Weighted Inequilities

(1) A_p Conditions:

Next, we will find $w \geq 0$, L^1_{loc} s.t. $Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f|$ is bdd on $L^p(w)$.

First, consider weighted weak-(p,p) Inequi.:

$$w\{Mf > \lambda\} \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx$$

1°) Set $f \geq 0$. $f(Q) = \int_Q f > 0$ for some cube Q.

Fix $0 < \lambda < f(Q)/|Q| \Rightarrow Q \subseteq \{M(f\chi_Q) > \lambda\}$.

So: $w(Q) \leq \frac{C}{\lambda^p} \int_Q |f|^p w\,dx$

Let $\lambda \to f(Q)/|Q|$. $\Rightarrow$ $w(Q)\,(f(Q)/|Q|)^p \leq C \int_Q |f|^p w$

2°) $\forall S \subseteq Q$ measurable. Set $f = \chi_S$

$\Rightarrow w(Q) \left(\frac{|S|}{|Q|} \right)^p \leq C\, w(S)$.

Rmk: i) $w \equiv 0$ or $w > 0$ a.e.

Pf: If S measurable, $|S| > 0$, $w(S) = 0$

Assume S is bdd (Or approxi it by bdd set)

Then $\forall Q \supseteq S \Rightarrow w(Q) = 0$.

ii) $w \in L^1_{loc}(\mathbb{R}^n)$ or $w = \infty$ a.e.

Pf: If $\exists Q$ cube $w(Q) = \infty$.

Then $\forall \tilde{Q} \supseteq Q$. $w(\tilde{Q}) = \infty$.

$\forall S \subseteq Q$. $w(S) = \infty$ if $|S| > 0$

Next, we separate it into two cases:

i) $p=1$: $w(Q)/|Q| \le C\, w(S)/|S|$.

Set $a = \text{essinf}\{w(x) \mid x \in Q\}$. $\exists S_\varepsilon \subset Q$. s.t. $|S_\varepsilon| > 0$.

$\forall x \in S_\varepsilon$. $w(x) \le a + \varepsilon$. $\Rightarrow \frac{w(Q)}{|Q|} \le C(a+\varepsilon)$.

Let $\varepsilon \to 0$. $\therefore w(Q)/|Q| \le C \operatorname*{essinf}_{Q} w(x)$.

$\Rightarrow w(Q)/|Q| \le C\, w(x)$. a.e. $x \in Q$. $(*)$

Rmk: $(*)$ is called A_1 condition. Actually, it's equi.: $Mw(x) \le C\, w(x)$. a.e. x.

Pf: $(\Leftarrow)$ is trivial. For $(\Rightarrow)$:

If $\exists x$. $Mw(x) > C\, w(x)$. Then $\exists$ Q with rational vertice s.t. $\frac{w(Q)}{|Q|} > C\, w(x)$ $\Rightarrow x \in N_Q \subset Q$. $m(N_Q) = 0$

Take union over such cubes.

$\Rightarrow \{Mw > Cw\}$ has measure 0.

ii) $1 < p < \infty$: Set $f = w^{1-p'} \chi_Q$ in 1°)

We have: $\left(\int_Q w/|Q|\right)\left(\int_Q w^{1-p'}/|Q|\right)^{p-1} \le C$

which's called A_p - condition.

Rmk: For the inequi. makes sense, we can first set $\hat{w} = \min\{w, n\}$. Whatever, it implies

$W^{1-p'} \in L^1_{loc}$. if we let $n \to \infty$. by MCT.

<u>Thm.</u> For $1 \le p < \infty$. weak-(p,p): $W\{Mf > \lambda\} \le \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f|^p w \, dx$

holds $\iff$ $W \in A_p$. (i.e. satisfies A_p-condition)

<u>Pf</u>: ($\Rightarrow$) part is proved above. For ($\Leftarrow$):

1°) $p=1$. we have proved: $W\{Mf > \lambda\} \le \frac{C}{\lambda} \int f \, Mw \, dx$

combined with $Mw \lesssim w$ a.e.

2°) $p>1$: $\left(\frac{1}{|Q|} \int_Q |f| \right)^p \le \left(\frac{1}{|Q|} \int_Q |f|^p w \right) \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1}$

$$\lesssim \frac{1}{|Q|} \int_Q |f|^p w \cdot \left(\frac{|Q|}{w(Q)} \right)$$

Next. assume $f \in L^p(w) \cap L^1_{loc}(\mathbb{R}^n)$. $f \ge 0$.

(Or we can set $f \chi_{B_K} \uparrow f$)

By above: $W(Q) \left(\frac{|S|}{|Q|} \right)^p \lesssim w(S)$ in 2°) holds.

Apply C-Z Decompose on f at height λ.

By Lemma before $\{Mf \ge \lambda\} \le \cup 3Q_j$

$\Rightarrow W\{Mf \ge \lambda\} \le \sum w(3Q_j)$

$$\lesssim 3^{np} \sum w(Q_j). \text{ (inequi. in 2°)},$$

$$\lesssim 3^{np} \sum \left| \frac{|Q_j|}{f(Q_j)} \right|^p \int_{Q_j} |f|^p w$$

$$\lesssim 3^{np} \sum \left(\frac{4^n}{\lambda} \right)^p \int_{Q_j} |f|^p w \underset{n,p}{\lesssim} \|f\|^p_{L^p(w)}$$

<u>Prop.</u> i) $A_p \subset A_q$. $1 \le p < q$ ii) $W \in A_p \iff W^{1-p'} \in A_{p'}$.

iii) $W_0, W_1 \in A_1 \Rightarrow W_0 W_1^{1-p} \in A_p$.

Pf: i) $p>1$ is from Hölder inequility.

$p=1$: $\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1} \leq \sup_Q w^{-1} = (\inf_Q w)^{-1}$

$$\lesssim \left(\frac{w(Q)}{|Q|}\right)^{-1}$$

ii) is easy to check (Dual statement)

iii) is follows from: $w_i^{-1} \leq \sup_Q w_i^{-1} \lesssim \left(\frac{w_i(Q)}{|Q|}\right)^{-1}$ $x\in Q$

Subtitute negative exponent elements in inequi.

Rmk: Converse of iii) is true: i.e.

$w_0, w_1 \in A_1 \Leftrightarrow w_0 w_1^{1-p} \in A_p$

Cor. $w \in A_p \Leftrightarrow w = w_0 w_1^{1-p}$. $w_0, w_1 \in A_1$

(Factorization Thm of A_p-weight, P. Jones)

(2) Strong Type Weighted Inequi.

① First note that if $w \in A_q$. for $1<q<\infty$.

then $L^\infty(w) = L^\infty(\mathbb{R}^n)$ from: $w(Q)\left(\frac{|S|}{|Q|}\right)^q \lesssim w(S)$

(So: $|S|=0 \Leftrightarrow w(S)=0$.)

$\Rightarrow \|Mf\|_{L^\infty(w)} \leq \|f\|_{L^\infty(w)}$. Apply Interpolation:

We have: $\|Mf\|_{L^p(w)} \leq C_p \|f\|_{L^p(w)}$. $\forall p>q$.

Claim: $\|Mf\|_{L^q(w)} \leq C_q \|f\|_{L^q(w)}$ also holds.

Thm (Reverse Hölder Inequility)

If $w \in A_p$. $1 \le p < \infty$. Then $\exists\ C. \varepsilon > 0$. depend on p and A_p-const. of w. st. $\forall\ Q$. cube.

$$\left(\frac{1}{|Q|} \int_Q w^{1+\varepsilon} \right)^{1/1+\varepsilon} \le \frac{C}{|Q|} \int_Q w$$

Lemma. If $w \in A_p$. $1 \le p < \infty$. Then: $\forall \alpha$. $0 < \alpha < 1$. $\exists\ \beta$. $0 < \beta < 1$. st. given a cube Q. and $S \subseteq Q$. with $|S| \le \alpha |Q|$. $\Rightarrow w(S) \le \beta w(Q)$.

Pf: Note: $w(Q) \left(1 - \frac{|S|}{|Q|} \right)^{p} \lesssim w(Q) - w(S)$.

Pf: Fix cube Q. Apply C-Z decompose of w. w.r.t Q. at height $w(Q)/|Q| = \lambda_0 < \lambda_1 \cdots < \lambda_k \cdots$

(k^{th} decompose is on $(Q_{k-1,j})_j$. $(k-1)^{th}$ decompose's cubes. Note $\Omega_k = \bigcup Q_{k,j}$. $\Rightarrow \Omega_k \subset \Omega_{k-1}$)

1°) Fix Q_{k,j_0}. $\Rightarrow Q_{k,j_0} \cap \Omega_{k+1} = \bigcup Q_{k+1,i_t^{j_0}}$.

$$|Q_{k,j_0} \cap \Omega_{k+1}| = \Sigma |Q_{k+1,i}| \le \frac{1}{\lambda_{k+1}} \sum_i \int_{Q_{k+1,i}} w.$$

$$\le \int_{Q_{k,j_0}} w / \lambda_{k+1} \le 2^n \lambda_k |Q_{k,j_0}| / \lambda_{k+1}$$

2°) Choose λ_k. st. $2^n \lambda_k / \lambda_{k+1} = \alpha \in (0,1)$ fix.

By Lemma. $\exists\ \beta \in (0,1)$. $w(Q_{k,j_0} \cap \Omega_{k+1}) \le \beta w(Q_{k,j_0})$

Sum over: $w(\Omega_{k+1}) \le \beta w(\Omega_k)$

$\Rightarrow w(\bigcap_i^n \Omega_k) \to 0$. i.e. $|\bigcap^\infty \Omega_k| = 0$.

3°) $$\frac{1}{|Q|} \int_Q w^{1+\varepsilon} = \frac{1}{|Q|} \left(\int_{Q/\Omega_0} + \Sigma \int_{\Omega_k/\Omega_{k+1}} \right)$$

$$\lesssim \lambda_0^{\varepsilon} \frac{w(Q)}{|Q|} + \frac{1}{|Q|} \sum_k \lambda_{k+1}^{\varepsilon} w(Q_k)$$

$$\lesssim \lambda_0^{\varepsilon} \frac{w(Q)}{|Q|} + \frac{1}{|Q|} \sum_k (2^n \alpha^{-1})^{(k+1)\varepsilon} \lambda_0^{\varepsilon} \beta^k w(Q_0)$$

$$\lesssim \lambda_0^{\varepsilon} \frac{w(Q)}{|Q|} = \left(\frac{w(Q)}{|Q|} \right)^{1+\varepsilon}$$

follows from fix $\varepsilon > 0$, s.t. $(2^n \alpha^{-1})^{\varepsilon} \beta < 1$.

Cor. i) $A_p = \bigcup_{q<p} A_q$. $\forall\, 1 < p < \infty$.

ii) $w \in A_p$. $1 \le p < \infty \Rightarrow \exists\, \varepsilon > 0$. s.t. $w^{1+\varepsilon} \in A^p$.

iii) $w \in A^p$. $1 \le p < \infty \Rightarrow \exists\, \delta > 0$. s.t. given a cube Q. and $S \subset Q$. Then: we have.

$$w(S)/w(Q) \lesssim (|S|/|Q|)^{\delta} \quad (\Delta).$$

Rmk: We call (Δ) as A_∞ condition

Pf: i) $w \in A_p \Rightarrow w^{1-p'} \in A_{p'}$. By reverse Hölder:

$$\exists\, \varepsilon > 0.\ \left(\frac{1}{|Q|} \int_Q w^{(1-p')(1+\varepsilon)} \right)^{1/1+\varepsilon} \lesssim \frac{1}{|Q|} \int_Q w^{1-p'}$$

$$\Rightarrow \exists\ q < p.\quad q' - 1 = (p' - 1)(1+\varepsilon)$$

So: $w \in A_q$

ii) $p > 1$: Choose $\varepsilon > 0$. s.t. w. $w^{1-p'}$ both satisfy reverse Hölder inequility.

$p = 1$: $\frac{1}{|Q|} \int_Q w^{1+\varepsilon} \lesssim \left(\frac{1}{|Q|} \int_Q w \right)^{1+\varepsilon}$

$$\lesssim w^{1+\varepsilon}(x)$$

for a.e. $x \in Q$

iii) $w(S) = \int_Q w \chi_S \le C \left(\int_Q w^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} |S|^{\varepsilon/1+\varepsilon}$

$$\lesssim w(Q) \left(\frac{|S|}{|Q|} \right)^{\varepsilon/1+\varepsilon}.$$

Thm. For $1<p<\infty$. M is bdd on $L^p(w) \Leftrightarrow w \in A_p$.

Pf: By Cor i) above. with argue initially.

Thm. (Characterization of A_∞)

$w \in A_\infty \Longleftrightarrow$ one of following conditions hold:

i) $\exists\ 0<\gamma, \delta<1$. $\forall\ Q$ cube in $\mathbb{R}^n$. s.t.

$|\{x \in Q \mid w(x) \le \gamma\, w(Q)/|Q|\}| \le \delta |Q|$

ii) $\exists\ 0<\alpha, \beta<1$. s.t. $\forall$ cube Q. $A \subseteq Q$ measurable

$|A| \le \alpha |Q| \Rightarrow w(A) \le \beta\, w(Q)$

iii) Reverse Hölder inequality holds for w.

iv) $\exists\ 0<\alpha', \beta'<1$. s.t. $\forall$ cube Q. $A \subseteq Q$ measurable

$w(A) \le \alpha' w(Q) \Rightarrow |A| \le \beta' |Q|$.

Rmk: It's also easy to see M is bdd on

$L^\infty(w) \Leftrightarrow w \in A_\infty$ by definition (Δ).

Cor. $A_\infty = \bigcup_{1\le p<\infty} A_p$.

Pf: $\forall\ p<\infty$. $A_p \subset A_\infty$. Conversely. prove:

$\exists\ C, \varepsilon>0$. s.t. for every cube Q. we have.

$\left(\frac{1}{w(Q)} \int_Q w^{-1-\varepsilon} w\,dx\right)^{1/1+\varepsilon} \le C \int_Q w^{-1} w / w(Q)$.

By analogous arguement in pf of reverse

Hölder inequality: (Note it's for w^{-1}. $w\,dx$)

1') Lemma = Charac. Thm iv)

2') Apply C-Z decompose on w^{-1} w.r.t measure $w\,dx$ (It's easy to prove for $w \in A_\infty$ since $w(kQ) \lesssim_k w(Q)$)

<u>Thm.</u> (Reverse Jensen Inequility)

$w \in A_\infty \iff \exists C > 0$. st. $w(Q)/|Q| \le C \exp(\int_Q \log w / |Q|)$

<u>Pf</u>: ($\Rightarrow$) $\exists p$. $w \in A_p$. $\Rightarrow \forall q > p$. $w \in A_q$. Set $q \to \infty$ in A_q condition.

($\Leftarrow$) is proved by Garcia-Cuerva.

Next. we generalize the consequence to pair of weights : (u, v). For achieving :

$u\{ |Mf| > \lambda \} \lesssim \int |f|^p v / \lambda^p$. (weak-(p,p) inequility)

$$\Rightarrow \begin{cases} A_1 \text{ condition}: & Mu(x) \le C v(x) \text{ . a.e. } x. \\ A_p \text{ condition}: & (u(Q)/|Q|) (\int_Q v^{1-p'} / |Q|)^{p-1} \le C \end{cases}$$

<u>Thm.</u> For that weak-(p,p) inequility holds $\iff$ $(u, v) \in A_p$. (for $1 \le p < \infty$)

<u>Rmk</u>: $(u, v) \in A_p$ is necessary but not sufficient for strong-(p.p) holds.

<u>Pf</u>: By contradiction :

$(u, Mu) \in A_1 \subset A_p \Rightarrow ((Mu)^{1-p'}, u^{1-p'}) \in A_{p'}$. $\forall p$.

(Note $(u, v) \in A_p \Leftrightarrow (v^{1-p'}, u^{1-p'}) \in A_{p'}$)

$\Rightarrow$ Strong-(p', p') holds for $((Mu)^{1-p'}, u^{1-p'})$.

Set $u = |f|$. $\Rightarrow M$ is bdd on L^1.

Contradiction!

<u>Thm</u> For $1 < p < \infty$. M is bdd from $L^p(v)$ to $L^p(u)$.

$\Leftrightarrow$ $\forall Q$. cube : $\int_Q M(v^{1-p'}\chi_Q)^p u\,dx \lesssim \int_Q v^{1-p'} < \infty$.

<u>Rmk</u>: It's equi: M is bdd on the family:

$\{v^{1-p'}\chi_Q\}_Q$. test functions.

(3) <u>A_1 weight and Extrapolation:</u>

<u>Thm.</u> i) $f \in L^1_{loc}$ s.t. $Mf(x) < \infty$ a.e. For $0 \le \delta < 1$.

Then $w = (Mf)^\delta \in A_1$. the const only

depends on δ. (rather than f!)

ii) For $w \in A_1$. Then $\exists f \in L^1_{loc}(\mathbb{R}^n)$. $\delta < 1$.

and $K, K^{-1} \in L^\infty$. s.t. $w = K(Mf)^\delta$.

<u>Pf</u>: i) By def : prove : $\frac{1}{|Q|}\int_Q (Mf)^\delta \underset{\delta}{\lesssim} Mf(x)^\delta$. a.e. $x \in Q$.

1) Decompose $f = f_1 + f_2$. $f_1 = f\chi_{2Q}$.

$Mf \le Mf_1 + Mf_2 \overset{\delta<1}{\Rightarrow} (Mf)^\delta \le (Mf_1)^\delta + (Mf_2)^\delta$.

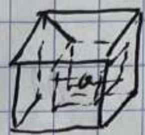
M is weak-(1,1). By Kolmogorov inequality:

$$\frac{1}{|Q|}\int_Q (Mf_1)^\delta \le \frac{C_\delta |Q|^{1-\delta}}{|Q|}\|f_1\|_1^\delta \le C_\delta \left(f(2Q)/|Q|\right)^\delta$$

$$\le 2^{n\delta} C_\delta (Mf)^\delta .$$

2) To estimate Mf_2:

For $y \in Q$. R is cube s.t. $y \in R$. $\int_R |f_2| > 0$.

(Require: $l_R > \frac{1}{2} l_Q$.  $\exists$ $\frac{1}{2}l_Q$, l_Q, $\frac{1}{2}l_Q$.)

Besides. $\exists C_n$ s.t. $x \in Q \Rightarrow x \in C_n R$ $(\forall y, R)$

$$\frac{1}{|R|}\int_R |f_2| \le \frac{C_n^n}{|C_nR|}\int_{C_nR} |f| \le C_n^n Mf(x).$$

$\Rightarrow$ $Mf_2(y) \le C_n^n Mf(x)$. $\forall y \in Q$. $x \in Q$.

So: $\frac{1}{|Q|}\int_Q (Mf_2)^\delta \le C_n^{n\delta} Mf^\delta$.

ii) By reverse Hölder inequality and A_1 condition:

$M(w^{1+\varepsilon})^{1/1+\varepsilon} \lesssim w(x)$. a.e. x.

Set $f = w^{1+\varepsilon}$. $\delta = 1/1+\varepsilon$. So:

$w(x) \overset{(LDT)}{\le} Mf^\delta \le C\, w(x)$. Let $k = Mf^\delta / w$.

Cor. Replace f by finite Borel measure μ in

i). s.t. $M\mu(x) < \infty$ a.e. it still holds.

Rmk: Set $\mu = \delta$ $\Rightarrow$ $M\delta(x) = \sup_{B \ni x} \frac{\delta(B)}{|B|} = c|x|^{-n}$

So: $|x|^a \in A_1$ for $-n < a < 0$.

Pf: $M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B_r)}\int_{B_r} |f(x-y)|\, d\mu(y)$ is weak(1,1) of μ.

Thm: (Extrapolation)

Fix $r \in (1,\infty)$. If T is bdd on $L^r(w)$ for $\forall w \in A_r$.

$\|T\|$ only depends on const. of A_r const. of w.

Then: T is bdd on $L^p(v)$. $1<p<\infty$. $\forall v \in A_p$.

pf: 1°) Prove: for $1<q<r$. $w \in A_1 \Rightarrow T$ is bdd on $L^q(w)$.

Note $Mf^{(r-q)/(r-1)} \in A_1$. $w(Mf)^{q-r} \in A_r$

$$\int_{\mathbb{R}^n} |Tf|^q w = \int |Tf|^q \, Mf^{-(r-q)q/r} \, Mf^{(r-q)q/r} w$$

$$\le \left(\int |Tf|^r w (Mf)^{q-r}\right)^{q/r} \left(\int (Mf)^q w\right)^{r-q/r}$$

$$\lesssim \left(\int |f|^r |f|^{q-r} w\right)^{\frac{q}{r}} \left(\int |f|^q w\right)^{r-q/r}$$

follow from $r-q<0$. T is bdd on $L^r(w(Mf)^{q-r})$.

2°) Prove: $\forall$ fix p. $1<p<\infty$. $\forall q \in (1, \min\{p,r\})$

If $w \in A_{r/q}$. Then: T is bdd on $L^p(w)$

Note $\exists u \in L^{(p/q)'}(w)$. s.t. $\| |Tf|^q \|_{L^{p/q}(w)} =$

$|\langle |Tf|^q, u\rangle|$. i.e. $\left(\int |Tf|^p w\right)^{q/p} = \int |Tf|^q wu$

$$\Rightarrow \int |Tf|^q wu \le \int |Tf|^q M((wu)^s)^{\frac{1}{s}} \quad (s>1)$$

$$\overset{\text{(By 1°)}}{\lesssim} \int |f|^q M((wu)^s)^{\frac{1}{s}}$$

$$= \int |f|^q w^{q/p} \,\square\, w^{-q/p}.$$

$$\lesssim \left(\int |f|^p w\right)^{q/p} A^{1/(p/q)'}$$

$$A = \int M((wu)^s)^{(p/q)'/s} \, w^{1-(p/q)'}$$

Choose s close to 1. s.t. $w^{1-(p/q)'} \in A_{(p/q)'/s}$

$\Rightarrow A \lesssim \int w u^{c(p/2)'} w^{1-c(p/2)'} = C$

3') Note $\forall w \in A_2$. $\exists p>1$. s.t. $w \in A_{2/p}$

By 2') T is bdd on $L^2(w)$.

<u>Cor.</u> For some $s>1$. and $r>s$.

If T is bdd on $L^r(w)$. $\forall w \in A_{r/s}$

Then for $p>s$. T is bdd on $L^p(w)$

for $\forall w \in A_{p/s}$.

<u>Pf:</u> Set $\tilde{r} = r/s$. $\tilde{p} = p/s$.

$\Rightarrow T$ is bdd on $L_{s\tilde{r}}(w)$.

Replace 1') by: $1<q<r$. $w \in A_1$

$\Rightarrow T$ is bdd on $L_{sq}(w)$. (consider $M(f^s)$)

2') Prove T is bdd on $L_{sp}(w)$.

(4) <u>Weighted inequi</u>

<u>for singular integral:</u>

<u>Lemma.</u> If T is C-Z operator. Then for $\forall s>1$.

$M^{\#}(Tf)(x) \le C_s M(|f|^s)(x)^{\frac{1}{s}}$. where $M^{\#}$

is sharp max operator: $M^{\#}f(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f - f_Q|$.

<u>Pf:</u> Prove: Fix $s>1$. for x and $Q \ni x$. $\exists a$

s.t. $\frac{1}{|Q|}\int_Q |Tf - a| \lesssim M(|f|^s)(x)^{\frac{1}{s}}$

Decompose $f = f_1 + f_2$. $f_1 = f\chi_{2Q}$. set $a = Tf_2(x)$.

LHS $\le \frac{1}{|Q|}\int_Q |Tf_1| + \frac{1}{|Q|}\int_Q |Tf_2(y) - Tf_2(x)|\,dy$.

i) $\frac{1}{|Q|}\int_Q |Tf_1| \lesssim_{s,n} M(|f|^s)(x)^{\frac{1}{s}}$ is direct.

follow from T is bdd on $L^s(\mathbb{R}^n)$. $s>1$.

2) Second term $\le \frac{1}{|Q|}\int_Q \left|\int_{\mathbb{R}^n/2Q} [K(x,z) - K(y,z)] f(z)\,dz\right|$

$\lesssim \frac{1}{|Q|}\int_Q \int_{\mathbb{R}^n/2Q} \frac{|y-x|^\delta}{|x-z|^{n+\delta}} |f(z)|\,dz\,dy$.

$\lesssim \frac{l_Q^\delta}{|Q|}\sum_{k=1}^{\infty}\int_Q \int_{2^k l_Q < |x-z| < 2^{k+1} l_Q} \frac{|f|}{|x-z|^{n+\delta}}\,dz\,dy$

$\lesssim M(|f|^s)(x)^{\frac{1}{s}}$.

<u>Lemma.</u> For $1 \le p_0 \le p < \infty$. If $w \in A_p$. f satisfies $M_d f \in L^{p_0}(w)$.

Then. $\|M_d f\|_{L^p(w)} \lesssim \|M^\# f\|_{L^p(w)}$.

<u>Pf:</u> It's weighted version of: $\|M_d f\|_{L^p} \lesssim \|M^\# f\|_{L^p}$.

follows from: $w\{M_d f > 2\lambda,\ M^\# f \le \gamma\lambda\} \lesssim \gamma^\delta w\{M_d f > \lambda\}$.

by A^∞ codition and C-Z decompose on $\{M_d f > \lambda\}$.

<u>Thm</u> For T a C-Z operator. $\forall w \in A_p$. $1<p<\infty$.

$\Rightarrow T$ is bdd on $L^p(w)$.

<u>Pf:</u> Assume $f \in C_c^\infty$ which is dense in $L^p(w)$.

First, prove $Tf \in L^p(w)$. (Suppose $\mathrm{supp} f \subseteq B(0,R)$)

i) $\int_{|x|<2R} |Tf|^p w\,dx \lesssim \left(\int_\square w^{1+\varepsilon}\right)^{1/1+\varepsilon} \left(\int_\square |Tf|^{\frac{p(1+\varepsilon)}{\varepsilon}}\right)^{\varepsilon/1+\varepsilon}$.

with T is bdd on L^p. $p>1$.

2') For $|x| > 2R$:

$$|Tf(x)| = \left| \int_{\mathbb{R}^n} k(y,x) f(y) \right|$$

$$\lesssim \int_{|y|<R} |f(y)| / |x-y|^n \, dy$$

$$\lesssim \|f\|_1 / |x|^n .$$

$$\int_{|x|>2R} |Tf| w \lesssim \sum_{k=1} \int_{2^k R < |x| < 2^{k+1} R} w / |x|^{np}$$

$$\lesssim \sum (2^k R)^{-np} w(B(0, 2^{k+1} R))$$

Note $\exists\, q < p$ s.t. $w \in A_q$ which lets the last equation converges.

3') Denote $q = p/s$, $s > 1$. $w \in A_{p/s}$.

$$\| Tf \|_{L^p(w)}^p \le \| M_s(Tf) \|_{L^p(w)}^p$$

$$\overset{(Lem)}{\lesssim} \| M^\#(Tf) \|_{L^p(w)}^p$$

$$\overset{(Lem)}{\lesssim} \int M(|f|^s)^{p/s} w \lesssim \|f\|_{L^p(w)}^p$$

<u>Thm.</u> T is C-Z operator. $w \in A_1$. Then: T is weak-(1,1) w.r.t w: $w\{|Tf| > \lambda\} \lesssim \frac{c}{\lambda} \int |f| w \, dx$.

<u>Pf.</u> Decompose f at height λ: $f = g + b$.

1') $w\{|Tg| > \lambda\} \underset{n}{\lesssim} \frac{1}{\lambda} \int |g| w \overset{(*)}{\lesssim} \frac{1}{\lambda} \int |f| w$.

prove $(*)$: On each Q_i:

$$\int_{Q_i} |g| w \le \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(y)| \, dy \, w(x) \, dx$$

$$= \int_{Q_i} |f| \frac{w(Q_i)}{|Q_i|} \overset{(A_1)}{\lesssim} \int_{Q_i} |f| w .$$

with $f = g$ on $\mathbb{R}^n / \cup Q_i$.

2°) $w(\cup Q_i^*) \leq \sum w(Q_i^*) \overset{Q_i^* = 2Q_i}{\lesssim} \sum w(Q_i)$

By $w(Q_i) \overset{C|Q_i| \leq \square}{\lesssim} \frac{1}{\lambda} \int_{Q_i} f \quad \frac{w(Q_i)}{|Q|} \overset{(A_1)}{\lesssim} \frac{1}{\lambda} \int_{Q_i} fw$

3°) Denote C_i is center of Q_i:

$w\{x \in \mathbb{R}^n / \cup Q_i^* \mid |Tb(x)| \geq \lambda\} \lesssim$

$\frac{1}{\lambda} \sum \int_{\mathbb{R}^n / Q_i^*} |Tb_i(x)| w(x) dx \lesssim$

$\frac{1}{\lambda} \sum \int_{Q_i} \left(\int_{\mathbb{R}^n / Q_i^*} \frac{|y - C_i|^\delta}{|x - C_i|^{n+\delta}} |b_i(y)| w(x) dx \right) dy.$

$\lesssim \frac{1}{\lambda} \sum \int_{Q_i} |b_i(y)| \, M w(y)$

$\lesssim \frac{1}{\lambda} \sum \int_{Q_i} |b_i| w \lesssim \frac{1}{\lambda} \int (|f| + |g|) w$

__Cor__. For T is C-Z operator. $1 < p < \infty$. Then:

$T^*(\cdot) = \sup_{\varepsilon > 0} |T_\varepsilon(\cdot)|$ is bdd on $L^p(w)$ if

$w \in A_p$. T^* is weak-(1.1) w.r.t. $w \iff w \in A_1$

__Pf__: It's similar as before:

1°) Cotlar's inequi: $T^* f \lesssim_\nu M(|Tf|^\nu)^{\frac{1}{\nu}} + Mf$.

2°) M is weak-(1.1) w.r.t. $w\,dx$.

3°) Kolmogorov Lemma hold for $w\,dx$.

4°) $w\{M(|Tf|^\nu)^{\frac{1}{\nu}} > \lambda\} \lesssim_n w\{M_d(|Tf|^\nu) > \frac{\lambda^\nu}{4^n}\}$.

(for the form is finite measure).

__Rmk__: Note $w \in A_p$ is a sufficient condition for C-Z operator to hold strong-(p.p). weak-(1.1).

A_p is necessary in sense: each of Riesz Transf is weak-(p.p) w.r.t w. $1 \leq p < \infty \Rightarrow w \in A_p$.

__Cr__. H is weak-(p.p) w.r.t. $w \iff w \in A_p$.