H^1 and BMO.

(1) Real Hardy space H_r^1:

① <u>Def</u>: i) A bdd measurable real-valued function $a(x)$ on $\mathbb{R}^n$ is a atom associated with a ball $B \subset \mathbb{R}^n$ if:

(a) $\operatorname{supp} a \subseteq B$. $\|a\|_\infty \le 1/|B|$.

(b) $\int_B a = 0$.

<u>Rmk</u>: $\|a\|_{L^1(\mathbb{R}^n)} \le 1$. by (a).

ii) $H_r^1(\mathbb{R}^n) = \{ f \in L^1 \mid \exists (\lambda_k).\ (a_k)$ atoms. s.t. $f = \sum_1^\infty \lambda_k a_k.\ \sum |\lambda_k| < \infty \}$.

<u>Rmk</u>: $\sum_1^n \lambda_k a_k \xrightarrow[L^1]{n\to\infty} f$. if $f \in H_r^1$.

<u>Rmk</u>: i) $H_r^1(\mathbb{R}^n)$ is another important subtitute for $L^1(\mathbb{R}^n)$. When strong-(p,p) inequi. will break down if $p=1$.

<u>Rmk</u>: You may use the $LS: \{ f \mid |\{|f|>\lambda\}| \le \frac{A}{\lambda}.\ \exists A \in \mathbb{R}_+ \}$. But there's no nontrivial BLF on it.

ii) original def of $H_r^1(\mathbb{R}^n)$ when $n=1$:

Def: $H^p = \{ F \in \mathcal{O}(\mathbb{R}^2_+) \mid \sup_{\eta>0} \int_{-\infty}^{+\infty} |F(x+i\eta)|^p dx < \infty \}$.

Rmk: It can be shown: $F \in H^p$. $p<\infty$

$\Rightarrow F_0 = \lim_{\eta\to 0} F(x+i\eta)$ exists in L^p

Besides: $\|F_0\|_{L^p}^p = \sup_{\eta>0} \int_{-\infty}^{+\infty} |F(x+i\eta)|^p dx$.

Prop. $F \in H^p \iff \exists f \in L^p(\mathbb{R}^1)$. s.t. $2F_0 = f + iHf$.

$\Rightarrow$ For $p=1$, define: $f \in H^1_r(\mathbb{R}^1) \iff f \in L^1(\mathbb{R}^1)$ and $Hf \in L^1(\mathbb{R}^1)$ in "weak-sense". which is for: $F \in H^1$.

Thm. $f \in H^1_r(\mathbb{R}^1) \iff f \in L^1(\mathbb{R}^1)$. $Hf \overset{*}{\in} L^1(\mathbb{R}^1)$ in sense of: $\exists g \in L^1(\mathbb{R}^1)$. s.t. $\int g\varphi = \int f H(\varphi)$.

$\forall \varphi \in S$ (i.e. $g = Hf$ in weak-sense)

iii) $\|f\|_{H^1_r(\mathbb{R}^n)} = \inf \{ \sum |\lambda_k| \mid f = \sum \lambda_i a_i \in H^1_r \}$.

Prop. i) H^1_r is a Banach space. (equipped with $\|\cdot\|_{H^1_r}$)

ii) $f \in H^1_r(\mathbb{R}^n) \Rightarrow f \in L^1$. $\|f\|_{L^1} \le \|f\|_{H^1_r}$.

Pf: ii) $\|f\|_{L^1} = \|\sum \lambda_i a_i\|_{L^1} \le \sum |\lambda_i| \, \|a_i\|_{L^1} \le \sum |\lambda_i|$

i) (f_n) Cauchy in $H^1_r \Rightarrow$ Cauchy in L^1 by ii).

$\Rightarrow \exists f \in L^1$. s.t. $f_n \xrightarrow{L^1} f$. So $\int f = 0$.

$\exists (n_k)$. s.t. $\|f_{n_k} - f_{n_{k+1}}\|_{H^1_r} \le 1/2^k$. $f_{n_k} \xrightarrow{a.e.} f$

$\Rightarrow f_{n_k} \to f$ in H^1_r. So: $f_n \to f$ in H^1_r

where $f = \sum f_{n_k} - f_{n_{k+1}} + f_0 \in H^1_r$

Prop (*): $f \in L^p(\mathbb{R}^n)$, $p > 1$. $\mathrm{supp}(f)$ is bdd. Then:

$$f \in H^1_r(\mathbb{R}^n) \iff \int f\,dx = 0$$

Rmk: i) For $f \in L^1$. $\int f = 0$. it's the necessary cond. for away from $f \in H^1_r$

ii) Actually. drop the requirement: $\int a = 0$ for atoms. Then $f = \Sigma \lambda_k a_k$. represents arbitrary func. in $L^1(\mathbb{R}^n)$ if $\Sigma |\lambda_k| < \infty$.

Pf: ($\Rightarrow$) is trivial. For ($\Leftarrow$).

Suppose $\mathrm{supp} f \subseteq B(0,1)$. by normality.

Apply 2^{nd}-method of C-Z decompose on $\{Mf > \alpha\} = E_\alpha$: $f = g + b$.

Decompose at height $\alpha = 2^k$. $k \in \mathbb{Z}^+$.

obtain g_k and $(A_j^k)_j$ supp on (Q_j^k).

Note $|E_{2^k}| \xrightarrow{k \to \infty} 0 \Rightarrow b_k = \sum_j A_j^k \to 0$.

$f = g_0 + \Sigma(g_{k+1} - g_k)$

i) $g_{k+1} - g_k = \Sigma b_j^k - \Sigma b_i^{k+1} = \sum_j (b_j^k - \sum_{i: Q_i^{k+1} \subset Q_j^k} b_i^{k+1})$

$=: \sum_j A_j^k$. $\mathrm{supp}(A_j^k) \leq Q_j^k$.

$|g_{k+1} - g_k| \leq C_n(2^{k+1} + 2^k) \underset{n}{\lesssim} 2^k$

So $|A_j^k| \leq C_n 2^k$. ($\mathrm{supp}(A_j^k) \cap \mathrm{supp}(A_i^k) = \emptyset$)

2) From $f = g_0 + \sum_{k,j} A_j^k$.

set $\lambda_j^k = c' 2^k m(a_j^k)$. $A_j^k = \lambda_j^k a_j^k$.

a_j^k is atom associates :  B_j^k.

Smallest ball B_j^k contain a_j^k.

$\sum \lambda_i^k = c' \sum 2^k m(\{ Mf > 2^k \})$

$\lesssim \int_0^\infty m\{ Mf > a \} da = \| Mf \|_{L^1} \le \tilde{C}$

g_0 is multiple of an atom.

② Alternative Def:

Def: For $p > 1$. a p-atom associated with a ball.

B is a real-valued measurable func. a. st.

i) $supp(a) \subseteq B$. $\|a\|_{L^p} \le m(B)^{1/p - 1}$.

ii) $\int a \, dx = 0$.

Rmk: i) $\forall$ atom is a p-atom.

ii) Set $p = \infty$ is def of atom.

Thm. Fix $p > 1$. $\forall$ p-atom a. $a \in H_r^1$. st. $\exists C_p > 0$.

indept of a. st. $\|a\|_{H_r^1} \le C_p$.

Rmk: $C_p = O(1/_{p-1})$ $(p \to 1^+)$.

pf: 1°) Rescale:

Note $a_r(x) = r^n a(rx)$. $supp(a_r) \subset \frac{1}{r} B$.

$\|a_r\|_{L^p} = r^{n - n/p} \|a\|_{L^p} \le m(\frac{1}{r} B)^{1 - 1/p}$

$\Rightarrow a_r \in H^1$. WLOG, suppose $r(B) = 1$.

2) Apply the pf of prop.(*) above.

<u>Cor.</u> If $f = \sum \lambda_k a_k$, with p-atom a_k, and $\sum |\lambda_k| < \infty$. Then $f \in H^1$ and

$$\|f\|_{H^1} \leq C_p \sum_1^\infty |\lambda_k| .$$

<u>Rmk:</u> We replace "atom" in definition of H^1 by "p-atom", $p > 1$, which has equi. norm.

③ Application:

Next, we will see H^1 is good substitute of L^1.

<u>Thm.</u> If $f \in H^1(\mathbb{R}^1)$. Then: $H_\varepsilon f \in L^1(\mathbb{R}^1)$. $\forall \varepsilon > 0$.

Besides: $H_\varepsilon f \xrightarrow[\varepsilon \to 0]{L^1} Hf$, satisfies: $\|Hf\|_{L^1} \lesssim \|f\|_{H^1}$.

<u>Rmk:</u> $(C_c^\infty(\mathbb{R}^1) \underset{dense}{\subseteq} (H^1, \|\cdot\|_1)) \Rightarrow H_\varepsilon f \xrightarrow{a.e.} Hf$.

So: H is bdd from H^1 to L^1.

<u>Pf.</u> It suffices to prove: $\forall$ atom a.

We have: $\|H_\varepsilon(a)\|_{L^1} \leq A$. (indept of ε, a.)

1') Note: $H(a_r) = r H(a)(rx)$.

$$\|rF(r(x+h))\|_{L^1} = \|F\|_{L^1}.$$

WLOG. suppose a is atom associated with interval $|I| \leq \frac{1}{2}$.

2') $\int_{|x|\le 1} |H_\varepsilon(a)| \lesssim \| H_\varepsilon(a)\|_{L^2}$

$$\lesssim \|a\|_{L^2} \le 1.$$

3') $\int_{|x|>1} |H_\varepsilon(a)| = \frac{1}{\pi}\int_{|x|>1} \left| \int_{|x-t|>\varepsilon} \frac{a(t)\,dt}{x-t} \right| dx$

$$= \frac{1}{\pi}\int_{|x|>1} \left| \int_{|x-t|>\varepsilon} a(t)\left(\frac{1}{x-t} - \frac{1}{x}\right) \right|$$

$$\lesssim \int_{|x|>1} 1/|x|^2 \, dx$$

4') For $f = \sum_1^\infty \lambda_k a_k \in H^1_r$.

$\| H_\varepsilon(f)\|_{L^1} \le \sum |\lambda_k| \, \| H_\varepsilon(a_k)\|_{L^1} \lesssim \sum |\lambda_k|$.

5') For L^1 converge. truncate f: $f_N = \sum_1^N \lambda_k a_k$.

<u>Cor.</u> H maps $H^1_r(\mathbb{R}^1)$ to $H^1_r(\mathbb{R}^1)$ actually.

<u>Rmk:</u> For general C-Z operator T.

T is bdd on $H^1_r(\mathbb{R}^n)$. So T

is bdd: $H^1_r(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ (proved in (2))

<u>Thm.</u> $\phi \in C^1_c(\mathbb{R}^d)$. $M(f)(x) = \sup_{\varepsilon>0} |f * \phi_\varepsilon|(x)$ is bdd

from $H^1_r(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$. i.e. $\|Mf\|_{L^1} \le A \|f\|_{H^1_r}$.

<u>Rmk:</u> $|f(x)| \le Mf(x) \lesssim f^*(x)$. a.e. x.

<u>Pf:</u> prove: $\|Ma\|_{L^1} \le A$. indep of a. $\forall a$ atom.

WLOG. a associates with a ball $B(0,1)$

1') Note $|Ma| \le C$. $\Rightarrow \int_{|x|\le 2} Ma \le \tilde{C}$.

2') $a * \phi_\varepsilon = \int_{\mathbb{R}^d} \frac{1}{\varepsilon^d}\, a(y)\left(\phi\left(\frac{x-y}{\varepsilon}\right) - \phi\left(\frac{x}{\varepsilon}\right)\right) dy$

Note $|x|\geq 2$. $|y|\leq 1 \Rightarrow |x|/2 \leq |x-y|$

$|\phi(x-y/\varepsilon) - \phi(x/\varepsilon)| \lesssim \frac{1}{\varepsilon}$

With $|\frac{x-y}{\varepsilon}|\leq A$. since $\mathrm{supp}\,\phi$ is cpt.

$\Rightarrow |a * \phi_\varepsilon| \leq c\,\varepsilon^{-d-1} \lesssim 1/|x|^{d+1}$.

$\therefore \int_{|x|\geq 2} Ma \lesssim \int 1/|x|^{d+1} \leq c < \infty$.

<u>Thm</u>: (Character of H^1_r)

$\phi\in\mathcal{S}$. $\int\phi\neq 0$. $M(f) =: \sup_{\varepsilon>0} |\phi_\varepsilon * f|$. Then:

$f\in H^1_r(\mathbb{R}^d) \iff M(f)\in L^1(\mathbb{R}^d)$.

<u>Rmk</u>: It means: $\|f\|_{H^1_r} \lesssim \|Mf\|_{L^1} \lesssim \|f\|_{H^1_r}$. $\forall f$

as well. reverse of Thm. above.

(2) <u>Hardy Space $H^1(\mathbb{R}^d)$</u>:

We replace "atom" in (1) by complex-valued measurable function a st. $\mathrm{supp}(a)\subset Q$ (cube)

$\int_Q a(x)\,dx = 0$. $\|a\|_\infty \leq 1/|Q|$.

<u>prop</u>.$^{(*)}$ T is bdd on L^2. with kernel K on $\mathbb{R}^d\times\mathbb{R}^d/\Delta$. st.

$\int_{|x-y|>2|y-z|} |K(x,y) - K(x,z)| \leq C$. and $Tf(x) =$

$\int K(x,y) f(y)\,dy$ if $f\in L^2$. $\mathrm{supp}(f)\not\ni x$ is cpt.

Then: $\exists\, C>0$. st. $\|Ta\|_1 \leq C$. $\forall$ atom a.

Pf: $a \in L^2 \Rightarrow Ta$ is well-def. Set Q^* have same center as Q. with $2\sqrt{n}$ side length of Q. (Denote center c_Q)

1') $\int_{Q^*} |Ta| \le |Q^*|^{\frac{1}{2}} (\int_{Q^*} |Ta|^2)^{\frac{1}{2}}$

$\lesssim |Q|^{\frac{1}{2}} \|a\|_2 \le C$

2') $\int_{\mathbb{R}^n/Q^*} |Ta(x)| = \int_{\mathbb{R}^n/Q^*} |\int_Q (K(x,y) - K(x, c_Q)) a|$

$\le \int_Q \int_{\mathbb{R}^n/Q^*} |K(x,y) - K(x,c_Q)||a|$.

$\lesssim \int_Q |a| \le 1$.

Def: Atom space H^1 defined by: $H^1_{at}(\mathbb{R}^n) = \{ \sum \lambda_i a_i \mid \lambda_i \in \mathbb{C}$, a_i is atom, $\sum |\lambda_i| < \infty \} \subseteq L^1(\mathbb{R}^n)$. with norm: $\|f\|_{H^1_{at}} = \inf \{ \sum |\lambda_i| \mid f = \sum \lambda_i a_i \}$.

prop. $H^1_{at}(\mathbb{R}^n)$ is Banach space.

Thm. T is Operator as in Prop $(*)$. Then T is bdd from H^1_{at} to L^1: $\|Tf\|_1 \lesssim \|f\|_{H^1_{at}}$.

Pf: It's direct by prop $(*)$.

Rmk: The Thms holds in $H^1(\mathbb{R}^n)$ also holds in $H^1_{at}(\mathbb{R}^n)$ parallely. (About (1))

Def: $H^1(\mathbb{R}^n) = \{ f \in L^1(\mathbb{R}^n) \mid R_j f \in L^1$. $1 \le j \le n$. R_j is Riesz transform $\}$. with norm: $\|f\|_{H^1} = \|f\|_1 + \sum_1^n \|R_j f\|_1$

Rmk: $H^1_{at}(\mathbb{R}^n) \subseteq H^1(\mathbb{R}^n)$ follows from

$R_i \in$ C-Z operator.

Thm $H^1(\mathbb{R}^n) = H^1_{at}(\mathbb{R}^n)$. $\|\cdot\|_{H^1} \sim \|\cdot\|_{H^1_{at}}$.

Thm (Interpolation)

T is sublinear. T is weak-(p_1, p_1) for some $p_1 \in (1, \infty]$. and bdd from H^1 to L^1. Then $\forall\ p \in (1, p_1)$. T is bdd on L^p.

Rmk: It's another example to replace weak-(1,1) cond. by H^1.

(3) Space BMO:

BMO is a natural subtitute for L^∞.

Def: i) For $f \in L^1_{loc}(\mathbb{R}^n)$. $f_Q =: \int_Q f / |Q|$.

ii) $BMO = \{ f \in L^1_{loc}(\mathbb{R}^n) \mid M^\# f \in L^\infty \}$. where $M^\#$ is sharp max function: $M^\# f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f - f_Q| dx$. with a norm:

$\|f\|_* = \| M^\# f \|_{L^\infty}$.

Rmk: i) $\|\cdot\|_*$ is a seminorm actually. But we consider $BMO = \{ f \in L^1_{loc} \mid M^\# f \in L^\infty \} / \{ f \equiv \text{const} \}$. which lets $\|\cdot\|_*$ a norm.

ii) BMO means "Bdd Mean Oscillation"

iii) $M^{\#}f(x) \leq C_n Mf(x)$. pointwise. So it's also strong-(p,p). $p>1$. weak-$(1,1)$.

① Next. we will prove: $H^1(\mathbb{R}^n)^* = BMO(\mathbb{R}^n, \mathbb{R})$.

$(H^1(\mathbb{R}^n)^* = BMO(\mathbb{R}^n, \mathbb{C})$ similarly$)$.

<u>Def</u>: $H_0^1 \subset H_r^1$ is space of finite linear combination of atoms.

<u>Rmk</u>: $\forall f \in H_0^1$. f is multiple of an atom.

<u>Thm.</u> $g \in BMO$. Then LF: $l(f) = \int_{\mathbb{R}^n} fg$ defined on H_0^1 has a unique extension $\tilde{l}$. st. $\|\tilde{l}\| \leq C_n \|g\|_*$. Conversely. $\forall l \in H_r^{1*}$. it can be written as: $l(f) = \int fg$. with $g \in BMO$. and $\|g\|_* \leq \tilde{C}_n \|l\|$.

<u>Rmk</u>: We consider H_0^1 rather H_r^1 at first because $\exists g \in BMO$. $f \in H_r^1$. st. $\int fg$ doesn't converge.

<u>Lemma.</u> (Equi. norm)

i) $\frac{1}{2}\|f\|_* \leq \sup\limits_{Q} \inf\limits_{a \in \mathbb{C}} \frac{1}{|Q|}\int_Q |f-a|\,dx \leq \|f\|_*$.

ii) $M^{\#}(|f|) \leq 2M^{\#}(f)$.

Pf: i) $\int_Q |f - f_Q| \le \int_Q |f-a| + \int_Q |a - f_Q|$

$$\le 2 \int_Q |f-a| .$$

ii) $\frac{1}{2} \fint_Q \big||f| - |f_Q|\big| \le \frac{1}{2} \fint_Q |f - f_Q|$

$$\le \fint_Q |f-a|\, dx$$

Cor. $L^\infty \subsetneq BMO$. $\|f\|_* \le 2 \|f\|_\infty$. $\forall f \in L^\infty$.

e.g. $f(x) = \log|x| \in BMO / L^\infty$.

Cor. BMO forms a lattice: $f, g \in BMO$.

Then: $f \wedge g$, $f \vee g \in BMO$. s.t.

$$\begin{cases} \|f \wedge g\|_* \le \frac{3}{2} (\|f\|_* + \|g\|_*) \\ \|f \vee g\|_* \le \frac{3}{2} (\|f\|_* + \|g\|_*) \end{cases}$$

Pf: Write $f \wedge g = \dfrac{-|f-g| + f+g}{2}$.

Rmk: i) $|f| \ge |g| \not\Rightarrow g \in BMO$ if $f \in BMO$.

ii) $|f| \in BMO \not\Rightarrow f \in BMO$. Actually.

absolute value reduces intense of oscillation.

e.g. $f(x) = \begin{cases} \log(1/|x|) , & |x| \le 1 \\ 0 , & |x| > 1 \end{cases}$ on $\mathbb{R}^1$.

$\operatorname{sgn}(x) f(x) \notin BMO$. But $|\operatorname{sgn}(x) f(x)| = f(x) \in BMO$.

Lemma. $f \in BMO$. $f^{(k)} = \min\{\max\{f, k\}, -k\}$.

truncation of f. Then: $\|f^{(k)}\|_* \xrightarrow{k\to\infty} \|f\|_*$.

Pf: 1°) $f_B |f - f_B| \overset{\text{Fatou's}}{\le} \varliminf_k f_B |f^{(k)} - f_B^{(k)}|$

$\le \varliminf_k \|f^{(k)}\|_*$.

2°) Another side is tricky.

Return to pf:

1°) Suppose $g \in BMO$. Consider $g^{(i)}$. bdd.

Note: $\int a_k g^{(i)} = \int a_k (g - g_{B_k}^{(i)})$ (supp $a_k = B_k$)

$|\int f g^{(i)}| \le \sum |\lambda_k| f_{B_k} |g^{(i)} - g^{(i)}_{B_k}|$

$\le \|f\|_{H^1} \|g^{(i)}\|_* \xrightarrow{i\to\infty} \|f\|_{H^1} \|g\|_*$

for $f = \sum_1^N \lambda_k a_k \in H_0^1$. LHS $\xrightarrow{DCT} |l(f)|$)

Then apply Hahn-Banach Thm.

2°) For converse:

Test with p-atoms. when $p=2$.

Fix a ball B. $L_B^2 = \{f \mid \|f\|_{L_B^2} = (\int_B f^2)^{\frac{1}{2}} < \infty\}$

$L_{B,0}^2 = \{\int f = 0\} \cap L_B^2$.

Assume $\|l\| \le 1$. (normalized)

3°) Note: $|l(f)| \le \|f\|_{H^1} \le \widetilde{C}_n |B|^{\frac{1}{2}} \|f\|_{L_{B,0}^2}$

for $f \in L_{B,0}^2$ ($\lambda = |B|^{\frac{1}{2}} \|f\|_{L_{B,0}^2}$)

By Riesz Representation. $\exists\, \eta \in L^2_{B,0}$.

s.t. $\int f \eta^B = \ell(f)$. $\forall f \in L^2_{B,0}$.

with $\|\eta^B\|_{L^2_{B,0}} = \|\ell\| \le \tilde{C}_n |B|^{\frac{1}{2}}$

4) If $B_1 \subset B_2$. Then $\eta^{B_1} - \eta^{B_2} = \text{const}$ on B_1

since $\ell(f) = \int_{B_1} f \eta^{B_1} = \int_{B_1} f \eta^{B_2}$. $\forall f \in L^2_{B_1,0}$.

Set $\tilde{\eta}^B = \eta^B + C_B$. s.t. $\int_{|x|\le 1} \tilde{\eta}^B = 0$.

$\Rightarrow \tilde{\eta}^{B_1} = \tilde{\eta}^{B_2}$ on B_1 if $\{|x|\le 1\} \subseteq B_1 \subset B_2$.

(if $\tilde{\eta}^{B_2} = c\chi_{B_1} + \tilde{\eta}_{B_1}$. $\int_{|x|\le 1} \tilde{\eta}^{B_2} = 0 = c$)

5) Set $\eta(x) = \tilde{\eta}^B(x)$. $\forall x \in B$. Well-def.

$⨍_B |\eta - C_B| \le |B|^{-\frac{1}{2}} \|\tilde{\eta}^B - C_B\|_{L^2_B}$

$\le |B|^{-\frac{1}{2}} \|\eta^B\|_{L^2_{B,0}} \le \tilde{C}_n$.

$\Rightarrow \|\eta\|_* \le \tilde{C}_n \|\ell\|$.

Note it holds for $H^1_0 \subseteq L^2_{B,0}$. Then

by continuity extenxison. and DCT

__Cor.__ BMO is Banach space.

② With singular Integral:

__Thm.__ T is bdd on L^2. with kernel K on $\mathbb{R}^n \times \mathbb{R}^n / \Delta$.

s.t. $Tf(x) = \int K(x,y) f(y)\, dy$. for $f \in L^2$ has

cpt support. if $x \notin \operatorname{supp}(f)$. and :

$\int_{|x-y|>2|x-w|} |K(x,y) - K(w,y)|\, dy \le C$.

Thm: f is bdd with a cpt support $\Rightarrow Tf \in BMO$

and $\|Tf\|_* \le c\|f\|_\infty$

Pf: Q^* is cube with $2\sqrt{n}$ times length of Q.

having same center C_Q as Q.

Decompose $f = f_1 + f_2$. $f_1 = f\chi_{Q^*}$.

$f_Q |Tf(x) - Tf_2(C_Q)| \le f_Q |Tf_1| + f_Q |Tf_2(x) - Tf_2(C_Q)|$

$\lesssim \|f\|_\infty$. (routine)

Rmk: i) bdd with cpt support functions are not

dense in L^∞. So it can't extend by

continuity of T directly.

Next, we redefine T: For $Q_0 = [-\frac{1}{2}, \frac{1}{2}]^n$, $f_1 = f\chi_{Q_0}$

Note $Tf_1 \in L^2$ by $f_1 \in L^2$. Well-def.

Set $\bar{T}f = Tf_1 + \int_{\mathbb{R}^n} (K(x,y) - K(0,y)) f_2(y) dy$, $x \in Q_0$

Rmk: Let $\bar{Q}$ contains Q_0. $\bar{f}_1 = f\chi_{\bar{Q}^*}$.

define $\tilde{T}f = T\bar{f}_1 + \int_{\mathbb{R}^n} (K(x,y) - K(0,y)) \bar{f}_2(y) dy$.

$\tilde{T}f - \bar{T}f = -\int_{\bar{Q}^*/Q^*} K(0,y) f(y) = \text{const}$.

$\Rightarrow \tilde{T}f = \bar{T}f$ in BMO.

So we can def on any cube Q.

$\Rightarrow$ Similarly, prove: $f \in L^\infty \Rightarrow \hat{T}f \in BMO$.

ii) In particular, Hilbert Transform is

from BMO to BMO.

iii) Indirect form to prove $\log|x|$ is BMO. Set $f(x) = \mathrm{sgn}(x) \in L^\infty$

$$Hf(x) = \frac{2}{\pi}(\log|x| - \log\pi) \in BMO.$$

③ Characterization of BMO:

For $b \in L^1_{loc}(\mathbb{R}^n)$. Define: $M_b f = b(x) f(x)$.

Set $[b, T] = M_b T - T M_b$. T is singular integral.

If $f \in C_c^\infty$. Then $[b,T]f = \int (b(x) - b(y)) K(x,y) f(y)\, dy$.

$x \notin \mathrm{supp}(f)$ for $T = K * f$.

Rmk: $b \in L^\infty \Rightarrow [b, T]$ is bdd on L^p. $1 < p < \infty$.

Thm. (Coifman. Rochberg. Weiss)

$[b, T]$ is bdd on L^p $1 < p < \infty$ for T is C-Z operator $\Leftrightarrow b \in BMO$.

④ VMO (Vanish mean Oscillation):

Def: $VMO = \{ f \in BMO \mid \fint_Q |f - f_Q| \xrightarrow{|Q| \to \infty / 0} 0. \}$

Rmk: i) $C_0 \subset VMO$.

ii) $\overline{C_0} = VMO$ in BMO.

e.g.
$$f(x) = \begin{cases} \log\log 1/|x|. & |x| < 1/e \\ 0 & |x| \geq \frac{1}{e}. \end{cases}$$

$f \in VMO$.

Thm. T is singular integral operator.

If $f \in C_0$. Then $Tf \in VMO$.

(4) Inequility:

① Interpolation:

Thm. T is linear operator. bdd on L^{p_0} for some $p_0 \in (1, \infty)$. and bdd from L^∞ to BMO.

Then $\forall\, p \in (p_0, \infty)$. T is bdd on L^p.

Lemma. If $f \in L^{p_0}$ for some $p_0 \in (1, \infty)$. Then $\forall\, \gamma > 0$ and $\lambda > 0$. $|\{M_d f > 2\lambda,\ M^{\#} f \le \gamma\lambda\}| \le 2^n \gamma\, |\{M_d f > \lambda\}|$

Pf: WLOG. $f \ge 0$. Apply C-Z Decompose on $\{M_d f > \lambda\}$. $\Rightarrow \bigcup Q_j$ disjoint.

Prove: $|\{x \in Q \mid M_d f > 2\lambda.\ M^{\#} f < \gamma\lambda\}| \le 2^n \gamma\, |Q|$

for one of (Q_j) cube Q.

If $\{\cdots\} = \emptyset \Rightarrow$ it's nothing to prove.

Assume $\exists\, x \in Q$. s.t. $M^{\#} f < \gamma\lambda$.

Note: On $\{\cdots\}$:

$M_d((f - f_{2Q})\chi_Q) \ge M_d(f\chi_Q) - f_{2Q} > 2\lambda - \lambda = \lambda$

$\Rightarrow |\{\cdots\}| \le |\{M_d((f - f_{2Q})\chi_Q) > \lambda\}|$

$$\overset{\text{weak-}(1,1)}{\le} \frac{1}{\lambda} \int_Q |f(x) - f_{2Q}|\, dx$$

$$\le \frac{2^n |Q|}{\lambda} \inf_{x \in Q} M^{\#} f \le 2^n \gamma\, |Q|.$$

Lemma. If $1 \le p_0 \le p < \infty$. $f \in L^{p_0}$. Then:

$$\|M_d f\|_p \lesssim \|M^\# f\|_p.$$

Pf: Truncate: $I_N = \int_0^{\frac{N}{2}} p 2^p \cdot \lambda^{p-1} |\{M_d f > 2\lambda\}| d\lambda$

$$\le 2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} (|\{\cdots\}| + |\{M^\# f > \lambda\}|$$

$$\overset{(\text{lem})}{\le} 2^{p+n} \int_0^{\frac{N}{2}} p\lambda^{p-1} (\gamma |\{M_d f > \lambda\}| + |\{M^\# f > \lambda\}|)$$

$$= 2^{p+n}\gamma I_N + \frac{2^{p+n}}{\gamma p} \int_0^{\gamma N/2} p\lambda^{p-1} |\{M^\# f > \lambda\}|$$

choose γ s.t. $2^{p+n}\gamma = \frac{1}{2}$.

Set $N \to \infty$. ($f \in L^{p_0}$ for $I_N \underset{p,p_0}{\lesssim} N^{p-p_0} \|f\|_{p_0}$)

Return to pf:

$M^\# \circ T$ is sublinear. bdd on L^{p_0}. L^∞.

Note: $\|M^\#(Tf)\|_\infty = \|Tf\|_* \le C\|f\|_\infty$.

Apply interpolation Thm. on $M^\# \circ T$.

Truncate f: $f^{(k)} = f \chi_{|f| \le k}$.

$$\int |Tf^{(k)}|^p \le \int |M_d(Tf^{(k)})|^p$$

$$\lesssim \int |M^\#(Tf^{(k)})|^p \lesssim \int |f^{(k)}|^p.$$

Set $k \to \infty$. $\|Tf\|_p \lesssim \|f\|_p$.

② John-Nirenburg Inequility:

Next, we consider the growth rate of BMO.

For $\lambda > 1$. $|\log(1/|x|) - (1-\log a)| > \lambda$ has measure $2a e^{-\lambda-1}$ ($\frac{1}{2a}\int_{-a}^{a} \log(1/|x|) = 1 - \log a$)

Actually. Logarithm growth (in sense of above) is the possible max rate for BMOs.

<u>Thm</u> (John - Nirenberg Inequality)

If $f \in BMO(\mathbb{R}^n)$. Then. $\exists\, C_1, C_2 > 0$ const. depend on n. st. $\forall Q \in \mathbb{R}^n$. cube. $\forall \lambda > 0$. We have:

$$|\{x \in Q \mid |f(x) - f_Q| > \lambda\}| \le C_1 e^{-C_2\lambda/\|f\|_*} |Q|.$$

<u>Rmk</u>: The idea is: Do C-Z decompose on the bad part of f to "Constantize".

<u>Pf</u>: 1) WLOG. $\|f\|_* = 1$. So: $⨍_Q |f - f_Q| \le 1$. $\forall Q$.

Apply C-Z Decompose of $f - f_Q$ on Q at height 2. obtain $(Q_{1,i})_i$.

Note: $⨍_{Q_{1,i}} |f - f_{Q_{1,i}}| \le 1$ ($\|f\|_* = 1$ again)

$$|f_{Q_{1,i}} - f_Q| = |⨍_{Q_{1,i}} (f - f_Q)| \le 2^{n+1}.$$

Apply C-Z decompose on each $f - f_{Q_{1,i}}$ on $Q_{1,i}$ again. obtain $(Q_{2,i})_i = \bigcup_i (Q_{1,i,k})_k$

We have: $\sum |Q_{1,i,k}| \le \int_{Q_{1,i}} \frac{1}{2} |f - f_{Q_{1,i}}| \le \frac{|Q_{1,i}|}{2}$.

$$\Rightarrow \sum |Q_{2,i}| \le \frac{1}{2} \sum |Q_{1,i}| \le \frac{1}{4} |Q|.$$

Besides. $\forall x \notin \bigcup_i Q_{2,i}$. $|f - f_Q| \le |f - f_{Q_{1,i}}| + |f_{Q_{1,i}} - f_Q|$

$$\le 2 + 2^{n+1} \le 2^{n+1} \cdot 2$$

Repeat the procedure. Obtain $(Q_{N,j})_j$

s.t. $\begin{cases} \sum_j |Q_{N,j}| \le 2^{-N}|Q| \\ |f(x) - f_Q| \le N \cdot 2^{n+1}. \ \forall x \notin \bigcup_j Q_{N,j} \end{cases}$

2°) Fix $\lambda \ge 2^{n+1}$. Let N be: $N \cdot 2^{n+1} \le \lambda < (N+1) \cdot 2^{n+1}$

$$|\{x \in Q \mid |f(x) - f_Q| > \lambda\}| \le \sum_j |Q_{N,j}| \le |Q|/2^N$$
$$= e^{-N\log 2}|Q|$$
$$\le e^{-c_2\lambda}|Q|.$$

where $c_2 = \log 2/2^{n+2}$

If $\lambda < 2^{n+1}$. Then $c_2\lambda < \log\sqrt{2}$. set $c_1 = \sqrt{2}$.

$$|\{\cdots\}| \le |Q| \le e^{\log\sqrt{2} - c_2\lambda}|Q| = \sqrt{2}\, e^{-c_2\lambda}|Q|.$$

Cor. Set $\|f\|_{*,p} = \sup_Q \left(⨍_Q |f(x) - f_Q|^p \right)^{\frac{1}{p}}$.

Then $\|\cdot\|_{*,p} \sim \|\cdot\|_*$. for $1 < p < \infty$.

Pf. Prove $\|f\|_{*,p} \underset{p}{\lesssim} \|f\|_*$.

$$⨍_Q |f - f_Q|^p = \int_0^\infty p\lambda^{p-1} |\{ |f - f_Q| > \lambda \}| \, d\lambda$$
$$\lesssim \int_0^\infty p|Q|\lambda^{p-1} e^{-c_2\lambda/\|f\|_*} d\lambda$$
$$= p c_1 c_2^{-p} \Gamma(p) \|f\|_*^p \cdot |Q|$$

Cor. For $f \in BMO$. $\exists \lambda > 0$. s.t. $\forall Q$. cube.

$$⨍_Q e^{\lambda|f(x) - f_Q|} dx < \infty$$

Pf: LHS $= \sum ⨍_Q \lambda^k |f - f_Q|^k dx$

$$= \sum_k p c_1 c_2^{-k} \lambda^k \Gamma(k) \|f\|_*^k$$

Choose λ small enough. for converge.

Cor. (Converse of John-Nirenberg Inequai)

For f. If $\exists C_1, C_2, K$. st. $\forall Q$. $\lambda > 0$.

$|\{x \in Q \mid |f(x) - f_Q| > \lambda\}| \leq C_1 e^{-C_2\lambda/K} |Q|$.

Then: $f \in BMO$.

Pf: $⨍_Q |f - f_Q|^p \leq p C_1 C_2^{-p} \Gamma(p) K^p$

$\Rightarrow \|f\|_{*,p} < \infty$. i.e. $f \in BMO$.

Cor. $BMO \subseteq L^p_{loc}$. for $1 < p < \infty$.

Pf: By above. $f - f_Q \in L^1_{loc}$. so $f \in L^p_{loc}$.

Rmk: L^∞ and BMO share this property.

③ A_p-weight and BMO:

Prop. For $f \in L^1_{loc}$. If $e^f \in A_2$. Then: $f \in BMO$.

Pf: By A_2: $\left(\frac{1}{|Q|}\int_Q e^f\right)\left(\frac{1}{|Q|}\int_Q e^{-f}\right) =$

$\left(⨍_Q e^{f - f_Q}\right)\left(⨍_Q e^{f_Q - f}\right) \leq C$

By Jensen: each factor $\in [1, C]$.

$\Rightarrow ⨍_Q e^{|f - f_Q|} \leq 2C$. i.e. $\frac{1}{|Q|}\int |f - f_Q| \leq 2C$

Cor. (Converse)

For λ small enough > 0. $f \in BMO \Rightarrow e^{\lambda f} \in A_2$.

Pf: By John-Nirenberg inequai. directly.

Rmk: In fact. $BMO = \{\lambda \log w \mid w \in A_2\}_{\lambda \in \mathbb{R}}$