Distributions

(1) Def and Properties:

Def: For $\Omega \subseteq \mathbb{R}^d$ open

i) $D(\Omega) =: C_c^\infty(\Omega)$ space of test functions.

ii) $\varphi_n \subseteq D(\Omega) \xrightarrow{D} \varphi \in D$ if $\forall n$, $\mathrm{supp}\,\varphi_n \subseteq K$ a common support, and $\forall \alpha$ multi-index, we have $\partial_x^\alpha \varphi_n \xrightarrow{u} \partial_x^\alpha \varphi$ as $n \to \infty$

iii) Distribution F on Ω is complexed-value $LF: \varphi \mapsto F(\varphi)$ defined on $D(\Omega)$ st. conti. i.e. $\varphi_n \xrightarrow{D} \varphi \Rightarrow F(\varphi_n) \xrightarrow{n\to\infty} F(\varphi)$

Denote the space of distributions by $D^*(\Omega)$.

① Operations:

i) Derivate: For $F \in D^*$, $(\partial_x^\alpha F)(f) =: (-1)^{|\alpha|} F(\partial_x^\alpha f)$.

ii) Product: For $F \in D^*$, $\psi \in C^\infty$, $(\psi \cdot F)(f) =: F(\psi f)$

iii) For $F \in D^*$, if $\tau_h f(x) =: f(x+h)$, $f_a(x) =: f(ax)$, $a>0$ and $L \in M^d$, $|L| \neq 0$, $f_L(x) =: f(Lx)$

$(\tau_h F)(\phi) =: F(\tau_{-h}\phi)$. $(F_a)(\phi) =: a^{-d} F(\phi_{a^{-1}})$

$(F_L)(\phi) =: |L|^{-1} F(\phi_{L^{-1}})$

iv) Convolution: set $\psi_x^\sim(y) =: \psi(x-y)$. $\psi^\sim = \psi_0^\sim$

For $F \in D^*$, $(F * \psi)(\phi) =: F(\psi^\sim * \phi)$

$\forall \psi \in D$.

Rmk: We can also define $F * \psi$ as a function: $F * \psi = F(\tilde{\psi_x})$

Prop. i) The two ways of defining $F * \psi$ coincides.

ii) Distribution $F * \psi \in C^\infty$

i), ii) hold if $F \in D^*$, $\psi \in D$.

Pf: i) Prove: $\int F(\tilde{\psi_x}) \varphi(x) dx = F(\tilde{\psi} * \varphi)$

By $\tilde{\psi} * \phi(x) = \lim\limits_{\varepsilon \to 0} \varepsilon^d \sum\limits_{n \in \mathbb{Z}^d} \tilde{\psi}(x - n\varepsilon) \phi(n\varepsilon)$

$\overset{\Delta}{=} \lim\limits_{\varepsilon \to 0} S_\varepsilon$. $S_\varepsilon \in D$.

It follows from conti. of F. approxi. by S_ε.

ii) Check: $F(\tilde{\psi_x}) \in C^\infty$. (It's routine)

Def: $(F_n) \subset D^* \to F \in D^*$ in weak sense if $\forall \phi \in D$.

$F_n(\phi) \xrightarrow{n \to \infty} F(\phi)$.

Thm. $F \in D^*(\mathbb{R}^d)$. Then $\exists (f_n) \in C^\infty \to F$ in weak sense.

Pf: Set $F_n = F * \varphi_n$. distributions. (φ_n) is approxi of id.

$F_n(\varphi) = F(\tilde{\varphi_n} * \varphi) \xrightarrow{n \to \infty} F(\varphi)$.

② Supports:

Def: $F \in D^*$ vanishes on U. if $\forall \phi \in D$. $\operatorname{supp} \phi \subset U$.

$\Rightarrow F(\phi) = 0$. supp F is the complement of max. open set that F vanishes.

Rmk: i) supp$(\partial_x^\alpha F)$. supp$(\psi \cdot F) \subseteq$ supp F.

ii) $F(\varphi) = 0$. when supp$(F) \cap$ supp $\varphi = \emptyset$.

prop. $F \in D^*$. $\varphi \in D$. $\Rightarrow$ supp$(F * \varphi) \subseteq$ supp F + supp ϕ.

Pf: $F * \psi(x) = F(\tilde{\psi_x}) \neq 0$. $\Leftrightarrow$ supp $F \cap (x -$ supp $\phi) \neq \emptyset$.

Def: $F. G \in D^*$. supp G is cpt. We can define convolution between them: $(F * G)(\phi) = F(G^\sim * \phi)$

Rmk: $G^\sim * \phi \in C_c^\infty$ so belongs to D. well-def.

$\varphi \mapsto (F * G)(\varphi)$ is conti. easy to check.

prop. i) $F. G \in D^*$. both have cpt supports. $\Rightarrow F * G = G * F$.

ii) $\delta * F = F * \delta = F$

iii) $F. G \in D^*$. supp F is cpt $\Rightarrow \forall \alpha$. $\partial_x^\alpha(F * G) = (\partial_x^\alpha F) * G = F * (\partial_x^\alpha G)$.

iv) $F. G \in D^*$. supp F is cpt $\Rightarrow$ supp$(F * G) \subseteq$ supp F + supp G

Pf: i). ii). iii) are trivial. For iv):

approxi. G by $G \cdot \varphi_n$. φ_n is bump function.

③ Tempered Dist.

Def: $(\|\cdot\|_N)_{N \geq 0}$ is family of semi-norms on S. def by $\|\varphi\|_N = \sup_{\substack{x \\ |\alpha|, |\beta| \leq N}} |x^\alpha \partial_x^\beta \varphi(x)|$ (But for single one. $\|\cdot\|_N$ is a norm on S)

Rmk: i) $S(\mathbb{R}^d)$ isn't normable (under topo $(\|\cdot\|_k)_k$)

Pf: By contradiction. $\exists$ norm $n(\cdot)$ on S

$\Rightarrow n(\phi) \le C\|\phi\|_N$. (by conti.) $\exists N$. $^{(*)}$

Besides. $\|\phi\|_k \le C_k n(\phi)$. $\forall k$. $\exists C_k$

since it defines a topology.

So: $\|\cdot\|_N \sim \|\cdot\|_m$. $\forall N, m$. contradict!

(Consider $e^{-|x|^2}\sin(Nx_1)/N^k \xrightarrow{N\to\infty} 0$ only

in the first k semi-norms $\|\cdot\|_i$)

ii) Note that $D \subsetneq S$. tempered dist. is a

dist. automatically. But converse fails.

e.g. On $\mathbb{R}'$. $f(x) = e^x$.

Actually D is dense in S.

Pf: $\forall \varphi \in S$. set $\varphi_k = \varphi\, \eta_k$. (η_k) is bump

function. $\eta_k = 1$ on $[-k, k]$.

Lemma. $^{(*)}$ $F \in S^*$. $\Rightarrow \exists N \in \mathbb{Z}^+$. $c > 0$. s.t. $|F(\varphi)| \le c\|\varphi\|_N$.

Pf: By contradict. $\exists (\varphi_k)$. $\|\varphi_k\|_k = 1$. and

$|F(\varphi_k)| \ge k\|\varphi_k\|_k$. Set $(\phi_k) = (\varphi_k/\sqrt{k})$

Note that $\|\phi_k\|_N \xrightarrow{k\to\infty} 0$. ($\|\cdot\|_k \ge \|\cdot\|_N$ if $k \ge N$)

But $|F(\phi_k)| \ge k^{\frac{1}{2}} \to \infty$.

Prop. i) $F \in D^*$. supp F is cpt $\Rightarrow$ $F \in S^*$.

ii) $F \in S^* \Rightarrow x^\alpha \partial_x^\beta F \in S^* \quad \forall \alpha, \beta$.

iii) $f \in L^1_{loc}(\mathbb{R}^n)$. $\exists N \geq 0$. s.t. $\int_{\{R \leq |x| \leq R+1\}} |f| = O(R^N) \ (R \to \infty)$

$\Rightarrow$ dist. correspond f is tempered

Pf: i) $F = F \cdot \eta$. η is bump on nbd of supp F

iii) $|\int f \varphi| \lesssim \|\varphi\|_1 + \sum_n \int_{n \leq |x| \leq n+1} |f| |\phi| \, dx$

$\lesssim \|\varphi\|_1 + \sum_n \|\phi\|_{N+2} \dfrac{n^N}{n^{N+2}}$

$\lesssim \|\varphi\|_{N+2}$.

Thm. For $F \in D^*$. Then $F \in S^* \Leftrightarrow \exists A > 0 . N \in \mathbb{Z}^+ . \forall R \geq 1$. st.

$|F(\varphi)| \leq A R^N \sup\{ |\partial_x^\alpha \varphi| \mid |x| \leq R . |\alpha| \leq N \}$ for

$\forall \varphi \in D$. supports on $|x| \leq R$.

Pf: $(\Rightarrow)$ By Lemma $(*)$.

$(\Leftarrow)$ Smoothrize $(\chi_{\{n \leq |x| \leq n+1\}}) \to (e_n)$. $e_n \in C_c^\infty$

$\chi_{\{n \leq |x| \leq n+1\}} = e_n$. $|x| \in (n, n+1)$. $e_n = 0$. $|x| < n-1 \,/> n+2$

$|F(\varphi)| \lesssim \|\varphi\|_N + \sum_{n \geq m} |F(\varphi e_n)|$

$\lesssim \|\varphi\|_N + \sum n^N \sup_{\substack{n \leq |x| \leq n+2 \\ |\alpha| \leq N}} |\partial_x^\alpha \varphi|$

$\lesssim \|\varphi\|_N + \sum n^{-2} \|\varphi\|_{N+2} \lesssim \|\varphi\|_{N+2}$

Prop. i) $F \in S^*$. $\psi \in S \Rightarrow F * \psi \in S$.

ii) $F \in S^*$. $G \in D^*$. supp G is cpt $\Rightarrow F * G \in S^*$.

Pf: i) By: $\varphi_1, \varphi_2 \in S \Rightarrow \varphi_1 * \varphi_2 \in S$.

ii) Lemma $F \in D^*$. with cpt supp. $\varphi \in S$.

$\Rightarrow F * \varphi \in S$.

Pf: $|x^\beta \partial_x^\alpha \int F \tilde{\varphi}_x d\eta|$

$\leq \int_{B_n} |F| \frac{|x|^N}{|x-\eta|^N} \|\varphi\|_N d\eta$

$|x|^N / |x-\eta|^N \leq (1 + |\eta| / |x-\eta|)^N$

$\leq (1 + |\eta|)^N$.

Since $F * \varphi \in C^\infty$. We only consider x is large enough.

$\Rightarrow \|F * \varphi\|_N \lesssim \|\varphi\|_N$

Consider $(F * h)(\varphi) = F(\tilde{h} * \varphi)$.

Thm. (Point Support Dist.)

$F \in D^*$. support on origin. $\Rightarrow F = \sum_{|\alpha| \leq N} a_\alpha \partial_x^\alpha \delta$.

finite sum ($\exists N \in \mathbb{Z}^+$). i.e. $F(\phi) = \sum_{|\alpha| \leq N} a_\alpha (-1)^{|\alpha|} \partial_x^\alpha \varphi(0)$.

Pf: Lemma. $F \in D^*$. support on origin. If $\exists N$. s.t.

i) $|F(\varphi)| \leq c \|\varphi\|_N$. $\forall \varphi \in D$.

ii) $F(x^\alpha) = 0$. $\forall |\alpha| \leq N$.

Then $F = 0$.

Pf: Set $\eta \in D$. $\eta = 0$. $|x| \geq 1$. $\eta = 1$. $|x| \leq \frac{1}{2}$

$\eta_\varepsilon(x) = \eta(x/\varepsilon) \Rightarrow F(\varphi) = F(\eta_\varepsilon \varphi)$.

$R(x) = \varphi(x) - \sum_{|\alpha| \leq N} \frac{\partial_x^\alpha \varphi(0)}{\alpha!} x^\alpha$

$\Rightarrow |R| \lesssim |x|^{N+1}$. $|\partial_x^\beta R| \lesssim_\beta |x|^{N+1-|\beta|}$

combined with $|\partial_x^\beta \eta_\varepsilon| \le c_\beta \varepsilon^{-|\beta|}$. $\partial_x^\beta \eta_\varepsilon = 0$. $|x| \ge \varepsilon$.

$\Rightarrow \|\eta_\varepsilon R\|_N \le C\varepsilon$.

$$|F(\varphi)| = |F(\eta_\varepsilon R)| \lesssim \|\eta_\varepsilon R\|_N \lesssim \varepsilon \to 0.$$

$\Rightarrow$ Set $\tilde{F} = F - \sum_{|\alpha| \le N} a_\alpha \partial_x^\alpha \delta$. N for $|F(\varphi)| \lesssim \|\varphi\|_N$.

and $a_\alpha = (-1)^{|\alpha|} F(x^\alpha)/\alpha!$ satisfies Lemma.

④ <u>Fourier Transf:</u>

We only define "$\wedge$". "$\vee$" on $\mathcal{S}^*$. $F^\wedge(\varphi) = F(\varphi^\wedge)$. $\forall \varphi \in \mathcal{S}$.

<u>Lemma.</u> i) $\|\hat{\varphi}\|_N \underset{N}{\lesssim} \|\varphi\|_{N+d+1}$. for $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

ii) $(\partial_x^\alpha F)^\wedge = (2\pi i x)^\alpha F^\wedge$. $((-2\pi i x)^\alpha F)^\wedge = \partial_x^\alpha (F^\wedge)$.

<u>prop.</u> $F \in \mathcal{S}^*$. $\psi \in \mathcal{S}$. $\Rightarrow F * \psi \in \mathcal{S}^*$. st. $(F*\psi)^\wedge = F^\wedge \cdot \psi^\wedge$.

<u>Pf:</u> $F(\tilde{\psi_x}) \in \mathcal{S}$. follows from the inequality:

$$\|\partial_x^\alpha \tilde{\varphi_x}\|_N \le c(1+|x|)^N \|\varphi\|_{N+|\alpha|}. \text{ in Lemma above.}$$

Check $(F*\psi)^\wedge(\varphi) = (F^\wedge \cdot \psi^\wedge)(\varphi)$.

<u>prop.</u> $F \in \mathcal{D}^*$ with cpt support. $\Rightarrow F^\wedge \in \mathcal{S}^*$.

moreover, as a function $F^\wedge(\xi) = F(e_\xi)$.

where $e_\xi = \eta e^{-2\pi i x \xi}$. $\eta \in \mathcal{D}$. $\eta = 1$. in

nbd of supp F.

<u>Pf:</u> 1°) Check: $\int F(e_\xi)\varphi(\xi)d\xi = F(\hat{\varphi})$. $\forall \varphi \in \mathcal{D}$.

$$\text{LHS} = \lim_{\varepsilon \to 0} F(S_\varepsilon). \quad S_\varepsilon = \varepsilon^d \sum_{n \in \mathbb{Z}^d} e_{n\varepsilon} \varphi(n\varepsilon).$$

$S_\varepsilon \xrightarrow{\varepsilon \to 0} \eta \hat{\varphi}$. Then by conti. of F.

Extend $\varphi \in D$ to $S \underset{\text{dense}}{\supseteq} D$.

2') $F(e_\xi) \in S^*$

Note $|F(e_\xi)| \lesssim \|e_\xi\|_N \lesssim (1+|\xi|)^N$.

$|\partial_\xi^\alpha F(e_\xi)| \lesssim \|\partial_\xi^\alpha e_\xi\|_N \lesssim (1+|\xi|)^{N+|\alpha|}$

(2) <u>Examples:</u>

① <u>Hilbert Transf:</u>

<u>Thm.</u> dist. $p.v.(\frac{1}{x})$ equals:

i) $\frac{d}{dx}\log|x|$. ii) $\frac{1}{2}\left(\frac{1}{x-i\cdot 0}+\frac{1}{x+i\cdot 0}\right)$

<u>Pf:</u> i) $\left(\frac{d}{dx}\log|x|\right)(\varphi) = -\int_{-\infty}^{+\infty}\log|x|\frac{d}{dx}\varphi$

defined by: $\lim_{\varepsilon\to 0} -\int_{|x|\geq\varepsilon}\log|x|\frac{d\varphi}{dx}dx$.

$=\lim_{\varepsilon}\int\frac{\varphi}{x}dx + \log\varepsilon\,(\varphi(\varepsilon)-\varphi(-\varepsilon))$

$= p.v.(\frac{1}{x})(\varphi)$, by $\varphi(\varepsilon)-\varphi(-\varepsilon) = O(\varepsilon)$.

ii) $\frac{1}{x\pm i\cdot 0} = \lim_{\varepsilon\to 0}\frac{1}{x\pm i\varepsilon}$ in sense of dist.

1') Prove: $x/x^2+\varepsilon^2 \xrightarrow{\varepsilon\to 0} p.v.(\frac{1}{x})$.

LHS $= \pi Q_\varepsilon$. we have proved before

2') $-1/iz = P_y(x) + iQ_y(x)$. Set $y=\varepsilon>0$

Note $P_y\to\delta$. Take complex conjugate:

$\frac{1}{x-i\varepsilon} = \pi Q_\varepsilon + i\pi P_\varepsilon \longrightarrow p.v.(\frac{1}{x}) + i\pi\delta$.

Similarly. take complex conjugate again

$\Rightarrow 1/x\pm i\cdot 0 = p.v.(\frac{1}{x}) \mp i\pi\delta$.

① Homogenuous Dist. :

Def: f defined on $\mathbb{R}^d/\{0\}$ is homogeneous of degree λ if $f_a(x) = f(ax) = a^\lambda f$. $a>0$.

Rmk: Denote $\varphi^a = a^{-d}\varphi_{a^{-1}}$ for φ dist. / func.

Then $F_a(\varphi) = F(\varphi^a)$. $F \in D^*$.

Say F is homogeneous of degree λ

if $F_a = a^\lambda F$. $\forall a>0$.

Lemma. $(\varphi^a)^\wedge = (\varphi^\wedge)_a$. $\forall \varphi \in S$.

Prop. $F \in S^*(\mathbb{R}^d)$ has homogeneous of degree λ.

$\Rightarrow F^\wedge$ has homogenous of degree $-d-\lambda$.

Pf: $(F^\wedge)_a(\varphi) = F((\varphi^a)^\wedge) = F((\varphi^\wedge)_a)$

$= F^\wedge(\varphi^a) = a^{-d} F_{a^{-1}}(\varphi^\wedge) = a^{-d-\lambda} F^\wedge(\varphi)$

Def: H_λ is correspond dist. of $|x|^\lambda$. $0>\lambda>-d$.

Rmk: When $0>\lambda>-d$. $|x|^\lambda \in L^1_{loc}$ and has homogeneous of degree λ.

Thm. $(H_\lambda)^\wedge = c_\lambda H_{-d-\lambda}$ $\quad c_\lambda = \dfrac{\Gamma(\frac{d+\lambda}{2})}{\Gamma(-\frac{\lambda}{2})}\pi^{-\frac{d}{2}-\lambda}$

Pf: Note $\int_{\mathbb{R}^d} e^{-\pi t|x|^2}\hat{\varphi}(x) = t^{-d/2}\int_{\mathbb{R}^d} e^{-\pi|x|^2/t}\varphi\,dx$

follows from $(\varphi_a)^\wedge = (\varphi^\wedge)_a$. $a = t^{\frac{1}{2}}$

multiple $t^{-\lambda/2-1}$. integrate $\int_0^\infty dt$. both sides.

Def: $K \in D^*$ is regular if $\exists k \in C^\infty(\mathbb{R}^n/\{0\})$

s.t. $K(\varphi) = \int k(x)\varphi(x)dx$. $\forall \varphi \in D$. whose

supports are disjoint from the origin.

Rmk: i) We say k is function associated to K.

ii) e.g. $|x|^\lambda \sim H_\lambda$. $\frac{1}{x} \sim p.v.(\frac{1}{x})$

prop. K has homogeneous of degree $\lambda \iff k$ has

homogeneous of degree λ.

Pf: $K_r(\varphi) = \int k(x)\varphi^r(x) = \int k_r \varphi = r^\lambda \int k\varphi$.

$\forall \varphi \in D$. $\operatorname{supp}\varphi \cap \{0\} = \emptyset$.

i) Characterize Fourier Transf on regular Dist:

prop. $K \in D^*$. regular with homogeneous of degree

$\lambda \iff \hat{K} \in D^*$. regular with homogeneous

of degree $-\lambda - d$.

Pf: $(\Rightarrow)$. Only prove $\hat{K}$ is regular:

Set $\eta \in D$. $\eta = 1$ $|x| \le \frac{1}{2}$. $\eta = 0$. $|x| \ge 1$.

$K = K_0 + K_1$. $K_0 = \eta K$. cpt supp $\Rightarrow \hat{K_0} \in C^\infty$

And $K_1 = (1-\eta)K$ associated to $(1-\eta)k$.

prove: $\hat{K_1} \in C^\infty$ away from origin

$\partial_\xi^\alpha(\hat{K_1}) = (\Delta^N((-2\pi i x)^\alpha K_1))^\wedge / (-4\pi^2|\xi|^2)^N$

1°) $|x| \geq 1$. $k_1 = k$. $\Rightarrow \partial_x^\beta(k_1)$ is bdd with degree $\lambda - |\beta|$. So $\sim O(|x|^{\lambda-|\beta|})$.

$\Rightarrow \Delta^N(x^\alpha k_1) \sim O(|x|^{\lambda+|\alpha|-2N})$.

2°) $|x| \leq 1$. $\Rightarrow \Delta^N(x^\alpha k_1)$ is bdd

$\Rightarrow$ By $\Delta^N(x^\alpha k_1) \in L^1(\mathbb{R}^n)$ for large N.

We have $(\Delta^N(x^\alpha k_1))^\wedge \in C(\mathbb{R}^n)$.

So $\partial_\xi^\alpha(k_1^\wedge)$ agree with a conti func. away from origin (Fix a nbhd). $\forall \alpha$.

($\Leftarrow$) Note: $k^\vee = (k^\wedge)^\sim$. Directly.

ii) Given func. k with degree λ. C^∞ away from the origin. When $\exists K \in D^*$. regular of degree λ st. K associated to k. Does K uniquely determined by k?

<u>Thm.</u> $k \in C^\infty(\mathbb{R}^n/\{0\})$ with degree λ.

(a) λ isn't form $-d-m$. $m \in \mathbb{Z}^{\geq 0}$ $\Rightarrow \exists$ unique $K \in D^*$. of degree λ st. associated k.

(b) $\lambda = -d-m$. $m \in \mathbb{Z}^{\geq 0}$. Then:

$\exists K \in D^*$ as in (a) $\Longleftrightarrow$ k satisfies cancell condition: $\int_{|x|=1} x^\alpha k(x)\, d\sigma = 0$. $\forall |\alpha| = m$.

<u>Rmk:</u> k satisfies (b) won't correspond unique $K \in D^*$. but K has form: $K + \sum_{|\alpha|=m} c_\alpha \partial_x^\alpha \delta$.

Cor. There's no $K \in D^*$ of degree $-d$ which agrees $1/|x|^d$ away from origin.

Pf: $\int_{|x|=1} 1/|x|^d \, d\sigma(x) = |S^{d-1}| \neq 0$

Rmk: For $\lambda = -d$. k satisfies $\int_{|x|=1} k \, d\sigma = 0$ the correspond dist. K is generalization of $p.v.(\frac{1}{x})$ on $\mathbb{R}^1$. denoted by $p.v.(k)$

We have $K(\varphi) = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} k(x) \varphi(x) \, dx$.

$= \int_{|x| \leq 1} k(x) (\varphi(x) - \varphi(0)) \, dx + \int_{|x|>1} k(x) \varphi(x) \, dx$

By $\int_{\varepsilon \leq |x| \leq 1} k(x) = \log 1/\varepsilon \int_{|x|=1} k \, d\sigma = 0$.

③ Fundamental Solutions:

Def: For $L = \sum_{|\alpha| \leq m} a_\alpha \partial_x^\alpha$ partial differential operator on $\mathbb{R}^d$. $a_\alpha \in \mathbb{C}$.

i) A fundamental solution of L is $F \in D^*$. st. $L(F) = \delta$.

Rmk: i) Fundamental solution isn't unique: since $\exists u$. st. $L(u) = 0$.

ii) $T: f \mapsto F * f$. from D to C^∞ is "inverse" to L. i.e. $LT = TL = I$.

ii) $p(\xi) = \sum_{|\alpha| \leq m} a_\alpha (2\pi i \xi)^\alpha$ is characteristic poly of L

Rmk: $(L(f))^\wedge = p(\xi)\cdot f^\wedge$. We can define F by $\hat{F}(\xi) = 1/p(\xi)$. which is hard to defined as a dist. due to zeros of P.

i) Laplacian:

$\Delta = \sum_i^d \partial^2/\partial x_j^2$ in $\mathbb{R}^d$. $\sim$ $p(\xi) = -4\pi^2|\xi|^2$.

Thm. For $d\geq 3$. $F(x) = c_d |x|^{-d+2} \in L^1_{loc}$ is fundamental solution for Δ. $c_d = -\Gamma(\frac{d}{2}-1)/4\pi^{\frac{d}{2}}$.

Pf: By $H_\lambda^\wedge = H_{-\lambda-d}$. Set $\lambda = -d+2 \in (-d,0)$

$(\Delta F)^\wedge = 1 \Rightarrow \Delta F = \delta$.

Thm. $d=2$. $F = \frac{1}{2\pi}\log|x|$ is fundamental solution of Δ.

Pf: It arises when considering $\lambda\to 0$.

1') From $\int_{\mathbb{R}^2} \hat{\varphi}(x)|x|^\lambda dx = c_\lambda \int_{\mathbb{R}^2} \varphi(\xi)|\xi|^{-\lambda-2} d\xi$

$c_\lambda = \pi^{-1-\lambda}\Gamma(1+\frac{\lambda}{2})/\Gamma(-\frac{\lambda}{2}) \sim -\frac{\lambda}{2\pi} + c'\lambda^2 \ (\lambda\to 0)$

Differentiate both sides at λ. Set $\lambda\to 0$.

$$\Rightarrow \frac{1}{2\pi}\int \hat{\varphi}\log|x| = -\frac{1}{4\pi^2}\left(\int_{|x|\leq 1}\frac{\varphi(x)-\varphi(0)}{|x|^2} + \int_{|x|>1}\varphi/|x|^2\right) - c'\varphi(0).$$

So: $\hat{F} = -\frac{1}{4\pi^2}[1/|x|^2] + (-c'\delta)$

2') $(\Delta F)^\wedge = -4\pi^2|x|^2 F^\wedge = |x|^2[1/|x|^2] + \tilde{c}|x|^2\cdot\delta$

$= 1 \Rightarrow \Delta F = \delta$.

ii) Heat Operator:

$L = \partial/\partial t - \Delta x$ in $\mathbb{R}^{d+1}$ $\sim 2\pi i t + 4\pi^2 |x|^2$.

Recall: $\begin{cases} Lu = 0 & t > 0 \\ u|_{t=0} = f \end{cases}$ is solved by

$u(x,t) = H_t * f$. where $\hat{H_t} = e^{-4\pi^2 |\xi|^2 t}$

$H_t(x)$ is approxi. of id. $LH_t = 0$.

Thm. $F(x,t) = H_t(x) \chi_{\{t>0\}} \in L^1_{loc}$ is fundamental solution of L.

Pf: From $LF(\varphi) = F(L'\varphi)$. $L' = -\frac{\partial}{\partial t} - \Delta x$.

$\Rightarrow LHS = \lim\limits_{\varepsilon\to 0} \int_{t\geq\varepsilon} \int_{\mathbb{R}^d} F(-\frac{\partial}{\partial t} - \Delta_x)\varphi$

$= -\lim\limits_{\varepsilon\to 0} \int_{\mathbb{R}^d} \int_{t\geq\varepsilon} (H_t \frac{\partial\varphi}{\partial t} + \frac{\partial H_t}{\partial t}\varphi)$

$= -\lim\limits_{\varepsilon\to 0} \int_{\mathbb{R}^d} H_\varepsilon(x)\varphi(x,\varepsilon)\,dx$

follows from integrate by part.

$H_\varepsilon(x)\varphi(x,\varepsilon) = H_\varepsilon\varphi(x,0) + O(\varepsilon)$. by $\varphi \in S$.

So: $RHS = \varphi(0,0)$. ($H_t(x)$ is A of id.)

iii) General Case:

For general diff. operator L on $\mathbb{R}^d$. Candidate for a fundamental solution is $F = \int e^{2\pi i \xi \cdot x}/p(\xi)\,d\xi$.

$\Rightarrow F(\varphi) = \int_{\mathbb{R}^d} \frac{\hat{\varphi}(-\xi)}{p(\xi)}\,d\xi$. by Fubini.

Thm. Every const. coefficient linear partial diff. operators L on $\mathbb{R}^d$ has a fundamental solution.

Rmk: To circumvent the obstacle of zeros of $p(\xi)$. We shift the line of integration in ξ_1-variable to avoid zeros of $p(z,\xi')$. $\xi' = (\xi_2 \cdots \xi_d)$.

Cor. $\forall f \in C_c^\infty(\mathbb{R}^d)$. $\exists u \in C^\infty(\mathbb{R}^d)$. st.

$Lu = f$. for const.-coeff. LPDO. L.

Pf: Take $u = F * f$. F is fundamental Sol.

iv) Parametrix and Regular:

Def: i) For differential operator L with constant coefficients. $Q \in \mathcal{D}^*$ is parametrix for L if $LQ = \delta + r$. $r \in \mathcal{S}$. ("error")

We say Q is regular if it agrees with a C^∞ function away from the origin.

ii) $L = \sum_{|\alpha| \le m} a_\alpha \partial_x^\alpha$ is elliptic if its characteristic func. $p(\xi)$ satisfies: $\exists c > 0$. $|p(\xi)| \ge c|\xi|^m$. for large enough ξ.

Rmk: It's equi: P_m is principle part of P. i.e. homo part of degree m. $P_m(\xi) = 0 \iff \xi = 0$.

$(p(\xi) = |\xi|^m p(\xi'))$.

Thm. Every elliptic LPDO has a regular parametrix.

Pf: suppose $|P(\xi)| \geq C_0|\xi|^m$. for $\forall |\xi| \geq C$.

Next, we induct on $|\alpha| = k$:

$(\frac{\partial}{\partial \xi})^\alpha (1/P(\xi)) = \sum_{0 \leq l \leq k} q_l(\xi) / P(\xi)^{l+1}$

where q_l is poly with degree $\leq lm-k$.

1) Set: $\psi = 1$ for large ξ. $\mathrm{supp}\,\psi \subseteq \{|x| \geq C\}$.

and $\psi \in C^\infty$.

$Q \in \mathcal{S}^*$. s.t. $\hat{Q} = \psi/P$.

From: $((-4\pi^2|x|^2)^N \partial_x^\beta Q)^\wedge = \Delta_\xi^N ((2\pi i\xi)^\beta (\psi/P))$

$|\partial_\xi^\alpha (\psi/P)| \leq A_\alpha |\xi|^{-m-|\alpha|}$.

$\Rightarrow |RHS| \leq \tilde{A}_\alpha |\xi|^{-m-2N+|\beta|}$. $|\xi| \geq 1$.

So for large N. RHS $\in L^1$.

Argue as before: $\partial_x^\beta Q$ agree with a conti. func. away from origin.

2) $(LQ)^\wedge = P \cdot \psi/P = \psi$

Note: $\psi - 1 \in D^*$. $\Rightarrow \psi - 1 = \hat{r}$. $r \in \mathcal{S}$.

So: $(LQ)^\wedge = 1 + \hat{r} \Rightarrow LQ = \delta + r$.

Cor. $\forall \varepsilon > 0$. elliptic LPDO has a regular parametrix Q_ε supporting on $\{|x| \leq \varepsilon\}$.

Pf: Set $\eta_\varepsilon \in D$. $\mathrm{supp}\,\eta_\varepsilon \subseteq \{|x| \leq \varepsilon\}$. $\eta_\varepsilon = 1$. $|x| \leq \frac{\varepsilon}{2}$.

Set $Q_\varepsilon = \eta_\varepsilon Q$. check on it:

Note $L(\theta_\varepsilon) - \eta_\varepsilon L(\theta)$ support on $\{|x| \le \varepsilon\}$. C^∞.

$\eta_\varepsilon L(u) = \eta_\varepsilon(\delta + r) = \delta + \eta_\varepsilon r$.

$\Rightarrow L(\theta_\varepsilon) = \delta + r_\varepsilon$. $r_\varepsilon \in S$. support on $\{|x| \le \varepsilon\}$

Def: A LPDO. L with const. coefficient is hypo-elliptic if it has a regular parametrix.

Thm. $L = \sum_{|\alpha| \le m} a_\alpha \partial_x^\alpha$. where $a_\alpha \in \mathbb{C}$. on $\mathbb{R}^d$.

Then L is hypo-elliptic $\Leftrightarrow$ $\dfrac{\partial_\xi^\alpha P(\xi)}{P(\xi)} \xrightarrow{|\xi| \to \infty} 0$. $L \overset{\text{charac.}}{\sim} P$.

Rmk: It means that exists LPDO that's not hypo-elliptic. e.g. wave operator.

Thm. For hypo-elliptic operator L. If $u \in D^*(\Omega)$. $\Omega \underset{\text{open}}{\subseteq} \mathbb{R}^d$. and $L(u) = f \in C^\infty(\Omega)$. Then:

$u \in C^\infty(\Omega)$. as well.

Pf: Fix a ball $\overline{B} \subseteq \Omega$. Prove: u agree with a C^∞ function on every such B.

1') Set $\overline{B} \subset B_1 \subset \overline{B_1} \subset \Omega$. $\eta \in D$. cut-off func.

s.t. $\eta = 1$ in nbd of B_1. $\operatorname{supp} \eta \underset{\text{cpt}}{\subseteq} \Omega$.

Let $u_1 = u\eta \Rightarrow Lu_1 = f_1$. cpt. support.

f_1 agree $f \in C^\infty$ in nbd of $\overline{B}$.

2') Note the cor. above holds for hypo-elliptic.

$\exists Q_\varepsilon$ parametrix supports on $\{|x| \le \varepsilon\}$. of L.

From: $Q_\varepsilon * L(u) = L(Q_\varepsilon) * u = u_1 + r_\varepsilon * u_1$

$r_\varepsilon * u_1 = u_1 * r_\varepsilon \in C_c^\infty(\mathbb{R}^n)$

$Q_\varepsilon * L(u) = Q_\varepsilon * f + Q_\varepsilon * (f_1 - f) \in C^\infty$. in $\bar{B}$

follow from $Q_\varepsilon * (f_1 - f) = 0$ in $\bar{B}$

$\Rightarrow u_1 = Q_\varepsilon * L(u) - r_\varepsilon * u_1 \in C^\infty(\bar{B})$.

(3) Calderón - Zygmund Dist.

Prop. i) $K \in D^*$. regular. associated with k satisfies: $|\partial_x^\alpha k(x)| \le C_\alpha |x|^{-d-|\alpha|}$. $\forall \alpha$ and $\sup_{r>0} |K(\varphi_r)|$ $\le A$. for $\forall \varphi \in C_c^\infty$. $\operatorname{supp}\varphi \subseteq \overline{B(0,1)}$ and $\sup_x |\partial_x^\alpha \varphi| \le 1$. $\forall |\alpha| \le n$. for fix n. (A indept of φ)

ii) $K \in D^*$. regular. $\sim k$. s.t. $|\partial_x^\alpha k(x)| \le C_\alpha |x|^{-d-|\alpha|}$. $\forall \alpha$ and $K^\wedge$ is bdd

iii) $K \in S^*$. $m = K^\wedge$ is func. $\in C^\infty$ away from the origin. s.t. $|\partial_\xi^\alpha m(\xi)| \le \tilde{C}_\alpha |\xi|^{-|\alpha|}$. $\forall \alpha$.

We have i). ii). iii) equi.

Rmk: We refer to kernel K satisfies these equi. properties as Calderón - Zygmund dist. Set $T(f) = K * f$. it's a Calderón - Zygmund operator (easy to check) So weak-(1,1). strong-(p,p). $1<p<\infty$.