Singular Integrals.

(1) Definition and prop.

Consider the operator: $Tf(x) = \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} \frac{\Omega(y')}{|y|^n} f(x-y)\,dy$.

where $y' = y/|y|$. Ω defined on S^{n-1}. $\Omega \in L^1(S^{n-1})$.

Rmk: $Tf = \text{p.v.} \frac{\Omega(x')}{|x|^n} * f$. If $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$:

$$\text{p.v.} \frac{\Omega(x')}{|x|^n}(f) = \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{\Omega(x')}{|x|^n} f(x)\,dx.$$

$$= \int_{|x|<1} \frac{\Omega(x')}{|x|^n}(f(x)-f(0)) + \int_{|x|\geq 1} \frac{\Omega(x')}{|x|^n} f(x).$$

exists for $\forall f \in S$. $\Rightarrow \text{p.v.} \frac{\Omega(x')}{|x|^n} \in S^*$.

Actually, it's also necessary condition.

Pf: It equi. with existence of $Tf(x)$. $f \in S$.

Set $f \in S(\mathbb{R}^n)$. $f=1$. $\forall |x| \leq 2$.

$$\Rightarrow Tf(x) = \int_{|y|>1} \frac{\Omega(y')}{|y|^n} f(x-y)\,dy + \lim_{\varepsilon\to 0} \int_{\varepsilon<|y|<1} \frac{\Omega(y')}{|y|^n}\,dy.$$

$= A + B$. A converges.

For $B = \lim_{\varepsilon\to 0} \int_{S^{n-1}} \Omega(y')\,d\sigma(y') \log(1/\varepsilon)$

converges $\Leftrightarrow \int_{S^{n-1}} \Omega(y')\,d\sigma(y') = 0$.

e.g. i) $n=1$. $S^0 = \{\pm 1\}$. $\Omega(x')$ take opposite values on this two points $\Rightarrow \forall\, T$ has form above must be multiple of Hilbert transform.

ii) Newton potential. Logarithmic potential.

Thm: If $\Omega \in L^1(S^{n-1})$. $\int_{S^{n-1}} \Omega \, d\sigma = 0$. Then:

$m = (\text{p.v. } \Omega(x')/|x|^n)^\wedge$ is homogeneous of

degree 0. $m(\xi) = \int_{S^{n-1}} \Omega(u) [\log(1/|u\cdot\xi'|) -$

$i\frac{\pi}{2} \operatorname{sgn}(u\cdot\xi')] \, d\sigma(u)$ is its form.

Pf: 1°) p.v. $\Omega(x')/|x|^n$ has degree $-n$.

$\Rightarrow m$ has degree 0. So WLOG. set $|\xi|=1$.

2°) $$m(\xi) = \lim_{\varepsilon\to 0} \int_{\varepsilon<|\eta|<1/\varepsilon} \frac{\Omega(\eta')}{|\eta|^n} e^{-2\pi i \eta\cdot\xi} \, d\eta.$$

$$= \lim_{\varepsilon\to 0} \int_{S^{n-1}} \Omega(u) \Big[\int_\varepsilon^1 (e^{-2\pi i r u\cdot\xi} - 1) \, dr/r$$

$$+ \int_1^{1/\varepsilon} e^{-2\pi i r u\cdot\xi} \, dr/r \Big] \, d\sigma(u)$$

separate into Re. Im. part. Apply DCT.

Rmk: $m(\xi) = \int_{S^{n-1}} \Omega(u) (A - B) \, d\sigma(u)$.

A is even. not integrable. ($A^q \in L^1$. $q>1$)

B is odd. bdd as well.

Def: For Ω defined on S^{n-1}. decompose into two parts:

$\Omega_e(u) = \frac{\Omega(u)+\Omega(-u)}{2}$ (even) $\Omega_o(u) = \frac{\Omega(u)-\Omega(-u)}{2}$ (odd)

Prop. For Ω in S^{n-1}: $\int_{S^{n-1}} \Omega \, d\sigma = 0$. If $\Omega_e \in L^q(S^{n-1})$.

for some $q>1$. $\Omega_o \in L^1(S^{n-1})$. Then $m(\xi)$

$= (\text{p.v. } \Omega(x')/|x|^n)^\wedge$ is bdd.

Rmk: Replace "$\Omega_e \in L^q(S^{n-1})$" by:

$\int_{S^{n-1}} |\Omega_e(w)| \log^+ |\Omega_e(w)| \, d\sigma(w) < \infty$. It still holds. (weaker condition.)

Pf: Note $AB \le A\log A + e^B$, $A \ge 1$, $B \ge 0$.

By the Rmk above: decompose $\Omega = \Omega_e + \Omega_o$.

Rmk: It gives a sufficient condition that T is bdd on L^2.

(2) Boundness on L^p:

① Method of Rotation:

If $T \in L^p(\mathbb{R}^1)$, $u \in S^{n-1}$. Then we can define a bdd operator T_u on $\mathbb{R}^n$:

Set $L_u = \{\lambda u \mid \lambda \in \mathbb{R}^1\}$. $L_u^\perp$ is its orthogonal comple.

$\forall x \in \mathbb{R}^n$, $\exists x_1 \in \mathbb{R}^1$, $\bar{x} \in L_u^\perp$, s.t. $x = x_1 u + \bar{x}$.

$\Rightarrow$ Define: $T_u f(x) = Tf(\cdot u + \bar{x})(x_1)$

Prop: T_u is bdd on $L^p(\mathbb{R}^n)$ uniform with u.

Pf: $$\int_{\mathbb{R}^n} |T_u f|^p dx = \int_{L_u^\perp} \int_{\mathbb{R}^1} |Tf(\cdot u + \bar{x})(x_1)|^p dx_1 \, d\bar{x}$$
$$\le C_p^p \int_{L_u^\perp} \int_{\mathbb{R}^1} |f(\cdot u + \bar{x})(x_1)|^p$$
$$= C_p^p \|f\|_p^p$$

Rmk: We can obtain directional Hardy-Littlewood Max func.: $M_u f(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x-tu)| dt$.

directional Hilbert Transf: $H_u f(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|t|>\varepsilon} f(x-tu) \, dt$.

prop. $T \in L^p(\mathbb{R}^1)$ with const. C_p. T_u is directional operator from T. If $\Omega \in L^1(S^{n-1})$. Then $T_\Omega f(x) = \int_{S^{n-1}} \Omega(u)\, T_u f(x)\, d\sigma(u)$ is bdd on $L^p(\mathbb{R}^n)$ with const. $\leq C_p \|\Omega\|_{L^1(S^{n-1})}$.

Pf: It's direct by prop. above.

Def: For $\Omega \in L^1(S^{n-1})$. $M_\Omega(f)(x) = \sup_{R>0} \frac{1}{|B(0,R)|} \int_{B(0,R)} |\Omega(y')| |f(x-y)| dy$.

prop. M_Ω is bdd on $L^p(\mathbb{R}^n)$. $1 < p \leq \infty$.

Pf: $M_\Omega f(x) = \sup_{R>0} \frac{1}{|B_1| R^n} \int_{S^{n-1}} |\Omega(u)| \int_0^R |f(x-ru)| r^{n-1} dr\, d\sigma$.

$$\leq \frac{1}{|B_1|} \int_{S^{n-1}} |\Omega(u)| M_u f(x)\, d\sigma(u).$$

prop. If Ω is odd integrable on S^{n-1}. Then $Tf(x) = \text{p.v.} \frac{\Omega(x')}{|x|^n} * f$ is bdd on L^p. $1 < p < \infty$

Pf: $Tf(x) = \lim_{\varepsilon\to 0} \int_{S^{n-1}} \Omega(u) \int_\varepsilon^\infty f(x-ru) \frac{dr}{r}\, d\sigma(u)$

$$= \lim_{\varepsilon\to 0} \frac{1}{2} \int_{S^{n-1}} \Omega(u) \int_{|r|\geq\varepsilon} f(x-ru) \frac{dr}{r}\, d\sigma$$

$$\overset{\text{z.A.}}{=} \frac{1}{2} \int \Box \int_{|r|\geq 1} + \frac{1}{2} \lim_{\varepsilon\to 0} \int \Box \int_{\varepsilon<|r|<1} (f(x-ru) - f(x))$$

$$\overset{\text{DCT}}{=} \frac{\pi}{2} \int_{S^{n-1}} \Omega(u) H_u f(x)\, d\sigma(u).$$

Rmk: i) For $f \in L^p$. Define Tf as a limit in L^p.

ii) Define $T^* f(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} \frac{\Omega(y')}{|y|^n} f(x-y)\, dy \right|$

$$\le \frac{\pi}{2} \int_{S^{n-1}} \Omega(y') H_y^* f(x)\, d\sigma(y')$$

$\Rightarrow$ Since H_n^* is strong-(p,p), $1<p<\infty$

So as T^* does. We can define Tf as a.e. limit in i).

Def: Riesz Transform: $R_j f(x) = C_n \text{ p.v.} \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} f(x-y)\, dy$.

for $1 \le j \le n$. $C_n = \Gamma(\frac{n+1}{2}) \pi^{-(n+1)/2}$

Rmk: i) It has an odd kernel.

ii) For $f \in L^2$: $(R_j f)^\wedge(\xi) = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi)$.

$\Rightarrow \sum_1^n R_j^2 = -I$ on L^2. (Actually, $\mathcal{S}$ dense in L^p. So it holds in L^p).

Pf: $(\text{p.v.} \frac{x_j}{|x|^{n+1}})^\wedge = (\frac{1}{1-n} \frac{\partial}{\partial x_j} |x|^{-n+1})^\wedge$

$$= \frac{2\pi i \xi_j}{1-n} \hat{H}_{-n+1}$$

② Even kernel:

For Ω is even. We can't represent Tf in terms of Hilbert Transf.

But: $Tf = -\sum R_j^2 (Tf) = -\sum R_j (R_j T(f))$

We have $R_j T$ is odd.

Def: Ω is even. Zero average on $L^q(S^{n-1})$ for some $q>1$

and $\varepsilon > 0$. $K_\varepsilon(x) = \dfrac{\Omega(x')}{|x|^n} \chi_{|x|\geq\varepsilon}$

Rmk: i) $K_\varepsilon \in L^r$. $1 < r \leq 2$.

ii) $\forall f \in C_c^\infty(\mathbb{R}^n)$. $R_j(K_\varepsilon * f) = (R_j K_\varepsilon) * f$.

can be proved by Fourier Transf.

Lemma. $\exists \widetilde{K_j}$ odd, homo with degree $-n$. s.t.

$\lim_{\varepsilon\to0} R_j K_\varepsilon = \widetilde{K_j}$ in L^∞ norm on every

cpt set that doesn't contain origin.

Pf: 1°) Prove: $(R_j K_\varepsilon)_\varepsilon$ is Cauchy

Fix $x \neq 0$. $0 < \varepsilon < \nu < |x|/2$. Then:

$R_j K_\varepsilon(x) - R_j K_\nu(x) = C_n \int_{\varepsilon<|y|<\nu} \left(\dfrac{x_j - y_j}{|x-y|^{n+1}} - \dfrac{x_j}{|x|^{n+1}} \right) \dfrac{\Omega(y')}{|y|^n}$.

$\underset{n}{\overset{\text{Mean Value}}{\lesssim}} \dfrac{1}{|x|^{n+1}} \int_{\square} \dfrac{|\Omega(y')|}{|y|^{n-1}} dy$

$\leq \dfrac{C\|\Omega\|_{L^1(S^{n-1})}}{|x|^{n+1}} \nu \to 0 \quad (\nu \to 0)$

$\Rightarrow \exists K_j^*$. $K_j^* = \lim_{\varepsilon\to0} R_j K_\varepsilon$. a.e. sense.

modify a set of measure 0. K_j^* is odd.

2°) Check: $R_j K_\varepsilon(\lambda x) = \lambda^{-n} R_j K_{\varepsilon/\lambda}(x)$.

$\Rightarrow K_j^*(\lambda x) = \lambda^{-n} K_j^*(x)$. a.e. x.

$D = \{(x,\lambda) \in \mathbb{R}^n \times \mathbb{R}^{>0}.\ K_j^*(\lambda x) \neq \lambda^{-n} K_j^*(x)\}$.

By Fubini. since $m(D) = 0 \Rightarrow \exists S \cap D$

has measure 0. $S = \partial B(0,\varepsilon)$.

3') Def: $\tilde{k}_j(x) = \begin{cases} (\ell/|x|)^n k_j^*\left(\frac{\ell x}{|x|}\right). & \text{if } x \neq 0,\ \frac{\ell x}{|x|} \in D \cap S \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow \tilde{k}_j = k_j^*$ a.e. (Modify $\Rightarrow \tilde{k}_j$ is degree $-n$)

Cor. $\int_{S^{n-1}} |\tilde{k}_j(w)|\, d\sigma(w) \le C_q \|\Omega\|_{L^q(S^{n-1})}$

Moreover, set $\tilde{k}_{j,\varepsilon} = \tilde{k}_j \chi_{\{|x| \ge \varepsilon\}}$. $\Delta_\varepsilon = R_j K_\varepsilon - \tilde{k}_{j,\varepsilon}$

$\Rightarrow \|\Delta_\varepsilon\|_{L^1(\mathbb{R}^n)} \le \tilde{C}_q \|\Omega\|_{L^q(S^{n-1})}$

Pf: $\tilde{k}_j(x) = \tilde{k}_j\left(|x| \cdot \frac{x}{|x|}\right) = |x|^{-n} \tilde{k}_j(x')$

1') $\int_{S^{n-1}} |\tilde{k}_j(w)|\, d\sigma(w) = \int_{1<|x|<2} |\tilde{k}_j(x)| / \log 2 \, dx$

$\lesssim \int_\square |\tilde{k}_j - R_j K_{\frac{1}{2}}| + \int_\square |R_j K_{\frac{1}{2}}|$

$\lesssim_q \int_\square \frac{\|\Omega\|_{L^1(S^{n-1})}}{|x|^{n+1}} + \|R_j K_{\frac{1}{2}}\|_{L^q}$ (Estimate above

$\lesssim_q \|\Omega\|_q + \|K_{\frac{1}{2}}\|_q \le C_q \|\Omega\|_q$.

2') Show $\varepsilon = 1$. Since $\Delta_\varepsilon(x) = \varepsilon^{-n} \Delta(\varepsilon^{-1} x)$.

$\|\Delta_1\|_1 \le \int_{|x|>2} |\Delta_1| + \int_{|x|<2} |R_j K_1| + \int_{1<|x|<2} |\tilde{k}_j|$

$\lesssim \int_{|x|>2} \frac{\|\Omega\|_{L^1(S^{n-1})}}{|x|^{n+1}} + \|R_j K_1\|_q + \|\Omega\|_q$

$\lesssim_q \|\Omega\|_q$.

Thm. Ω on S^{n-1} with zero average, s.t. $\Omega_e \in L^q(S^{n-1})$ for some $q > 1$, and $\Omega_o \in L^1(S^{n-1})$. Then:

$T\cdot = \text{p.v.} \frac{\Omega(x')}{|x|^n} * \cdot$ is bdd on $L^p(\mathbb{R}^n)$, $1 < p < \infty$

Pf: Only consider Ω is even. For $f \in C_c^\infty(\mathbb{R}^n)$,

Note $Tf = \lim_{\varepsilon \to 0} K_\varepsilon * f := \lim_{\varepsilon \to 0} -\sum_1^n R_j((R_j K_\varepsilon) * f)$,

From $(R_j K_\varepsilon) * f = \widetilde{K_{j,\varepsilon}} * f + \Delta_\varepsilon * f$.

By Lemma above. with R_j is bdd on L^p.

$\Rightarrow \|K_\varepsilon * f\|_p \lesssim \|\Omega\|_2 \|f\|_p$. $\Rightarrow$ Apply Fatou's. ✓.

(3) An operator algebra:

① Def: i) Λ is squared root of positive operator $-\Delta$.

if $(\Lambda f)^\wedge(\xi) = 2\pi|\xi| \hat{f}(\xi)$.

ii) For a LPDO. $L = \sum_{|\alpha|=m} a_\alpha \partial^\alpha/\partial x^\alpha \sim P(\xi) = \sum a_\alpha (2\pi i\xi)^\alpha$. $a_\alpha \in \mathbb{C}$. homogenous poly of degree m. Set: $(Tf)^\wedge(\xi) = i^m \frac{P(\xi)}{|\xi|^m} \hat{f}(\xi)$.

Rmk: i) From ii). We have $Lf = T(\Lambda^m f)$.

ii) Note: $P(\xi)/|\xi|^m$ has homo of degree 0.

T isn't singular integral as before. Since $P(\xi)$ may not have zero average on S^{n-1}.

Thm.$^{(*)}$ $m \in C^\infty(\mathbb{R}^n/\{0\})$ has homo of degree 0.

T_m is operator def by: $(T_m f)^\wedge = m\hat{f}$.

$\Rightarrow \exists\, a \in \mathbb{C}$. and $\Omega \in C^\infty(S^{n-1})$ with zero average. st. $Tf = af + \text{p.v.} \frac{\Omega(x')}{|x|^n} * f$. $f \in S$.

Lemma. For m in Thm. If m has zero average on S^{n-1}. Then $\exists\ \Omega \in C^\infty(S^{n-1})$ with zero average. s.t. $\hat{m}(\xi) = \text{p.v.}\ \Omega(\xi')/|\xi|^n$

Pf. 1°) Note $\frac{\partial^n m}{\partial x_i^n} = \text{p.v.}\ \frac{\partial^n m}{\partial x_i^n} + \sum_{|\alpha| \le N} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}\delta$

"$\wedge$"

$\Rightarrow\ \xi_i^n\ \hat{m}(\xi) = (\text{p.v.}\ \frac{\partial^n m}{\partial x_i^n})^\wedge + \sum_{|\alpha| \le N} \tilde{a}_\alpha\ \xi^\alpha$

So: $N = 0$. (by homo degree)

By it holds for $1 \le i \le n$.

$\Rightarrow\ \hat{m}$ agree with a $C^\infty(\mathbb{R}^n/\{0\})$ with degree $-n$.

Set $\Omega = \hat{m}(\xi)|_{S^{n-1}}$

2°) Prove Ω have zero average on S^{n-1}:

Set radial func. $\phi \in \mathcal{S}$. $\text{supp}\,\phi = \{1 \le |x| \le 2\}$

$\hat{m}(\phi) = \int_{\mathbb{R}^n} \hat{m}(\xi)\phi(\xi)\,d\xi = \int_{\mathbb{R}^n} \frac{\Omega(\xi')}{|\xi|^n}\phi(\xi)$

$= c\int_{S^{n-1}} \Omega(\xi')\,d\sigma(\xi')$. $c > 0$

Note: $\hat{m}(\phi) = m(\hat{\phi}) = c\int_{S^{n-1}} m(w)\,d\sigma(w)$

$= 0$. (zero average)

3°) Prove: $\hat{m} = \text{p.v.}\ \Omega(x')/|x|^n$.

Note $\hat{m} - \text{p.v.}\ \Omega(x')/|x|^n$ support on $\{0\}$

Take "$\vee$" on both sides $\Rightarrow$ It must be 0.

Return to pf. Set $\tilde{m} = m - \int_{S^{n-1}} m(w)\,d\sigma(w)/|S^{n-1}|$.

has zero average on S^{n-1}.

Thm. The set $\mathcal{A}$ of operators defined in Thm(*) is a commutative algebra.

$T_m \in \mathcal{A}$ is invertible $\iff m \neq 0$ on S^{n-1}.

Pf: 1) Note $T_{m_1} \circ T_{m_2} = T_{m_1 m_2}$

2) $T_{\frac{1}{m}}$ is inverse of T_m. from:

$m \neq 0$ on $\mathbb{R}^n / \{0\} \iff m \neq 0$ on S^{n-1}.

Rmk: Elliptic operator is invertible. which has homo of degree m.

② Consider more generally. $L(x, \partial/\partial x) = \sum_{|\alpha|=m} a_\alpha(x) \partial^\alpha/\partial x^\alpha$

Similarly. define: $Tf(x) = \int_{\mathbb{R}^n} \sigma(x,\xi) \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$.

$\sigma(x,\xi) = P(x, i\xi)/|\xi|^m$. $P(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha$.

$\Rightarrow L(x, \partial/\partial x) f = T(\Delta^m f)$.

Rmk: Subtitute $\hat{f}(\xi) = \int f(x) e^{-2\pi i \xi\cdot x}$ in Tf

$\Rightarrow Tf(x) = \int_{\mathbb{R}^n} k(x, x-y) f(y) dy$. where

$k(x,z) = \int_{\mathbb{R}^n} \sigma(x,\xi) e^{2\pi i z\cdot\xi} d\xi$.

Prop. $\exists\, a(x)$. $\mu(x,\cdot) \in C^\infty(S^{n-1})$ with zero average on S^{n-1}. st. $k(x,z) = a(x)\delta(z) + \text{p.v.} \dfrac{\mu(x,z')}{|z|^n}$

Pf: $\sigma(x,\xi)$ has degree 0 on ξ, Fix x.

Thm. For $\Omega(x,\eta)$ has homo of degree 0 on η.

If
i) $\Omega(x,-\eta) = -\Omega(x,\eta)$
ii) $\Omega^*(\omega) = \sup_x |\Omega(x,\omega)| \in L^1(S^{n-1})$

Then, $Tf(x) = \lim_{\varepsilon\to 0} \int_{|\eta|>\varepsilon} \frac{\Omega(x,\eta)}{|\eta|^n} f(x-\eta)\, d\eta$ is bdd

on L^p. $1<p<\infty$.

Pf: Similarly, $Tf(x) = \frac{\pi}{2} \int_{S^{n-1}} \Omega(x,\omega) H_\omega f(x)\, d\sigma(\omega)$.

for $\forall f \in S$. $\Rightarrow |Tf| \lesssim \int_{S^{n-1}} \Omega^*(\omega) |H_\omega f(x)|\, d\sigma(\omega)$

Cor. Replace ii) by ii)* $\sup_x \|\Omega(x,\cdot)\|_{L^q(S^{n-1})} < \infty$

for some q. $1<q<\infty \Rightarrow T$ is bdd on L^p.

where $q' \le p < \infty$. (Not mean: $\Omega^* \in L^q$!)

Pf: $|Tf| \lesssim \sup_x \|\Omega(x,\cdot)\|_{L^q(S^{n-1})} \|H_\omega f(x)\|_{L^{q'}(S^{n-1})}$

Integrate p-th power ($p \ge q'$). By Minkowski.

(4) Calderón - Zygmund Theory :

Next, we will consider one kind of singular integral whose kernels have essential properties as Hilbert transform one.

Thm. (Calderón - Zygmund)

$K \in S^*(\mathbb{R}^n)$, agree with a L^1_{loc} func on $\mathbb{R}^n/\{0\}$.

st. $|\hat{K}| \le A$. $\int_{|x|>2|\eta|} |K(x-\eta) - K(x)|\, dx \le B$. $\forall \eta \in \mathbb{R}^n$

(Hörmander Condition)

Thm: for $1<p<\infty$. $\|K*f\|_p \le C_p \|f\|_p$.

and $|\{x\in\mathbb{R}^n \mid |K*f|\ge\lambda\}| \le \frac{c}{\lambda}\|f\|_1$

Rmk: $\forall x\ne 0$. $|\nabla K(x)| \le \frac{c}{|x|^{n+1}}$ (Gradient Cond.)

$\Rightarrow$ Hörmander Condition.

Pf: By $|K(x-\eta)-K(x)| \overset{MVT}{\le} |\nabla K(x+\xi\eta)||\eta|$

$$\lesssim |\eta|/|x+\xi\eta|^{n+1}$$

Rmk: Recall def of C-Z dist. before. Note: it satisfies gradient condition. (To conditions in Thm above)

Pf: 1°) For $Tf=K*f$ is weak-(1,1).

Apply C-Z decompose on f at height λ.

As argue above. prove: $\int_{\mathbb{R}^n/Q_j^*}|Tb_j| \lesssim \int_{Q_j}|b_j|$

Q_j^* is cube with same center of Q_j and sides $2\sqrt{n}$ times long. (Denote c_j. center)

$Tb_j(x)=\int_{Q_j}K(x-\eta)b_j(\eta)d\eta=\int_{Q_j}(K(x-\eta)-K(x-c_j))\square$

$\Rightarrow \int_{\mathbb{R}^n/Q_j^*}|Tb_j| \le \int_{Q_j}|b_j(\eta)|\left(\int_{\mathbb{R}^n/Q_j^*}|K(x-\eta)-K(x-c_j)|dx\right)d\eta$

with $\mathbb{R}^n/Q_j^* \subseteq \{x\in\mathbb{R}^n \mid |x-c_j|\ge 2|\eta-c_j|\}$

2°) We have T is strong-(2,2). by $\hat{K}$ bdd

By interpolation and dual arguement.

Rmk: For $k(x) = \Omega(x')/|x|^n$. $\Omega(x') \in C^1(S^{n-1})$ is sufficient for k holds Hörmander condition.

Define: $w_\Omega(t) = \sup\{|\Omega(u_1) - \Omega(u_2)| \mid |u_1-u_2| \le t,\ u_1, u_2 \in S^{n-1}\}$. We have a weaker condition:

Prop. If Ω satisfies $\int_0^1 \frac{w_\Omega(t)}{t} < \infty$ Then $k(x)$ satisfy Hörmander condition.

Pf: 1) Ω is bdd.

Note $\exists\, t_0 \in (0,1)$. $w_\Omega(t_0) \le C < \infty$. $|S^{n-1}| < \infty$

Fix a point $x_0 \in S^{n-1}$. $|\Omega(x)| \le |\Omega(x_0)| + mC$

2) $$|k(x-\eta) - k(x)| = \left| \frac{\Omega((x-\eta)')}{|x-\eta|^n} - \frac{\Omega(x')}{|x|^n} \right|$$

$$\le \frac{|\Omega((x-\eta)') - \Omega(x')|}{|x-\eta|^n} + |\Omega(x')| \left| \frac{1}{|x-\eta|^n} - \frac{1}{|x|^n} \right|$$

$$\le \frac{w_\Omega(4|\eta|/|x|)}{(|x|/2)^n} + \frac{C|\eta|}{|x|^{n+1}}$$

follows from on $\{|x| > 2|\eta|.\}$. $|(x-\eta)' - x'| \le \frac{4|\eta|}{|x|}$

① Truncated Integral:

Def: For $k \in L^1_{loc}(\mathbb{R}^n/\{0\})$. $k_{\varepsilon,R}(x) = k(x)\chi_{\{\varepsilon<|x|<R\}}$.

Thm. If $k \in L^1_{loc}(\mathbb{R}^n/\{0\})$. satisfies:

i) $\forall\, 0<a<b<\infty$. $\left|\int_{a<|x|<b} k(x)\right| \le A$.

ii) $a>0$. $\int_{a<|x|<2a} |k(x)| \le B$.

iii) $\eta \in \mathbb{R}^n$. $\int_{|x|>2|\eta|} |k(x-\eta) - k(x)|\, dx \le C$

Then, $\forall \xi \in \mathbb{R}^n$. $|\widehat{K_{\varepsilon,R}}| \le C$. C is indept of ε. R.

Rmk: Condition ii) $\Leftrightarrow$ $\int_{|x|\le r} |x||K(x)| dx \le \widetilde{B} r$.

Pf: ($\Rightarrow$) $\int_{|x|<r} |K(x)| dx = \sum_{k=0}^{\infty} \int_{2^{-k-1} r<|x|<2^{-k} r} |x||K(x)| dx$

$\le \sum rB/2^k$

($\Leftarrow$) $\int_{r<|x|<2r} |K(x)| dx \le \int_{|x|<2r} \frac{|x||K(x)|}{r} dx \le 2\widetilde{B}$

Pf: Fix ξ. st. $\varepsilon < |\xi|^{-1} < R$.

$\widehat{K_{\varepsilon,R}}(\xi) = \int_{\varepsilon<|x|<|\xi|^{-1}} + \int_{|\xi|^{-1}<|x|<R} K(x) e^{-2\pi i x\cdot\xi} dx.$

$= I_1 + I_2.$

(For $|\xi|^{-1} \le \varepsilon$. or $|\xi|^{-1} > R$. Consider I_2 or I_1)

1) For I_1: $|I_1| = |\int_\square K(x) + \int_\square K(x)(e^{2\pi i x\cdot\xi} - 1) dx|$

$\le 1.\int_\square |K(x)| + 2\pi|\xi| \int_\square |x||K(x)|$

2) Set $z = \frac{1}{2}\xi|\xi|^{-2}$. $\Rightarrow e^{2\pi i z\cdot\xi} = -1$.

$\therefore 2I_2 = \int_{|\xi|^{-1}<|x|<R} K(x) e^{-2\pi i x\cdot\xi} dx - \int_{|\xi|^{-1}<|x-z|<R} K(x-z) e^{-2\pi i x\cdot\xi}$.

Set $\hat{A} = \{|\xi|^{-1} < |x| < R\}$. $\hat{B} = \{|\xi|^{-1} < |x-z| < R\}$.

$\Rightarrow \hat{A} \cap \hat{B} \subseteq \{|\xi|^{-1} < |x| < R\} \cap \{\frac{1}{2}|\xi|^{-1} < |x| < R + \frac{|\xi|^{-1}}{2}\}$

$\subseteq \{|\xi|^{-1} \le |x|\}$.

$\hat{A} - \hat{B} \subseteq \hat{A} \cap (\{|x| < \frac{3}{2}|\xi|^{-1}\} \cup \{R - \frac{|\xi|^{-1}}{2} < |x|\})$

$\subseteq \{|\xi|^{-1}/2 < |x| < \frac{3}{2}|\xi|^{-1}\}$

$\hat{B} - \hat{A} \subseteq \hat{B} \cap (\{|x| \le |\xi|^{-1}\} \cup \{|x| > R\})$

$\subseteq \{R < |x| < R + \frac{1}{2}|\xi|^{-1}\}$.

$$\delta_0 \cdot 2|I_2| \leq \int_{\hat{A}\cap\hat{B}} |k(x)-k(x-z)|\,dx + \int_{\hat{B}-\hat{A}} |k| + \int_{\hat{A}-\hat{B}} |k|.$$

$$\leq C + 2B. \quad (R + \tfrac{1}{2}|\xi|^{-1} \leq 2R, \ |z| = \tfrac{1}{2}|\xi|^{-1})$$

<u>Cor.</u> For k satisfies i). ii). iii) above. Then:

$T_{\varepsilon,R}(f) = k_{\varepsilon,R} * f$ is weak-(1,1). strong-(p,p)

for $1 < p < \infty$. with C_p indept of ε, R.

<u>Pf:</u> $p = 2$ is from Thm above.

Note $k_{\varepsilon,R}$ satisfies Hörmander condition.

<u>Rmk:</u> When $k(x) = \Omega(x')/|x|^n$. Then:

condition i). ii) $\Leftrightarrow$ $\Omega \in L^1(S^{n-1})$. $\int_{S^{n-1}} \Omega\, d\sigma = 0$.

If Ω satisfies $\int_0^1 \frac{w_\Omega(t)}{t}\,dt < \infty$ additionally.

Then $k * f$ is weak-(1,1). strong-(p,p).

Note $k_{\varepsilon,R} \in L^1$. $k_{\varepsilon,R} * f$ is well-def for $f \in L^p$.

For $Tf(x) = \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} k_{\varepsilon,R} * f(x)$. may not exist for $f \in S$.

But. we can first let $R \to \infty$. Consider extence of

$$p.v.\ k(\phi) = \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} k(x)\,\phi(x)\,dx. \quad \phi \in S.$$

<u>prop.</u> For k satisfies: $\int_{R<|x|<2R} |k| \leq B$. Then:

$p.v.\ k \in S^*$. exists. $\Leftrightarrow$ $\lim_{\varepsilon\to 0} \int_{\varepsilon<|x|<1} k(x)\,dx$ exists.

<u>Pf:</u> ($\Rightarrow$) Fix $\phi \in S$. $\phi = 1$ on $B(0,1)$.

$$p.v.\ k(\phi) = \lim_{\varepsilon\to 0} \int_{\varepsilon<|x|<1} k(x) + \int_{|x|>1} k\phi.$$

Note: $\int_{|x|>1} |k\phi| \le \|x\phi\|_\infty \sum_{k=0}^{\infty} 2^{-k} \int_{2^k<|x|<2^{k+1}} |k(x)| dx$

($\Leftarrow$) suppose $\lim_{\varepsilon\to 0} \int_{\varepsilon<|x|<1} k(x) dx = L$ exists.

Then: $p.v. k(\phi) = \phi(0)L + \int_{|x|<1} k(x)(\phi(x)-\phi(0)) + \int_{|x|>1} k\phi$.

$|\int_{|x|<1} k(x)(\phi(x)-\phi(0))| \le \int_{|x|<1} |k(x)||x| \|\nabla\phi\|_\infty dx$.

Rmk: Additionally suppose k satisfies condition i). ii). iii). $\Rightarrow$ p.v. k is bdd on L^p. for $1<p<\infty$. and weak-(1.1).

Pf: Strong-(p.p) is from Fstor's.

weak-(1.1) we have proved.

e.g. For $k(x) = |x|^{-n-it} \in L^1_{loc}(\mathbb{R}^n/\{0\})$

it satisfies condition i). ii). iii) and Hörmander cond.

So: strong-(p.p). $1<p<\infty$. weak-(1.1)

However p.v. k doesn't exist since $\int_{\varepsilon<|x|<1} dx/|x|^{n+it}$

$= |S^{n-1}| \frac{1-\varepsilon^{-it}}{-it}$'s limit doesn't exist. when $\varepsilon\to 0$.

Actually. define p.v. $\frac{1}{|x|^{n+it}}(\phi) = \int_{|x|<1} \frac{\phi(x)-\phi(0)}{|x|^{n+it}} + \int_{|x|>1} \frac{\phi(x) dx}{|x|^{n+it}}$

but it's not homo of degree $-n-it$.

Rmk: For $Re z<n$. $|x|^{-z} \in L^1_{loc}$. can be defined as a tempered dist. More generally:

$$\frac{1}{|x|^z}(\phi) = \int_{|x|<1} \frac{\phi(x)-\phi(0)}{|x|^z} + \int_{|x|>1} \frac{\phi(x)}{|x|^z} + \phi(0)\int_{|x|<1} \frac{1}{|x|^z}$$

$$= \int_{|x|<1} \phi(x)-\phi(0)/|x|^z + \int_{|x|<1} \frac{\phi(x)}{|x|^z} + \frac{|S^{n-1}|}{n-z}$$

holds for $\mathrm{Re}\, z < n+1$. except $z = n$. Besides. it's homo of degree $-z$.

① Generalized C-Z operator:

Denote: $\Delta = \{ (x, x) \mid x \in \mathbb{R}^n \} \subseteq \mathbb{R}^n \times \mathbb{R}^n$.

Thm. T is bdd on $L^2(\mathbb{R}^n)$. k is a func. on $\mathbb{R}^n \times \mathbb{R}^n / \Delta$. st. for $f \in L^2(\mathbb{R}^n)$. has cpt support. $Tf(x) = \int_{\mathbb{R}^n} k(x,y) f(y)\, dy$ if $x \notin \mathrm{supp}(f)$.

If k satisfies: i) $\int_{|x-y| > 2|y-z|} |k(x,y) - k(x,z)|\, dx \le C$

ii) $\int_{|y-x| > 2|x-z|} |k(x,y) - k(z,y)|\, dy \le C$

Then T is weak-$(1,1)$. strong-(p,p). $1 < p < \infty$.

Pf: Apply C-Z decomposition as before.
i) is for weak-$(1,1)$. ii) is for dual argue.

Def: i) $k : \mathbb{R}^n \times \mathbb{R}^n / \Delta \to \mathbb{C}$ is a standard kernel.
if $\exists\, \delta > 0$ st. $|k(x,y)| \le C / |x-y|^n$

$$|k(x,y) - k(x,z)| \le C \frac{|y-z|^\delta}{|x-y|^{n+\delta}}. \text{ if } |x-y| > 2|y-z|$$

$$|k(x,y) - k(w,y)| \le C \frac{|x-w|^\delta}{|x-y|^{n+\delta}}. \text{ if } |y-x| > 2|x-w|$$

Rmk: A standard kernel satisfies Hörmander condition i) ii) in Thm. above.

ii) T is a generalized C-Z operator if

(a) T is bdd on $L^2(\mathbb{R}^n)$

(b) $\exists$ standard kernel K. st. for $f \in L^2(\mathbb{R}^n)$ with cpt support. $Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)$ if $x \notin \operatorname{supp}(f)$.

Rmk: A C-Z operator is strong-(p,p) for $1 < p < \infty$ and weak-$(1,1)$.

③ C-Z Singular Integral:

Note for C-Z operator T. $Tf(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} K(x,y) f(y)\,dy$ holds for $f \in S(\mathbb{R}^n)$. Set $T_\varepsilon f = \int_{|x-y|>\varepsilon} K(x,y) f(y)\,dy$.

It's not neccessary K is standard kernel for $\lim_{\varepsilon\to 0} T_\varepsilon$ exists.

Prop. $\lim_{\varepsilon\to 0} T_\varepsilon f(x)$ exists a.e for $f \in C_c^\infty(\mathbb{R}^n) \iff$

$\lim_{\varepsilon\to 0} \int_{\varepsilon<|x-y|<1} K(x,y)\,dy$ exists a.e.

Pf: $(\Rightarrow)$ Let $f \in C_c^\infty$. $f = 1$ on $\{|x| \le 1\}$.

$g \in C_c^\infty$. $g \ge 0$. $g = 1$ on $\operatorname{supp} f$.

$(\Leftarrow)$ Similarly argue as before

Rmk: Existence $\not\Rightarrow$ $\lim_{\varepsilon\to 0} T_\varepsilon f = Tf$. e.g. $K = 0$

$I(f) = f(x) = 0$ if $x \notin \operatorname{supp} f$. But $\lim_{\varepsilon\to 0} T_\varepsilon f = 0 \ne I(f)$. for $f \in C_c^\infty$. $(T = I)$.

Prop. If two C-Z operators are associated a same kernel. Then their difference is a pointwise multiplication operator ($Tf(x) = a(x) f(x)$. $a \in L^\infty$)

Pf: Fix x. Their diff. supports on $\{0\}$ with homo of degree 0. (limit of degree n function has degree n)

Def: $T^* f(x) = \sup_{\varepsilon > 0} | T_\varepsilon f(x) |$

Thm. T is C-Z operator. Then T^* is weak-(1,1) and strong-(p,p). $1 < p < \infty$.

Lemma. S is weak-(1,1). $0 < \nu < 1$. $|E| < \infty$.

Then $\exists\, C(\nu)$. s.t. $\int_E |Sf(x)|^\nu dx \le C(\nu) |E|^{1-\nu} \|f\|_1^\nu$.

Rmk: We can replace Lebesgue measure with a general Borel measure. μ

Pf:
$$LHS = \nu \int_0^\infty \lambda^{\nu-1} \mu\{x \in E \mid |Sf| > \lambda\} d\lambda$$
$$\le \nu \int_0^\infty \lambda^{\nu-1} \min\{\mu(E), \tfrac{C}{\lambda}\|f\|_1\} d\lambda$$
$$= \nu \int_0^{C\|f\|_1/\mu(E)} \lambda^{\nu-1} \mu(E) + \nu \int_\square^\infty C\|f\|_1 \lambda^{\nu-2} d\lambda$$
$$= C(\nu)\, \mu(E)^{1-\nu} \|f\|_1^\nu$$

Lemma. T is C-Z operator $\Rightarrow$ For $0 < \nu \le 1$. $\forall f \in C_c^\infty$

$$T^* f(x) \le C(\nu) \left(M(|Tf|^\nu)(x)^{\frac{1}{\nu}} + Mf(x) \right)$$

Pf: By translation. Consider $x = 0$. $|T_\varepsilon f(0)| > 0$

1°) Fix $\varepsilon > 0$. $Q = B(0, \frac{\varepsilon}{2})$. $f_1 = f \chi_{2Q}$.

$f_2 = f - f_1$. $\Rightarrow Tf_2(0) = T_\varepsilon f(0)$.

For $z \in 2Q$. $|Tf_2(z) - Tf_2(0)|$

$\leq C|z|^\delta \int_{|\eta|>\varepsilon} |f(\eta)| / |\eta|^{n+\delta} d\eta$

$\lesssim \varepsilon^\delta \sum_0^\infty \int_{2^k\varepsilon < |\eta| < 2^{k+1}\varepsilon} |f| / |\eta|^{n+\delta} d\eta$

$\leq C_\delta Mf(0)$.

$\Rightarrow |T_\varepsilon f(0)| \leq C Mf(0) + |Tf(z)| + |Tf_1(z)|$

2°) Fix λ. s.t. $0 < \lambda < |T_\varepsilon f(0)|$.

Set: $\begin{cases} Q_1 = Q \cap \{|Tf(z)| > \frac{\lambda}{3}\} \\ Q_2 = Q \cap \{|Tf_1(z)| > \frac{\lambda}{3}\}. \\ Q_3 = Q \text{ if } CMf(0) > \frac{\lambda}{3}, \ \varnothing. \text{ otherwise} \end{cases}$

$\Rightarrow Q = Q_1 \cup Q_2 \cup Q_3$

If $Q_3 = Q$. Then $\lambda \lesssim Mf(0)$

If $Q_3 = \varnothing$. by weak-(1.1) of T

Then: $\lambda \lesssim Mf(0) + M(Tf)(0)$

Set $\lambda \to |T_\varepsilon f(0)|$. We proved $\nu = 1$. case

3°) For $0 < \nu < 1$. By C_r-inequ:

$|T_\varepsilon f(0)|^\nu \lesssim_\nu Mf(0)^\nu + |Tf(z)|^\nu + |Tf_1(z)|^\nu$.

integrate on z over Q. divide $|Q|$.

$\Rightarrow |T_\varepsilon f(0)| \lesssim_\nu Mf(0) + \cdots$ (raise to power $1/\nu$)

Return to pf: 1°) $\nu = 1$ in Lemma. We have proved T^* is strong-(p,p).

2') For showing T^* is weak-(1,1). By Lemma:

$|\{T^*f > \lambda\}| \le |\{Mf \ge \lambda/2c\}| + |\{M(|Tf|^\nu)^{\frac{1}{\nu}} \ge \lambda/2c\}|$

set $E = \{M(|Tf|^\nu)^{\frac{1}{\nu}} \ge \lambda/2c\}$. $|E| < \infty$ if $f \in C_c^\infty$.

Note $|E| \underset{\nu}{\lesssim} \frac{1}{\lambda^\nu} \int_E |Tf|^\nu \underset{\nu}{\lesssim} \frac{1}{\lambda^\nu} |E|^{1-\nu} \|f\|_1^\nu$

follows from Lemma.

<u>Rmk</u>: We can define Tf for $f \in L^1$ as a.e. limit of $T_\varepsilon f$ $(\varepsilon \to 0)$

④ <u>Vector-Valued Extension:</u>

For A, B separable Banach. Denote $L^p(\mathbb{R}^n; B) = L^p(B)$.

<u>Def</u>: $K : \mathbb{R}^n \times \mathbb{R}^n / \Delta \longrightarrow \mathcal{L}(A, B)$. T is an operator which has K as its associated kernel if for $f \in L^\infty(A)$, has cpt support. then:

$Tf(x) = \int_{\mathbb{R}^n} K(x,y) \cdot f(y)\, dy$. $x \notin \mathrm{supp}(f)$.

<u>Thm</u>. T is bdd operator from $L^r(A)$ to $L^r(B)$. for some $r \in (1, \infty)$. with associated kernel K.

If K satisfies:

i) $\int_{|x-y|>2|y-z|} \|K(x,y) - K(x,z)\|\, dx \le C$

ii) $\int_{|y-x|>2|x-w|} \|K(x,y) - K(w,y)\|\, dy \le C$

Thm: T is bdd from $L^p(A)$ to $L^p(B)$.

$1<p<\infty$ and weak-(1.1). $|\{\|Tf\|_B \geq \lambda\}| \leq$
$\frac{c}{\lambda}\|f\|_{L^1(A)}$

<u>Pf</u>: 1°) We still have "C-Z decomposition" for $f \in L^1_{loc}(A)$. weak-(1.1) is similarly argued as before.

2°) By interpolation T is bdd in $1<p\leq r$.

To apply dual arguement:

3°) A is reflexive $\Rightarrow (L^p(A))^* = L^{p'}(A^*)$.

$T^* \sim \tilde{K}(x,y) = K^*(y,x) \in \mathcal{L}(B^*, A^*)$

also satisfies the conditions for K

$\Rightarrow T^*$ is bdd for $1<p\leq r'$

4°) A isn't reflexive. Consider $A_0 \underset{LS}{\subseteq} A$. st.

$\dim A_0 < \infty$. $T_0 = T|_{\{f(x) \in A_0, \forall x\}}$. $\sim$

$K_0 = K|_{A_0}$. $\Rightarrow \|K_0\| \leq \|K\|$.

Conditions i). ii) for K_0 holds also for const. C indept of A_0.

Argue as before. T_0 is bdd for $r<p<\infty$ with C_p indept of A_0.

5°) So T is also bdd for $r<p<\infty$. Since

$L^p \otimes A = \{\sum_1^N f_k a_i . N \in \mathbb{Z}^+\} \underset{dense}{\subseteq} L^p(A)$.