Hilbert Transform

(1) Background:

① Motivation:

i) $\forall f \in L^p(\mathbb{R}^1)$. Can we find $F_f(z) \in \mathcal{O}(\mathbb{R}^2_+)$ s.t. $Re(F_f)|_{\mathbb{R}^1} = f$?

Note that $\{Im z > 0\}$ is simple connected.

We find a harmonic function: $P_\eta * f(x) = \frac{\eta}{\pi}\int_{\mathbb{R}^1} f(t)/(x-t)^2+\eta^2 \, dt$ on upper plane.

Besides, $P_\eta * f \xrightarrow{\eta \to 0^+} f$, a.e.

Next. we find its conjugate harmonic func.

Note that $P_\eta * f(x) = Re\left(\frac{i}{\pi}\int_{\mathbb{R}^1} \frac{f(t)\,dt}{(x-t)+i\eta}\right)$

$= Re\left(\frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{f(t)}{z-t}\,dt\right)$

Set $F_f(z) = \frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{f(t)}{z-t}\,dt$

Rmk: $Im(F_f(z)) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(t)(x-t)}{(x-t)^2+\eta^2}\,dt$.

$= Q_\eta * f(x)$.

$Q_\eta(x) = \frac{1}{\pi}\frac{x}{x^2+\eta^2}$. $P_\eta + iQ_\eta = i/\pi z \in \mathcal{O}(\mathbb{R}^2_+)$

Q_η is called conjugate Poisson kernel.

ii) A natural question: $Q_\eta * f \xrightarrow{\eta \to 0} f$ a.e. ?

No. $(Q_t)_{t>0}$ isn't approxi. of id. and in fact, $Q_t \notin L^1$. Besides, $\lim_{t\to 0} Q_t(x) = 1/\pi x \notin L^1_{loc}(\mathbb{R})$

② Principle value of $1/x$:

Def: $p.v.\frac{1}{x}(\phi) = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \phi(x)/x\, dx$. $\phi \in S(\mathbb{R})$.

Rmk: $p.v.\frac{1}{x} \in S^*$. Note we can rewrite it:

$$p.v.\frac{1}{x}(\phi) = \int_{|x|<1}\frac{\phi(x)-\phi(0)}{x}dx + \int_{|x|>1}\frac{\phi(x)}{x}dx$$

$$\Rightarrow |LHS| \lesssim \|\phi'\|_\infty + \|x\phi\|_\infty$$

Prop. $\lim_{t\to 0} Q_t = \frac{1}{\pi}\, p.v.\frac{1}{x}$. in S^*.

Pf: $\psi_\varepsilon(x) = \chi_{\{|x|>\varepsilon\}}/x \xrightarrow{\varepsilon\to 0} p.v.\frac{1}{x}$ in S^*.

$\Rightarrow$ Prove: $\lim_{t\to 0}(Q_t - \frac{1}{\pi}\psi_t) = 0$. in S^*.

By: $(\pi Q_t - \psi_t)(\phi) = \int_{\mathbb{R}}\frac{x\phi(x)}{t^2+x^2} - \int_{|x|>t}\frac{\phi(x)}{x}dx$

$$= \int_{|x|\le t}\square - \int_{|x|>t}\left(\frac{x}{t^2+x^2} - \frac{1}{x}\right)\phi$$

Set $x = tu$. Apply DCT. Let $t\to 0$.

Def: $f\in S$. Hilbert Transf. of f is $Hf = \lim_{t\to 0} Q_t * f$

Rmk: i) Equivalent definitions:

$$Hf = p.v.\frac{1}{x} * f/\pi$$

$$Hf = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|y|>\varepsilon} f(x-y)/y\, dy$$

ii) Note by conti of "$\wedge$":

$$\hat{Q}_t(\xi) = -i\,\mathrm{sgn}(\xi)\, e^{-2\pi t|\xi|} \xrightarrow{t\to 0} \left(\frac{1}{\pi}\, p.v.\frac{1}{x}\right)^\wedge = -i\,\mathrm{sgn}(\xi)$$

$\Rightarrow (Hf)^\wedge(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)$. We have:

$\|Hf\|_2 = \|f\|_2$. $H(Hf) = -f$ (Period 4)

$$\int Hf \cdot g = -\int f \cdot Hg$$

Thm. f is bdd on $\mathbb{R}^1$ with cpt support. Then:

$Hf \in L^1(\mathbb{R}^1) \iff \int f \, dx = 0$.

Pf: Denote $a = \int f dx$.

$$Hf(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|t| > \varepsilon} \left(\frac{1}{x-t} - \frac{1}{x} \right) f(t) \, dt + \frac{a}{\pi x}$$

$$= \frac{1}{x^2} \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|t| > \varepsilon} \frac{t}{1 - t/x} f(t) \, dt + \frac{a}{\pi x}$$

$$= O\left(\frac{1}{x^2}\right) + \frac{a}{\pi x}. \quad (x \to \infty).$$

(2) Type-(p.p):

Thm. For $f \in S(\mathbb{R}^1)$. Then:

i) H is weak-(1.1): $|\{|Hf| > \lambda\}| \leq \frac{C}{\lambda} \|f\|_1$. $\lambda > 0$

ii) H is strong-(p.p): $1 < p < \infty$. $\|Hf\|_p \leq C_p \|f\|_p$.

Pf: i) Separate $f = f^+ - f^-$. WLOG. $f \geq 0$.

By C-Z. Decompose: $f = g + b$. $g \leq 2\lambda$.

$$|\{|Hf| \geq \lambda\}| \leq |\{|Hg| \geq \tfrac{\lambda}{2}\}| + |\{|Hb| \geq \tfrac{\lambda}{2}\}|$$

1°) $|\{|Hg| \geq \frac{\lambda}{2}\}| \leq \left(\frac{2}{\lambda}\right)^2 \int (Hg)^2 dx$

$$= \left(\frac{2}{\lambda}\right)^2 \|g\|_2^2 \leq \frac{8}{\lambda} \|g\|_1$$

$$= \frac{8}{\lambda} \int f(x) \, dx.$$

2°) $\Omega = \bigcup I_j$. Set $2I_j$ is interval with same center c_j as I_j but twice the length.

$\Omega^* = \bigcup 2I_j$. $|\Omega^*| \le 2|\Omega| \le \frac{2}{\lambda}\|f\|_1$

$\Rightarrow |\{|Hb| \ge \frac{\lambda}{2}\}| \le |\Omega^*| + |\{x \notin \Omega^* \mid |Hb| \ge \frac{\lambda}{2}\}|$

$\le \frac{2}{\lambda}\|f\|_1 + \frac{2}{\lambda}\int_{\mathbb{R}\setminus\Omega^*} |Hb|$

$\int_{\mathbb{R}\setminus\Omega^*} |Hb| \le \sum_j \int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)|\,dx$

where $Hb_j = \int_{I_j} \frac{b_j(\eta)}{x-\eta}\,d\eta$. well-def. since $x \notin 2I_j$.

$\int_{\mathbb{R}\setminus 2I_j} |Hb_j|\,dx = \int_{\mathbb{R}\setminus 2I_j} \left|\int_{I_j} b_j(\eta)\left(\frac{1}{x-\eta} - \frac{1}{x-c_j}\right)d\eta\right| dx$

$\le \int_{\mathbb{R}\setminus 2I_j}\int_{I_j} |b_j(\eta)| \frac{|\eta - c_j|}{|x-\eta||x-c_j|}\,d\eta\,dx$

$\le \int_{I_j} |b_j(\eta)| \left(\int_{\square} \frac{|I_j|}{|x-c_j|^2}\,dx\right) d\eta \le 2\int_{I_j} |b_j|$

$\le 4\int_{I_j} |f|$.

(It's alike $= E^2(X - E(X))$, $E(X) = 0$)

ii) By interpolation. since H is weak-(1,1), strong-(2,2).

We have H is strong-(p,p). $1<p<2$.

For $p>2$. $\|Hf\|_p = \sup\{|\int Hf\cdot g| \mid \|g\|_{p'} \le 1\}$

$= \sup\{|\int f\cdot Hg| \mid \|g\|_{p'} \le 1\}$

$\le \|f\|_p \sup_{\|g\|_{p'}\le 1} \|Hg\|_{p'}$

$\le C_{p'}\|f\|_p$. By Riesz. repre.

Rmk: i) $C_p = O(p)$ $(p\to\infty)$. $C_p = O(1/_{1-p})$ $(p \to 1)$

ii) We can extend H on $\mathcal{S}$ to L^p. $1 \le p < \infty$.

When $1<p<\infty$ is routine. for $p=1$.

Let $(f_n) \subset \mathcal{S} \xrightarrow{L^1} f$. Then $f_n \xrightarrow{m} f$.

By weak-(1,1). (Hf_n) is Cauchy in Leb-measure.

$\Rightarrow \exists$ subseq. a.e. converge to Hf

iii) Strong-(p,p) fails when $p=1,\infty$.

Set $f=\chi_{[0,1]}$. $Hf(x)=\frac{1}{\pi}\log\left|\frac{x}{x-1}\right|$

(3) <u>Truncated Form</u>:

<u>Def</u>: $H_\varepsilon f(x)=\frac{1}{\pi}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}dy$. for $f\in L^p$, $p\geq 1$.

<u>Thm</u>. H_ε is weak-$(1,1)$, strong-(p,p), $1<p<\infty$, $\varepsilon>0$

with const C_p^ε. uniform bdd with ε.

<u>Pf</u>: 1°) Weak-$(1,1)$ is identitial argue as H.

2°) Strong-$(2,2)$: Note that

$$\left(\frac{1}{y}\chi_{|y|>\varepsilon}\right)^\wedge(\xi)=\lim_{N\to\infty}\int_{\varepsilon<|y|<N}e^{-2\pi i y\xi}/y\,dy$$

$$=-2i\,\mathrm{sgn}(\xi)\lim_{N\to\infty}\int_{2\pi\varepsilon|\xi|}^{2\pi N|\xi|}\sin t/t\,dt$$

$\Rightarrow C_2^\varepsilon$ is uniform bdd.

3°) By interplation and dual argue

Solve $2<p<\infty$.

<u>Thm</u>: For $f\in L^p$, $1\leq p<\infty$. $H_\varepsilon f\xrightarrow{L^p}Hf$. if $p>1$.

$H_\varepsilon f\xrightarrow{m}Hf$ if $p=1$.

<u>Pf</u>: Approxi f by $(f_n)\subset S$.

$Hf=\lim_n Hf_n=\lim_n\lim_\varepsilon H_\varepsilon f_n=\lim_\varepsilon\lim_n H_\varepsilon f_n=\lim_\varepsilon H_\varepsilon f$

follows from the uniform bdd const. C_p.

Thm. $H^*f = \sup_{\varepsilon>0} |H_\varepsilon f|$ is weak-(1,1), strong-(p,p). $1<p<\infty$.

Lemma. $H^*f \le M(Hf) + cMf$. $\forall f \in S$.

Pf: Prove for each H_ε. c is indept with ε.

Fix $\phi \ge 0 \in S$. even. $\downarrow$ on $(0,\infty)$. $\operatorname{supp}\phi = \{|x|<\frac{1}{2}\}$

$\int \phi = 1$. Set $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$.

$$\frac{1}{y}\chi_{|y|>\varepsilon} = \phi_\varepsilon * \text{p.v.}\frac{1}{x} + \left(\frac{1}{y}\chi_{\{|y|>\varepsilon\}} - \square\right).$$

$$=: A + B$$

1') $|A(f)| \le \|\phi\|_1 M(Hf)$.

2') For B: Fix $\varepsilon = 1$.

If $|y|>1$. $|B| = \left|\int_{|x|<\frac{1}{2}} \phi(x)\left(\frac{1}{y} - \frac{1}{y-x}\right)dx\right|$

$$\lesssim 1/|y|^2.$$

If $|y|<1$. $|B| = \left|-\lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{\phi(y-x)}{x}dx\right|$

$$\le \left|\int_{|x|<2} \frac{\phi(y-x)-\phi(y)}{x}dx\right|$$

$$\le c$$

$\Rightarrow |B| \le c/_{1+y^2}$. satisfies radial.. condition

So $|B(f)| \lesssim Mf$.

Return to pf:

By Lemma $\Rightarrow H^*$ is strong-(p,p). $1<p<\infty$.

For weak-(1,1):

Fix $f \in L^1$. suppose $f \ge 0$. C-Z Decompose f in height $\lambda > 0$:

$f = g + b = g + \Sigma b_j$

good part is trivial by strong-(2,2) of H^*.

Similarly, prove: $|\{x \notin \Omega^* \mid H^*b(x) > \lambda\}| \leq \frac{c}{\lambda}\|b\|_1$

$\Omega^* = \cup 2I_j$, $\Omega = \cup I_j$. c_j is center of I_j.

Fix $\varepsilon > 0$, $x \notin \Omega^*$.

(a) $(x-\varepsilon, x+\varepsilon) \cap I_j = I_j \Rightarrow H_\varepsilon b_j = 0$

(b) $(x-\varepsilon, x+\varepsilon) \cap I_j = \emptyset \Rightarrow |H_\varepsilon b_j| = |Hb_j| \leq \frac{|I_j|}{|x-c_j|^2}\|b_j\|_1$

(c) $x-\varepsilon \in I_j$ or $x+\varepsilon \in I_j \Rightarrow I_j \subset (x-3\varepsilon, x+3\varepsilon)$

$\forall \eta \in I_j$, $|x-\eta| > \frac{\varepsilon}{3}$.

$$\therefore |H_\varepsilon b_j(x)| \leq \frac{3}{\varepsilon}\int_{x-3\varepsilon}^{x+3\varepsilon} |b_j(\eta)| d\eta$$

$$\Rightarrow |H_\varepsilon b(x)| \leq \sum_j \frac{|I_j|}{|x-c_j|^2}\|b_j\|_1 + C\, Mb.$$

$$|\Omega^{*c} \cap \{H^*b \geq \lambda\}| \leq |\Omega^{*c} \cap \{\Sigma \square \geq \tfrac{\lambda}{2}\}| + |\{Mb \geq \tfrac{\lambda}{2c}\}|$$

$$\leq \frac{2}{\lambda}\sum_j \|b_j\|_1 \int_{\mathbb{R}/2I_j} \frac{|I_j|}{|x-c_j|^2} dx + \frac{c}{\lambda}\|b\|_1$$

$$\leq \frac{c}{\lambda}\|b\|_1 \leq \frac{2c}{\lambda}\|f\|_1$$

<u>Cor</u>: $H_\varepsilon f \xrightarrow{\varepsilon \to 0} Hf$ a.e. $\forall f \in L^p$, $1 \leq p < \infty$.

(4) <u>Multipliers</u>:

<u>Def</u>: $m \in L^\infty(\mathbb{R}^n)$ is multiplier of operator T_m

if $(T_m f)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$

When T_m can be extended to bdd operator on L^p. Then we say m is multiplier on L^p.

<u>Rmk</u>: T_m is bdd on $L^2(\mathbb{R}^n)$, with $\|T_m\| = \|m\|_\infty$.

<u>Pf</u>: $\|T_m f\|_2 \leq \|m\|_\infty \|f\|_2$. By Plancherel.

Conversely, set f: $\hat{f} = \chi_A$, $A = \{|m| > \|m\|_\infty - \varepsilon\}$

where I small enough. $m(A) < \infty$.

eg. i) $m(\xi) = -i\,\mathrm{sgn}(\xi)$. multiplier of Hilbert Transf.

ii) Def: $m_{a,b}(\xi) = \chi_{(a,b)}(\xi)$. multiplier of $S_{a,b}$.

recover $S_{a,b} = \frac{i}{2}(M_a H M_{-a} - M_b H M_{-b})$.

where $M_a f(x) = e^{2\pi i a x} f(x)$. bdd on L^p. $1 \le p \le \infty$.

Pf: Multiplier of $iM_a H M_{-a}$ is $\mathrm{sgn}(\xi - a)$.

prop. $S_{a,b}$ is strong-(p,p). $1 < p < \infty$. for $\forall$

$-\infty \le a < b \le \infty$

Cor. Set $a = b = R$. Then $S_R f = D_R * f$.

D_R is Dirichlet kernel. So:

$\| S_R f - f \|_p \xrightarrow{R\to\infty} 0$. $\forall f \in L^p$. $1 < p < \infty$.

RMK: $p = 1$. the converge only hold in measure.

prop. If m is func. of BV on $\mathbb{R}$. Then m is a multiplier on L^p. $1 < p < \infty$.

Pf. Note $\lim\limits_{N\to\infty} \int_{-N}^{N} |dm| = \lim\limits_{N\to\infty} TV_m[-N,N] < \infty$.

suppose $m(t) \xrightarrow{t\to-\infty} 0$. m is right-cont.

$\Rightarrow m(\xi) = \int_{-\infty}^{\xi} dm(t) = \int_{\mathbb{R}} \chi_{(t,\infty)}(\xi)\, dm(t)$.

Recover $T_m f(x) = \int_{\mathbb{R}} S_{t,\infty} f(x)\, dm(t)$.

So: $\| T_m f \|_p \overset{\text{minkowski}}{\le} (\int_{\mathbb{R}} |dm|) \| S_{t,\infty} f \|_p$

$\le C_p (\int_{\mathbb{R}} |dm|) \| f \|_p$.

KOKUYO

prop. If m is multiplier on $L^p(\mathbb{R}^n)$. Then:

$m(\xi+a)$, $m(\lambda\xi)$, $m(C\xi)$. $a \in \mathbb{R}^n$, $\lambda > 0$, $C \in O(n)$

are all multipliers on L^p, with same norm.

Pf: By property of Fourier Transf.

Rmk: m is multiplier on $L^p(\mathbb{R}^1)$. $\Rightarrow \tilde{m}(\xi) = m(\xi_1)$ is

multiplier on $L^p(\mathbb{R}^n)$. $T_{\tilde{m}} f(x) = T_m f(\cdot, x_2 \cdots x_n)(x_1)$.

By: $\int_{\mathbb{R}^n} |T_{\tilde{m}} f|^p = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^1} |T_m f(\cdot \cdots)(x_1)|^p$

$\leq C \|f\|_p^p$.

prop. $P \subset \mathbb{R}^n$, convex polyhedron that contains origin. Then $\|S_{\lambda P} f - f\|_p \xrightarrow{\lambda \to \infty} 0$. $\forall\, 1 < p < \infty$.

$S_{\lambda P}$ is operator with multiplier $\chi_{\lambda P}$.

Rmk: $n > 1$. the operator with multiplier $\chi_{B(0,1)}$ won't be bdd on L^p. $p \neq 2$.