|\le R} \hat{f}(\xi) e^{2\pi i \xi\cdot x} d\xi$. i.e.

S_R with a multiplier $\chi_{\{|\xi|\le R\}}$. it's not well-behaved when $n \geq 2$ in $\mathbb{R}^n$.

But we can consider a more regular operator

$\sigma_R f(x) = \frac{1}{R} \int_0^R S_t f(x) dt = \int_{|\xi|\le R} (1 - \frac{|\xi|}{R}) \hat{f}(\xi) e^{2\pi i \xi\cdot x} d\xi$.

$\Rightarrow$ Lead us to consider : $(S_R^\alpha f)^\wedge(\xi) = (1 - \frac{|\xi|}{R})_+^\alpha \hat{f}(\xi)$. $\alpha > 0$.

Def: $(T^\alpha f)^\wedge(\xi) = (1-|\xi|^2)_+^\alpha \hat{f}(\xi)$. is a operator associate Bochner - Riesz multiplier. $(\alpha > 0)$

Rmk: i) $(1-|\xi|^2)_+^\alpha = (1-|\xi|)_+^\alpha \, [(1+|\xi|)^\alpha \psi_1(|\xi|)]$

$(1-|\xi|)_+^\alpha = (1-|\xi|^2)_+^\alpha \, [(1+|\xi|)^{-\alpha} \psi_2(|\xi|)]$.

$\psi_1, \psi_2 \in C_c^\infty(\mathbb{R}')$. $\psi_1(|\xi|) = \psi_2(|\xi|) = 1$ if $|\xi| \le 1$. So: In $[\cdots]$. it's a Hörmander operator. We have:

T^α is bdd on $L^p \iff S_1^\alpha$ is. $(p \in (1,\infty))$.

ii) Singularity of Bochner - Riesz multiplier is on $|\xi| = 1$. when integrating:

$T^\alpha f(x) = (1-|\xi|^2)_+^{\alpha\,\vee} * f(x)$. This lead us to decompose B-R multiplier.

<u>Denote</u>: i) $\phi_k \in C^\infty(\mathbb{R}^1)$. Support on $(1-2^{1-k}, 1-2^{-1-k})$.

s.t. $0 \le \phi_k \le 1$. $|D^\beta \phi_k| \le C_\beta 2^{k|\beta|}$. $\forall \beta$. and

$$\sum_{k\ge 1} \phi_k(t) = 1 \quad \text{if} \quad 0 \le t < \frac{1}{2}.$$

Define $\phi_0(t) = \begin{cases} 1 - \sum \phi_k(t). & 0 \le t < \frac{1}{2} \\ 0 & . \quad t > \frac{1}{2}. \end{cases}$

<u>Rmk</u>: $(1-|\xi|^2)_+^\alpha = \sum_{k\ge 0} (1-|\xi|^2)^\alpha \phi_k(|\xi|)$.

On $\mathrm{Supp}(\phi_k)$. $(1-|\xi|^2)^\alpha \simeq 2^{-k\alpha}$.

$\Rightarrow$ Set $\tilde{\phi}_k(|\xi|) = 2^{k\alpha}(1-|\xi|^2)^\alpha \phi_k(|\xi|)$

So: $(1-|\xi|^2)_+^\alpha = \sum 2^{-k\alpha} \tilde{\phi}_k(|\xi|)$.

ii) Set $(T_k f)^\wedge = \tilde{\phi}_k(|\xi|) \hat{f}(\xi)$. $T^\alpha = \sum 2^{-k\alpha} T_k$.

<u>Thm</u>. Bochner - Riesz multipliers T^α satisfies:

i) $\alpha > \frac{n-1}{2} \Rightarrow T^\alpha$ is bdd on $L^p(\mathbb{R}^n)$. $1 \le p \le \infty$.

ii) $\alpha \in (0, \frac{n-1}{2}] \Rightarrow T^\alpha$ is bdd in $L^p(\mathbb{R}^n)$ if

$|\frac{1}{p} - \frac{1}{2}| < \frac{\alpha}{n-1}$. isn't bdd if $|\frac{1}{p} - \frac{1}{2}| \ge \frac{2\alpha+1}{2n}$.

<u>Lemma</u>. $\delta \in (0,1)$. $0 \le \phi \le 1$. $\mathrm{supp}\,\phi = \{1-4\delta < t < 1-\delta\}$.

and $|D^\beta \phi| \le C \delta^{-|\beta|}$. $\forall \beta$. $(\phi = \phi(|\xi|))$

Set T_δ is operator associates $\phi(|\xi|)$.

Then: $\forall \varepsilon > 0$. $\|T_\delta f\|_p \le C_\varepsilon \delta^{-(\frac{n-1}{2}+\varepsilon)|\frac{2}{p}-1|} \|f\|_p$

<u>Pf</u>: 1°) Set $\hat{K}(\xi) = \phi(|\xi|)$. $\alpha > 0$. even.

$$\|(1+|x|^\alpha)k\|_2 \lesssim \|(I+(-\Delta)^{\frac{\alpha}{2}})\phi\|_2$$

$$\lesssim \delta^{\frac12}(1+\delta^{-\alpha}) \lesssim \delta^{\frac12-\alpha}$$

2) Extend to arbitrary $\alpha>0$. Fix s. s.t. αs is even

$$\|(1+|x|^\alpha)k\|_2 \underset{s}{\overset{(\text{Hölder})}{\lesssim}} \|(1+|x|^{\alpha s})^{\frac1s}k\|_2$$

$$\lesssim_s \|(1+|x|^{\alpha s})k\|_2^{\frac1s}\,\|k\|_2^{\frac1{s'}} \lesssim_s \delta^{\frac12-\alpha}$$

3) Set $\alpha = \frac{n}{2}+\varepsilon$ in 2):

$$\|T_\delta f\|_p \le \|k\|_1\|f\|_p$$

$$\le \|(1+|x|^\alpha)k\|_2\,\|(1+|x|^\alpha)^{-1}\|_2\,\|f\|_p.$$

$$\lesssim_\varepsilon \delta^{-(\frac{n-1}{2}+\varepsilon)}\|f\|_p.$$

holds for $p=1,\infty$. Discuss two case: $p>2$ or $p\le 2$. interpolation with $p=2$. (trivial case). $1 - p=2 - \infty$.

<u>Lemma.</u> If m is a multiplier with cpt support on L^p for some $p\in(1,\infty)$. Then $\hat{m}\in L^p$.

<u>Pf:</u> Set $f\in S$. s.t. $\hat{f}=1$. on $\mathrm{supp}(m)$.

$$\Rightarrow (T_m f)^\wedge = m\hat{f} = m. \in L^p.$$

<u>Lemma.</u> $(1-|\xi|^2)_+^{\alpha\,\wedge} = K_\alpha(x) = \pi^{-\alpha}\Gamma(\alpha+1)|x|^{-\frac n2-\alpha}J_{\frac n2+\alpha}(2\pi|x|)$

where $J_\mu(t) = (\frac t2)^\mu\int_{-1}^{1} e^{its}(1-s^2)^{\mu-\frac12}ds\,/\,\Gamma(\mu+\frac12)\Gamma(\frac12)$

<u>Rmk:</u> $J_\mu(t) \sim O(t^\mu)$ $(t\to 0)$

$J_\mu(t) \sim O(t^{-\frac12})$ $(t\to\infty)$

Return to pf:

1) Note ϕ_k satisfies Lemma. $\forall k$.

$$\Rightarrow \|T_k f\|_p \le C_\varepsilon 2^{k(\frac{n-1}{2}+\varepsilon)|\frac{2}{p}-1|}\|f\|_p.$$

$$S_0 = \|T^\alpha f\|_p \lesssim_\varepsilon \sum_{k\ge 0} 2^{k(\frac{n-1}{2}+\varepsilon)|\frac{2}{p}-1|-k\alpha}\|f\|_p$$

$$\lesssim_{n,\varepsilon,\alpha} \|f\|_p.$$ Fix $\varepsilon>0$ small.

2) By estimate of J_m:

$|K_\alpha(x)| \le C \quad (|x|\to 0)$. $|K_\alpha(x)| \sim O(|x|^{-(\frac{n-1}{2}+\alpha)}) \quad (|x|\to\infty)$

$$\Rightarrow K_\alpha \in L^p \iff p > \frac{2n}{n+1+2\alpha}.$$

By Dual argument. T^α is not bdd

if $|\frac{1}{p}-\frac{1}{2}| \ge 2\alpha+1/2n$.

($T^{\alpha *} \sim K_\alpha^*(x) = K_\alpha(-x)$ need bdd on $L^{p'}$)

(3) Apply on Singular Integral:

Lemma. K has cpt support. $\int K\,dx = 0 \Rightarrow |\hat{K}(\xi)| \le C|\xi|$.

Pf: Suppose $\mathrm{supp}(K) \subseteq B(0,R)$.

Fix $\xi_0>0$. $|\xi|>\xi_0$: $|\hat{K}(\xi)| \le \|K\|_1 \le \frac{\|K\|_1}{\xi_0}|\xi|$

For $|\xi| \le \xi_0$:

$$|\int K(x) e^{2\pi i x\xi}| = |\int K(x)(e^{2\pi i x\xi}-1)|$$

$$\lesssim \int_{B(0,R)} |K(x)||2\pi x\xi|\,dx$$

$$\lesssim_R \|K\|_1 |\xi|.$$

Thm. (D) $K \in L^1(\mathbb{R}^n)$ has cpt support. s.t. $\exists\, a>0$.

$|\hat{K}(\xi)| \lesssim |\xi|^{-a}$. and $\int K\,dx = 0$. Then:

for $K_j(x) = 2^{-jn} K(x/2^j)$. $Tf(x) = \sum_{\mathbb{Z}} K_j * f(x)$

is bdd on L^p. $1<p<\infty$.

Pf: 1) $p=2$:

$$\int |Tf|^2 \le \int \Big(\sum_{\mathbb{Z}} |\hat{K}_j(\xi)|\Big)^2 |\hat f(\xi)|^2 \, d\xi$$

$$\overset{(\text{Lem})}{\lesssim} \int \Big(\sum_j \min(|2^j\xi|^{-a}, |2^j\xi|)\Big)^2 |\hat f|^2$$

$$\lesssim \|f\|_2^2.$$

2) Fix $\varphi \in S(\mathbb{R}^n)$. radial. $\hat\varphi$ support on

$\{1/2 < |\xi| < 2\}$ and $\sum_{\mathbb{Z}} \hat\varphi(2^k\xi) = 1$. $\xi \ne 0$.

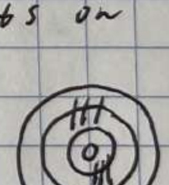
where $\varphi_k(x) = 2^{-kn}\varphi(x/2^k)$.

Note: $K_j = \sum_k K_j * \varphi_{j+k}$. Set $Tf = \sum_{\mathbb{Z}} \tilde T_k f$.

$\tilde T_k f = \sum_{j\in\mathbb{Z}} K_j * \varphi_{j+k} * f. = \sum_j (K*\varphi_k)_j * f$.

Rmk: $\tilde T_k \sim \sum_j \hat K_j \hat\varphi_{j+k}$. $\hat\varphi_{j+k}$ supports on

annuli: $\{2^{-j-k-1} < |\xi| < 2^{-j-k+1}\}$.



Each $\tilde T_k$ only share one

annuli part of each $\hat K_j$. $j\in\mathbb{Z}$.

Note: $\sum_k |\sum_j \hat K_j \hat\varphi_{j+k}| \le 3 \sum |\hat K_j| \overset{1)}{<} \infty$

(Fix one j. At most 3 k. s.t. $\tilde T_k$

will have a nonzero part of $\hat K_j$)

3°) Prove: $\|\widetilde{T}_k f\|_2 \lesssim 2^{-n|k|}\|f\|_2$

$$\|\widetilde{T}_k f\|_2^2 \le \sum_j \sum_{m=j-1}^{j+1} \int |\hat{k}_j \hat{k}_m| |\hat{\varphi}_{j+k} \hat{\varphi}_{m+k}| |\hat{f}|^2$$

$$\overset{(\text{Cont. Lem})}{\lesssim} \sum_j \int_{|\xi| \approx 2^{-j-k}} 2^{-2n|k|} |\hat{f}|^2.$$

$$\lesssim 2^{-2n|k|} \|f\|_2^2$$

4°) Check Hörmander condition on $\widetilde{T}_k$. Fix $\eta \ne 0$:

$$\int_{|x|>2|\eta|} \left| \sum_j (K*\varphi_k)_j(x) - (K*\varphi_k)_j(x-\eta) \right| dx$$

$$\le \int_{\square} \sum_j 2^{-jn} \left| K*\varphi_k\left(\frac{x-\eta}{2^j}\right) - K*\varphi_k\left(\frac{x}{2^j}\right) \right| dx$$

$$= \sum_j \int_{|x|>2^{1-j}|\eta|} |K*\varphi_k(x-2^{-j}\eta) - K*\varphi_k(x)| dx \triangleq \sum I_j$$

5°) Estimate I_j. (Note $\eta \mapsto 2\eta$. $\Rightarrow I_j \to I_{j-1}$.

WLOG. Set $1 \le |\eta| \le 2$. $\mathrm{Supp}(K) \subset B(0,1)$)

$$I_j \le \int |K(z)| \int |\varphi_k(\cdots) - \varphi_k(\cdots)| dx$$

$$\lesssim \|K\|_1 \, 2^{-j-k} \quad \text{by Mean Value Thm.}$$

$I_j \le C$. is direct.

$$I_j \le 2 \int_{|x|>2^{-j}} |K*\varphi_k(x)| dx$$

$$\lesssim \int_{|z|\le 1} |K(z)| dz \int_{|x|>2^{-j-k}} |\varphi(x-2^{-k}z)| dx$$

For $j<0 \Rightarrow |x| > 2|2^{-k}z| \Rightarrow |x-2^{-k}z| > |x|/2$.

$\Rightarrow I_j \lesssim \|k\|_1 \int_{|x|>2^{-j-k-1}} |\varphi(x)| dx$

$\lesssim \|k\|_1 \int_{\square} \|\varphi\|_{S_{N+1}} 1/|x|^{N+1} dx$

$\lesssim 2^{j+k}$

6') By estimate of I_j:

$\sum I_j \lesssim \left(\sum_{j\le -k} 2^{j+k} + \sum_{j\ge -k+1} 2^{-j-k} \right) \le C. \quad k \ge 0$

(the third estimate require $j < 0$).

$\sum I_j \lesssim \left(\sum_{j \ge |k|} 2^{-j-k} + \sum_{j=0}^{|k|-1} 1 + \sum_{j\le -1} 2^{j+k} \right) \le C(1+|k|)$

for $k < 0$.

$\Rightarrow |\{ |\tilde{T}_k f| > \lambda \}| \le \frac{C(1+|k|)}{\lambda} \|f\|_1$ (Recall pf in V.)

7') Interpolate the results in 3') and 6').

For $p \in (1,2)$. $\|\tilde{T}_k f\|_p \lesssim_p 2^{-n\theta|k|} (1+|k|)^{1-\theta} \|f\|_p$.

where $1/p = \theta/2 + 1 - \theta$.

$\Rightarrow \|Tf\|_p \le \sum_k \|\tilde{T}_k f\|_p \lesssim_p \|f\|_p \quad . \quad 1 < p < 2$

8') For $p > 2$. by dual argument.

Cor. If $\Omega \in L^q(S^{n-1})$. for some $q > 1$. $\int_{S^{n-1}} \Omega \, d\sigma = 0$.

Then $Tf(x) = \text{p.v.} \int \frac{\Omega(y')}{|y|^n} f(x-y) dy$ is bdd on $L^p(\mathbb{R}^n)$.

for $\forall \ 1 < p < \infty$.

Pf: Check: $|\hat{k}_0(\xi)| \leq C|\xi|^{-\alpha}$. $\forall \alpha < 1/2$.

for $k_0 = \Omega(x') \chi_{\{1<|x|<2\}} / |x|^n$.

So: $Tf = \sum k_j * f$. $k_j = 2^{-jn} k_0(x/2^j)$.

$$\hat{k}_0(\xi) = \int_{S^{n-1}} \Omega(w) \int_1^2 e^{-2\pi i r(w\cdot\xi)} \frac{dr}{r} \, d\sigma(w)$$

Set $I(t) = \int_1^2 e^{2\pi i r t} \frac{dr}{r}$. $|I(t)| \leq \min\{1, |t|^{-1}\}$.

Rmk: Extend similarly: $Tf = K*f$ where $K(x) = h(|x|) \frac{\Omega(x')}{|x|^n}$. $\Omega \in L^q(S^{n-1})$ for some $q > 1$. has zero average. h satisfies $\sup_{R>0} \frac{1}{R}\int_0^R |h|^2 < \infty$. Then T is bdd on L^p. $1<p<\infty$.

Cor. $\Omega \in L^\infty(S^{n-1})$. $\int_{S^{n-1}} \Omega \, d\sigma = 0$. Then:

T is bdd on $L^p(w)$ if $w \in A_p$.

for $1<p<\infty$. $T \sim \text{p.v.} \frac{\Omega(x')}{|x|^n}$.

Def: (r_j) is seq of Rademacher functions if

$r_j(t) = \operatorname{sgn}(\sin(2\pi \cdot 2^j t))$

Rmk: Rademacher Functions is o.n.b in $L^2[0,1]$.

Thm. (Khinchine Inequality)

For $1 \leq p < \infty$. $\exists C = C(p)$. s.t. $\forall N$. $(a_k)_1^N \subset \mathbb{C}$.

$$\left(\sum_1^N |a_j|^2\right)^{p/2} / C \leq \mathbb{E}\left|\sum_1^N a_j r_j\right|^p \leq C\left(\sum_1^N |a_j|^2\right)^{\frac{p}{2}}$$

Thm. If K, K_j as defined in Thm (∇).

Then: $g(f) = \left(\sum_{\mathbb{Z}} |K_j * f|^2 \right)^{\frac{1}{2}}$ is bdd on

L^p, $\forall\, 1 < p < \infty$.

Pf: 1) Note for $\varepsilon = (\varepsilon_j)$, $\varepsilon_j = \pm 1$. Then:

$T_\varepsilon f = \sum \varepsilon_j K_j * f$ is bdd on L^p for

$\forall\, 1 < p < \infty$, follows from Thm (∇).

2) For $F(t) = \sum a_j r_j(t) \in L^2[0,1]$, s.t. $\sum |a_i|^2 < \infty$.

$F(t) \in L^p[0,1]$, $\forall\, 1 < p < \infty$, and satisfies

$\|F\|_p \underset{p}{\lesssim} \|F\|_2 = \left(\sum |a_i|^2 \right)^{\frac{1}{2}} \underset{p}{\lesssim} \|F\|_p$.

3) $g(f)^p = \left(\sum |K_j * f|^2 \right)^{\frac{p}{2}} \overset{(2))}{\underset{p}{\lesssim}} \int_0^1 \left| \sum r_j(t) K_j * f(x) \right|^p dt$

Then by Fubini, integrate x first and with 1).

Cor. μ is finite Borel measure, with cpt support.

and $|\hat{\mu}(\xi)| \le C |\xi|^{-a}$, for some $a > 0$. Then,

$M f(x) = \sup_j \left| \int f(x - 2^j y)\, d\mu(y) \right|$ is bdd on

L^p, $1 < p \le \infty$.

Pf: Fix $\phi \in \overset{\text{radial}}{\mathcal{S}}(\mathbb{R}^n)$ with cpt supp, $\hat{\phi}(0) = 1$.

Let $\sigma = \mu - \hat{\mu}(0)\phi$ satisfies condition

of Thm (∇).

$\Rightarrow M f(x) \le \sup_j |\sigma_j * f(x)| + |\hat{\mu}(0)| \sup_j |f * \phi_j|$

$\le \left(\sum |\sigma_j * f|^2 \right)^{\frac{1}{2}} + |\hat{\mu}(0)|\, M f(x)$.

Cor. $\forall\ n\geq 2$. $\tilde{M}f(x) = \sup_j \int_{S^{n-1}} f(x-2^j y)\, d\sigma(y)$ is bdd on L^p. $1<p\leq\infty$

where σ is Lebesgue measure on S^{n-1}

Pf: $|\hat{\sigma}(\xi)| \lesssim |\xi|^{(1-n)/2}$. (nontrivial)

Rmk: Set $\bar{M}f(x) = \sup_{t>0} \left| \int_{S^{n-1}} f(x-ty)\, k\sigma(y) \right|$

$\Rightarrow$ We have a more general result:

$\bar{M}$ is bdd on $L^p(\mathbb{R}^n)$. if $p > n/(n-1)$

(4) Maximum Function along Parabolic:

Set: $Lu = \frac{\partial u}{\partial x_2} - \frac{\partial^2 u}{\partial x_1^2} = T_1(Lu) - T_2(Lu)$.

which is called parabolic operator.

where $\begin{cases} (T_1 f)^\wedge(\xi_1,\xi_2) = 2\pi i\xi_2 \hat{f}(\xi) / (2\pi i\xi_2 + 4\pi^2|\xi_1|^2) \\ (T_2 f)^\wedge(\xi_1,\xi_2) = 4\pi^2|\xi_1|^2 \hat{f}(\xi) / (\cdots) \end{cases}$

Note for multiplier $m(\xi_1,\xi_2)$ of L. satisfies:

$\begin{cases} m(\lambda\xi_1, \lambda^2\xi_2) = m(\xi_1,\xi_2). \\ \hat{m}(\lambda\xi_1, \lambda^2\xi_2) = \lambda^{-3}\hat{m}(\xi_1,\xi_2). \end{cases}$ $\forall \lambda>0$. but not degree 0.

It leads us to consider operator of such type:

Def: i) $k(\lambda x_1, \lambda^2 x_2) = \lambda^{-3} k(x_1,x_2)$. $\Omega(\theta) = k(\cos\theta, \sin\theta)$.

Rmk: $K(x_1, x_2) = r^{-3}\,\Omega(\theta)$. for $\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases}$.

ii) $Tf(x_1, x_2) = p.v. \int_{\mathbb{R}^2} K(\eta_1, \eta_2) f(x_1-\eta_1, x_2-\eta_2)\, d\eta_1 d\eta_2$

Rmk: $LHS = \int_0^{2\pi} (1+\sin^2\theta)\,\Omega(\theta)\, H_\theta f(x_1, x_2)\, d\theta$.

st. $H_\theta f(x_1, x_2) = p.v. \int_{\mathbb{R}} f(x_1 - r\cos\theta,\ x_2 - r\,\mathrm{sgn}(r)\sin\theta) \frac{dr}{r}$

if $K(x_1, x_2)$ is odd. (Method of Rotation)

So: $\Omega \in L^1(S^1) \Rightarrow H_\theta$ is uniform bdd on L^p. $1<p<\infty$

$\Rightarrow T$ is bdd on L^p.

iii) $H_I f(x_1, x_2) = p.v. \int f(x_1 - t, x_2 - t^2)\, dt/t$

$M_I f(x_1, x_2) = \sup_{h>0} \left| \frac{1}{2h} \int_{-h}^{h} f(x_1 - t, x_2 - t^2)\, dt \right|$

operators along $I(t) = (t, t^2)$.

Rmk: Decompose H_I, M_I:

Def: $\sigma_j(g) := \int_{2^j<|t|<2^{j+1}} g(t, t^2)\, dt/t$

$m_j(g) := \int_{2^j<|t|<2^{j+1}} g(t, t^2)\, dt/2^{j+1}$.

$\Rightarrow H_I f(x) = \sum_{\mathbb{Z}} \sigma_j * f$.

Lemma. $M_I f \leq 2 \sup_j m_j * |f|$.

Pf: $\left| \frac{1}{2h} \int_{-h}^{h} f(x_1 - t, x_2 - t^2)\, dt \right| \leq \frac{1}{2^{j+1}} \sum_{-\infty}^{j} 2^{i+1} m_i * |f|$

for $2^j \leq h < 2^{j+1}$.

Next, we cosider bddness of $M_I f$ and $H_I f$:

(H_θ, M_θ, are similar. Rotation doesn't effect the Pf)

Thm. $(\sigma_j)_{\mathbb{Z}}$ is seq of finite Borel measure

with $\|\sigma_j\|_{TV} \le c.\ \forall j.$ and $\exists\, \alpha > 0$. st.

$|\hat{\sigma}_j(\xi)| \lesssim \min(|2^j\xi_1|^{\alpha}, |2^j\xi_1|^{-\alpha})$. If

$\sigma^*(f) = \sup_j ||\sigma_j| * f|$ is bdd on L^q

for some $q > 1$. Then: $T(f) = \sum_{\mathbb{Z}} \sigma_j * f$

and $g(f) = \left(\sum_{\mathbb{Z}} |\sigma_j * f|^2\right)^{\frac{1}{2}}$ are bdd

on L^p if $|1/p - 1/2| < 1/2q$.

Pf: 1°) Set $(S_j f)^\wedge(\xi) = \chi_{\Delta_j}(\xi_1)\hat{f}(\xi)$.

$\Rightarrow Tf = \sum_{\mathbb{Z}} T_k f.\quad T_k f = \sum_j \sigma_j * S_{j+k} f$

2°) By $|\hat{\sigma}_j| \le c \min\{\cdots\}$

Argue as before: $\|T_k f\|_2 \lesssim 2^{-\alpha|k|}\|f\|_2$

3°) Fix p_0 st. $\frac{1}{2} - \frac{1}{p_0} = \frac{1}{2q}$.

prove: $\|(\sum|\sigma_j * g_j|^2)^{\frac{1}{2}}\|_{p_0} \lesssim \|(\sum|g_j|^2)^{\frac{1}{2}}\|_{p_0}$

Note $\exists\, u \in L^{?}.\ \|u\|_{?} = 1$. st.

$LHS = \int_{\mathbb{R}^n} \sum |\sigma_j * g_j|^2 u \, dx.$

$\overset{(\text{Hölder})}{\le} \|\sigma_j\|_{TV} \sum \int (|\sigma_j| * |g_j|^2) u$

$\lesssim \sum \int |g_j|^2 \sigma^*(u).$

$\le \|\sigma^*(u)\|_2 \|\sum \int |g_j|^2\|_{p_0/2}$

$\overset{(\text{cond})}{\lesssim} \|\sum \int |g_j|^2\|_{p_0/2}$

4°) $S_i S_j f \neq 0$ only when $i = j$.

$$\|T_k f\|_{p_0} \lesssim \|(\sum_{\mathbb{Z}} |\sigma_j * S_{j+k} f|^2)^{\frac{1}{2}}\|_{p_0}$$

$$\lesssim \|(\sum |S_{j+k} f|^2)^{\frac{1}{2}}\|_{p_0}$$

$$\lesssim \|f\|_{p_0}$$

5°) Interpolate $2 \leq p < p_0$. with dual argument for another side.

6°) For $g(f)$ is similar argue as before.

Thm. (μ_j) is seq of positive Borel measures on $\mathbb{R}^2$.

s.t. $\|\mu_i\|_{TV} \leq C$. $\forall i$. and $\exists \alpha > 0$. s.t.

$$\begin{cases} |\hat{\mu}_i(\xi)| \leq C|2^i \xi_1|^{-\alpha} \\ |\hat{\mu}_i(\xi) - \hat{\mu}_i(0,\xi_2)| \leq C|2^j \xi_1|^{\alpha} \end{cases}$$ Set $M_2 f(\xi_2) =$

$\sup_j |\mu_j^2 * f|$ where $\hat{\mu}_j^2(\xi) = \hat{\mu}_j(0,\xi_2)$.

If M_2 is bdd on $L^p(\mathbb{R}^1)$. $1 < p \leq \infty$. Then:

$Mf(x) = \sup_j |\mu_j * f(x)|$ is bdd on $L^p(\mathbb{R}^2)$. $1 < p \leq \infty$.

Pf: i) $\phi \in S(\mathbb{R}^1)$. s.t. $\hat{\phi}(0) = 1$.

Set $\hat{\sigma}_j(\xi) = \hat{\mu}_j(\xi) - \hat{\mu}_j(0,\xi_2)\,\hat{\phi}(2^j \xi_1)$.

satisfies condition of Thm. above. (MVT. cond.)

Set $\tilde{g} \sim \tilde{\sigma}^* = \sup_j ||\tilde{\sigma}_j| * \cdot|$

Argue as Cor. in (3):

So: $\mathcal{M}f(x) \leq \tilde{g}(f)(x) + C M_2 M_1 f(x)$

$\tilde{\sigma}^* f(x) \leq \mathcal{M}f(x) + C M_2 M_1 f(x)$.

where M_1 is Littlewood Max func on ξ_1.

2') $\tilde{g}(f)$ is bdd on L^2. (Plancherel) $\Rightarrow$ So $\mathcal{M}$

$\Rightarrow$ So $\tilde{\sigma}^*$ (by inequi. above with und.)

Apply Thm above $\Rightarrow$ $\tilde{g}(f)$ bdd on L^p.

for p : $4/3 < p < 4$.

3') By "Bootstraping" argue. repeatedly.

$\mathcal{M}$. $\tilde{\sigma}^*$. $\tilde{g}$ are bdd on L^p. $p \in (1, \infty)$.

<u>Cor.</u> H_I. M_I are bdd on $L^p(\mathbb{R}^2)$ for $p \in (1, \infty)$

and $p \in (1, \infty]$. respectively.

<u>Lemma.</u> (Van der Corput's)

For $I(a,b) = \int_a^b e^{ih(t)} dt$. Then:

i) If $|h'| \geq \lambda > 0$. h' is mono. Then:

$|I(a,b)| \leq 4/\lambda$.

ii) If $h \in C^k[a,b]$. $|h^{(k)}(t)| \geq \lambda > 0$.

Then: $|I(a,b)| \lesssim \lambda^{-1/k}$.

<u>Pf:</u> i) Integrate by part.

ii) For $k=2$: WLOG. $h^{(2)} \geq \lambda > 0$ (Conti)

$\Rightarrow h' \uparrow$. $h'(t_0) = 0$. at most 1 $t_0 \in (a,b)$

If such t_0 exists. Set $J = (t_0-\delta, t_0+\delta) \cap (a,b)$.

$\Rightarrow$ Note on $(a,b)/J$: $|h'(\lambda)| \geq \lambda\delta$. (MVT).

By i) : $\left|\int_{(a,b)/J} e^{ih(t)} dt\right| \leq 8(\lambda\delta)^{-1}$.

with $\left|\int_J e^{ih(t)}\right| \leq 2\delta$. choose $\delta = \lambda^{-\frac{1}{2}}$.

If $h'(t) \neq 0$. Set $t_0 = a$ or b.

For $k>2$. by induction:

$\left|\int_{(a,b)/J} \square\right| \lesssim (\lambda\tilde{\delta})^{-1/k}$. $\left|\int_J \square\right| \lesssim \tilde{\delta}$. Set $\tilde{\delta} = \lambda^{-\frac{1}{k+1}}$

<u>Lemma.</u> $I(b) = \int_1^b e^{i(t\xi_1 + t^2\xi_2)} dt$. Then for

$b \in (1,2)$. $|\xi_1| > 1$. $\Rightarrow$ $|I(b)| \leq C|\xi_1|^{-\frac{1}{2}}$.

<u>Pf:</u> Set $h(t) = t\xi_1 + t^2\xi_2 \in C^\infty$.

1) $|\xi_1| > 8|\xi_2|$. $|h'(t)| > \frac{|\xi_1|}{2}$

2) $|\xi_1| \leq 8|\xi_2|$ $|h''(t)| = 2|\xi_2|$.

Apply the Lemma above.

<u>Return to Pf:</u>

1°) For $|\xi_1| > 1$: $|\hat{m}_0(\xi)|$. $|\hat{\sigma}_0(\xi)| \lesssim |\xi_1|^{-\frac{1}{2}}$.

($\hat{\sigma}_0(\xi) = \int_{1<|t|<2} e^{it\xi_1 + it^2\xi_2}/t \, dt = \int_{\xi_0<|t|<2} e^{it\xi_1 + it^2\xi_2} dt$

by MVT of integral.)

2°) By 1°) : $|\hat{m}_j(\xi)|$. $|\hat{\sigma}_j(\xi)| \lesssim |2^j\xi_1|^{-\frac{1}{2}}$. for $|2^j\xi_1| > 1$.

with $|\hat{\sigma}_j(\xi)|$. $|\hat{m}_j(\xi) - \hat{m}_j(0,\xi_2)| \lesssim |2^j\xi_1|$ by def.

3°) $M_2 f(x_2) = \sup_j \left| \frac{1}{2^{j+1}} \int_{2^j<|t|<2^{j+1}} f(x_2 - t^2) dt \right| \lesssim Mf(x_2)$.

$\Rightarrow M_I$ is bdd on L^p. With $\sigma^* f \leq M_I f \Rightarrow$ So H_I