Hardy Littlewood Max Function

(1) (p,q)-type Operators:

① Approxi. of id.

Def: $\phi \in L^1(\mathbb{R}^n)$ s.t. $\int \phi = 1$. Set $\phi_t(x) = t^{-n}\phi(t^{-1}x)$

We say $(\phi_t)_{t>0}$ is approxi. of identity.

Rmk: $\phi_t(\varphi) = \int_{\mathbb{R}^n} \phi_t \varphi \, dx = \int \phi(x)\varphi(tx)\,dx$

$\xrightarrow{t\to 0} \varphi(0) = \delta(\varphi)$. $\varphi \in S$.

i.e. $\phi_t \underset{S^*}{\rightarrow} \delta$. in distribution sense.

Thm. $(\phi_t)_{t>0}$ is approxi. of id. Then:

i) $\| \phi_t * f - f \|_p \xrightarrow{t\to 0} 0$. $f \in L^p$. $1 \le p < \infty$

ii) $\phi_t * f \xrightarrow{u} f$ $(t \to 0)$. $f \in C_0(\mathbb{R}^n)$

Pf: i) By Minkowski. separate $\int_{|y|>\delta/t} dy + \int_{|y|\le \delta/t} dy$.

ii) Routine.

Cor $\exists (t_k) \xrightarrow{k\to\infty} 0$. $\phi_{t_k} * f \underset{a.e.}{\xrightarrow{k\to\infty}} f$. $f \in L^p$. $1 \le p < \infty$

② Weak-Type:

Def: (X,μ), (Y,ν) two measure spaces.

$T: L^p(X,\mu) \longrightarrow \{ f: Y \to \mathbb{C} \mid f \text{ is measurable}\}$

i) T is weak-(p,q) if:

$$\begin{cases} \nu\{y \in Y \mid |Tf(y)| > \lambda\} \leq \left(\dfrac{C\|f\|_p}{\lambda}\right)^q, & q < \infty \\ T \text{ is BLO from } L^p(X,\mu) \text{ to } L^\infty(Y,\nu), & q = \infty \end{cases}$$

ii) T is strong-(p,q) if T is BLO: $L^p(X) \to L^q(Y)$.

Rmk: i) Strong-$(p,q) \Rightarrow$ Weak-(p,q)

ii) $\|Tf\|_q^q = \int_{\mathbb{R}^+} q\lambda^{q-1}\nu\{\square\}$. Weak (p,q) may avoid $Tf \notin L^q(Y,\nu)$.

Thm. $(T_t)_{t\geq 0}$ LOs on $L^p(X,\mu)$. $T^*f(x) = \sup_t |T_t f(x)|$.

If T^* is weak-(p,q). Then: $\{f \in L^p(X,\mu) \mid \lim_{t\to t_0} T_t f(x) = f(x).\ \text{a.e.}\}$ is closed in $L^p(X,\mu)$.

Pf: Set $(f_n) \subset \{\cdots\}$ such set $\xrightarrow{L^p} f$. Prove: $f \in \{\cdots\}$.

$\mu\{\overline{\lim_{t\to t_0}} |T_t f(x) - f(x)| > \lambda\} =$

$\mu\{\overline{\lim_{t\to t_0}} |T_t(f-f_n)(x) - (f-f_n)(x)| > \lambda\}$

$\leq \mu\{T^*(f-f_n) \geq \frac{\lambda}{2}\} + \mu\{|f-f_n| \geq \frac{\lambda}{2}\}$.

$\leq \left(\frac{2C}{\lambda}\|f-f_n\|_p\right)^q + \left(\frac{2}{\lambda}\|f-f_n\|_p\right)^p \to 0 \quad (n\to\infty)$

$\Rightarrow \mu\{\overline{\lim}\ \square > 0\} \leq \sum_{\mathbb{Z}^+} \mu\{\overline{\lim}\ \square > \frac{1}{k}\} = 0$.

Cor. Under the assumption above. $\{f \in L^p(X,\mu) \mid \lim_{t\to t_0} T_t f(x)$ exists a.e.$\}$ is also closed in $L^p(X,\mu)$.

Pf: Show: $\mu\{\overline{\lim}\ T_t f - \underline{\lim}\ T_t f > \lambda\} = 0$

Similarly, with $\overline{\lim} - \underline{\lim}\ T_t f \leq 2T^*f$.

Rmk: i) For f is complex value. Separate Re. Im.

ii) Since $f \in S$. $\phi_t * f \xrightarrow{u} f$. $S \underset{dense}{\subseteq} L^p(\mu)$.

To extend a.e. convergence on $L^p(X,\mu)$.

Check: $\sup_{t>0} |\varphi_t * f|$ is weak-(p,q).

③ Interpolation (Marcinkiewicz):

Def: (X,μ) is measure space. $f: X \to \mathbb{C}$. measurable.

$a_f : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$, $a_f(\lambda) = \mu\{|f(x)| > \lambda\}$ is

called distribution func. of f. w.r.t μ.

prop. $\phi: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. $\phi \in C^1$. $\uparrow$. $\phi(0) = 0$. Then:

$$\int_X \phi(|f(x)|) \, d\mu = \int_0^\infty \phi'(\lambda) a_f(\lambda) \, d\lambda.$$

Pf: $LHS = \int_X \int_0^{|f(x)|} \phi'(\lambda) \, d\lambda \, d\mu$. By Fubini.

Cor. $\|f\|_p^p = p \int_0^\infty \lambda^{p-1} a_f(\lambda) \, d\lambda$

Rmk: Weak-Type measure the size of dist. Func.

Def: $T: \{f$ is (X,μ)-measurable$\} \to \{f$ is (Y,ν)-measurable$\}$

is sublinear if $\begin{cases} |T(f_0+f_1)| \leq |Tf_0| + |Tf_1| \\ |T(\lambda f)| = |\lambda| \, |Tf|. \ \lambda \in \mathbb{C}. \end{cases}$

Thm. $T: L^{p_0}(X,\mu) + L^{p_1}(X,\mu) \to \{f$ is (Y,ν)-measurable$\}$

is sublinear. weak-(p_0,p_0). weak-(p_1,p_1). $1 \leq p_0 < p_1 \leq \infty$.

Then: T is strong-(p,p). $p_0 < p < p_1$

Pf: For $f \in L^p$. decompose $f = f_0 + f_1$.

$$\begin{cases} f_0 = f\,\chi_{\{|f| > c\lambda\}} \\ f_1 = f\,\chi_{\{|f| \le c\lambda\}} \end{cases} \quad \lambda > 0. \quad \Rightarrow \quad \begin{matrix} f_0 \in L^{p_0} \\ f_1 \in L^{p_1} \end{matrix}$$

From sublinear $\Rightarrow$ $d_{Tf}(\lambda) \le d_{Tf_0}(\lambda/2) + d_{Tf_1}(\lambda/2)$.

1°) $p_1 = \infty$.

Set $c = 1/2A_1$. A_1 satisfies: $\|Tg\|_\infty \le A_1 \|g\|_\infty$. $g \in L^\infty$.

$$\begin{cases} d_{Tf_0}(\lambda/2) \le \left(\frac{2C_0}{\lambda}\|f_0\|_{p_0}\right)^{p_0} \\ d_{Tf_1}(\lambda/2) = 0 \end{cases}$$

Estimate: $\|Tf\|_p^p = \int_0^\infty p\lambda^{p-1} d_{Tf}(\lambda)\, d\lambda$.

2°) $p_1 < \infty$.

Note that $d_{Tf_i}(\lambda/2) \le \left(\frac{2C_i}{\lambda}\|f_i\|_{p_i}\right)^{p_i}$. $i = 0, 1$

Similarly. $\|Tf\|_p^p \le 2p^{\frac{1}{p}}\left(\frac{1}{p-p_0} + \frac{1}{p_1-p}\right)^{1/p} C_0^{1-\theta} C_1^{\theta} \|f\|_p$.

where $1/p = 1-\theta/p_0 + \theta/p_1$. (precise form)

(2) <u>Maximal Functions</u>:

① <u>Denote</u>: $B_r = B(0,r)$

<u>Def</u>: For $f \in L^1_{loc}(\mathbb{R}^n)$. The Hardy-Littlewood maximal function of f is $Mf(x) = \sup_{r>0} \frac{1}{|B_r|}\int_{B_r} |f(x-y)|\,dy$

<u>Rmk</u>: i) Set $\Phi = |B_1|^{-1} I_{B_1}$. $(\Phi_r)_{r>0}$ is approxi of id. $\Rightarrow$ $Mf(x) = \sup_{r>0} \Phi_r * |f|(x)$.

ii) Other forms:

$Q_r = [-r, r]^n$. $M'f(x) = \sup_{r>0} \frac{1}{(2r)^n}\int_{Q_r} |f(x-y)|\,dy$.

generally. $M''f(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|\,dy$.

<u>prop.</u> $M \overset{C_n}{\sim} M' \overset{C_n}{\sim} M''$. pointwise.

(i.e. $C_n |Af| \le |Bf| \le C_n |Af|$. $\forall f \in D(A) = D(B)$)

<u>Thm.</u> M is weak-(1,1). strong-(p,p). $1 < p \le \infty$.

<u>Rmk:</u> So for M'. M''.

<u>Pf:</u> We have proved weak-(1,1) in Lebesgue differentiate Thm. With: $\|Mf\|_\infty \le \|f\|_\infty$

Apply Interpolation (Marc...) Thm. (M sublinear)

<u>prop.</u> ϕ is positive. radial. $\downarrow$ on $(0,\infty)$. $\phi \in L^1(\mathbb{R}^n)$.

Then. $\sup_{t>0} |\phi_t * f(x)| \le \|\phi\|_1 Mf(x)$.

<u>Rmk:</u> $\phi_t = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-|x|^2/4t}$ satisfies condition.

<u>Pf:</u> Set $\phi_n = \sum a_j \chi_{B_j^n}(x) \nearrow \phi$

$\Rightarrow$ WLOG. $\phi = \sum_1^m a_j \chi_{B_j}$. $a_j > 0$.

So: $\phi * f(x) = \sum a_j |B_j| \cdot \frac{1}{|B_j|} \chi_{B_j} * f$

$\le \|\phi\|_1 Mf(x)$.

Note dilation won't change the integral.

<u>Cor.</u> $M_\phi f(x) = \sup_{t,y} \{ |\phi_t * f(y)| \mid |x-y| < t \} \le C M f(x)$.

for $\phi \in L^1(\mathbb{R}^n)$. bdd. positive. radial. $\downarrow$ on $(0,\infty)$

<u>Pf:</u> $M_\phi f(x) = \sup \{ |\int \phi_t(u + t_0 e) f(x-u) du| \mid \|e\| = 1$

$0 \le t_0 < t \}$.

WLOG. $\phi = \sum_1^N a_j \chi_{B_j}$. $a_j > 0$. $\sum a_j < \infty$.

$$\Rightarrow \left|\int \phi_t(u+t_0 e) f(x-u)\,du\right| \le$$

$$\sum_1^N \int \frac{1}{t^n} \chi_{B_j}\left(\frac{u+t_0 e}{t}\right) a_j\, |f(x-u)|\,du =$$

$$\sum_1^N \int_{\left|\frac{u+t_0 e}{t}\right| \le r_j} \frac{a_j}{t^n} |f(x-u)|\,du \qquad (B_j = B(0, r_j))$$

$$\le \sum_1^N \int_{|u| \le (r_j+1)t} \frac{a_j}{t^n} |f(x-u)|\,du.$$

$$= \sum_1^N a_j(|B_j|+1)\frac{|B(0,t(r_j+1))|}{(|B_j|+1)t^n} \cdot \frac{1}{|B(0,t(r_j+1))|}\int_{\square} |f(x-u)|\,du$$

$$\le C(\|\phi\|_1 + \|\phi\|_\infty)\, Mf(x).$$

<u>Cor.</u> If $|\phi| \le \psi$. a.e. ψ is radial. positive. $\downarrow$ on $(0,\infty)$ integrable. Then, $\sup_{t>0} |\phi_t * f(x)|$ is weak-$(1,1)$ and strong-(p,p). $1<p\le\infty$.

<u>Pf:</u> Set $Tf = \sup_{t>0} |\psi_t * f|$. $Tf \lesssim Mf$ by above.

$\Rightarrow$ T is weak-$(1,1)$, strong-(p,p). $1<p\le\infty$.

With : $|\phi_t * f| \le |\phi_t| * |f| \le \psi_t * |f|$

$\le \|\psi\|_1\, Mf$.

<u>Cor.</u> Under the assumptions of cor. above.

If $f\in L^p$. $1\le p<\infty$. or $f\in C_0$. Then :

$$\lim_{t\to 0} \phi_t * f = (\textstyle\int\phi)\cdot f. \text{ a.e.}$$

<u>Rmk:</u> Poisson kernel $P_t(x) = \dfrac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\dfrac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}$

Gauss-Weierstrass kernel. $W_t(x) = t^{-n} e^{-\pi|x|^2/t^2}$

Fejér kernel $\dfrac{\sin^2(\pi R x)}{R(\pi x)^2} \le \min\{1, 1/(\pi x)^2\}$

satisfy conditions.

③ Dyadic maximal Func :

Def: i) Q_0 is collection of cubes in $\mathbb{R}^n$ which are congruent to $[0,1)^n$ and whose vertice lie on lattice $\mathbb{Z}^n$.

Q_k is collection of similar cubes. but whose vertices lie on $(2^{-k}\mathbb{Z})^n$.

ii) $\bigcup_k Q_k$ contains cubes called dyadic cubes.

iii) For $f \in L^1_{loc}(\mathbb{R}^n)$.

$$E_k f(x) = \sum_{Q \in Q_k} \left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x).$$

Rmk: $E_k f$ is conditional expectation of f w.r.t σ-algebra generated by Q_k

It's also like discrete form of approxi of id.

iv) Dyadic maximal Func. of f is :

$$M_d f(x) = \sup_k |E_k f(x)|$$

Thm. i) $M_d f$ is weak-$(1,1)$

ii) $\lim_{k \to \infty} E_k f(x) = f(x)$. a.e.

Pf: i) For $f \in L^1$. suppose $f \geq 0$.

$\{x \in \mathbb{R}^n \mid M_d f > \lambda\} = \bigcup_k \Omega_k$.

$\Omega_k = \{x \in \mathbb{R}^n \mid E_k f(x) > \lambda.\ E_j f(x) \leq \lambda.\ \forall j < k\}$

1°) Ω_k is union of cubes in Q_k.

Note $f \geq 0 \Rightarrow E_k f \uparrow$ if $k \uparrow$ (Ω_k well-def. disjoint)

For $x \in \Omega_k$. $\exists\, Q^x \in \mathcal{Q}_k$. s.t. $x \in Q^x$.

It's clear that $Q^x \subset \Omega_k$. But not for larger one.

So $\Omega_k = \bigcup_{x \in \Omega_k} Q^x$. ($\in \sigma(\mathcal{Q}_k)$)

2) $|\{M_\Delta f > \lambda\}| = \sum |\Omega_k| \leq \frac{1}{\lambda} \sum \int_{\Omega_k} E_k f$

$$= \frac{1}{\lambda} \sum_k \int_{\Omega_k} f \leq \|f\|_1 / \lambda$$

ii) Note i) holds for $f \in C_c(\mathbb{R}^n) \underset{dense}{\subseteq} L^1(\mathbb{R}^n)$

By i) $\Rightarrow$ holds for $L^1(\mathbb{R}^n)$

To apply on L^1_{loc}. Set $\tilde{f} = f \chi_Q \in L^1$. $Q \in \mathcal{Q}_0$.

It holds a.e. $x \in Q$. So on the whole $\mathbb{R}^n$.

Thm. For $f \in L^1$. $f \geq 0$. $\lambda > 0$. There exists (Q_j) disjoint. dyadic cubes. St. i) $f \leq \lambda$. a.e. $x \notin \bigcup Q_j$.

ii) $|\bigcup Q_j| \leq \|f\|_1 / \lambda$.

iii) $\lambda < \frac{1}{|Q_j|} \int_{Q_j} f \leq 2^n \lambda$.

Rmk: "iii)" restricts the local mean in $Q_j \sim \lambda$.

Pf: Set (Q_j) is collection of cubes in $\bigcup \Omega_k$. above.

i) is by $M_\Delta f(x) \geq f(x)$ ii) we have proved.

iii) Set $\tilde{Q}_j$ is dyadic cube contain Q_j with twice side long. $\Rightarrow |\tilde{Q}_j|^{-1} \int_{\tilde{Q}_j} f \leq \lambda$. (By def of Ω_k)

So, RHS: $|Q_j|^{-1} \int_{Q_j} f \leq |\tilde{Q}_j| / |Q_j| \cdot |\tilde{Q}_j|^{-1} \int_{\tilde{Q}_j} f \leq 2^n \lambda$.

Cor. For $f \in L^1_{loc}(\mathbb{R}^n)$. There exists (Q_j). st. i). iii) holds.

Pf: Set $\tilde{f} = |f| \chi_Q$. $Q \in \mathcal{Q}_0$.

Cor. (Calderón - Zygmund Decomposition)

For $f \in L^1(\mathbb{R}^n)$. $\lambda > 0$. $\exists\, g, b$. s.t.

$f = g + b$ (good and bad parts). Satisfy:

i) $|g(x)| \le 2^n \lambda$ ii) $\int b(x)dx = 0$.

Pf: Set: $g(x) = \begin{cases} f(x). & \text{if } x \notin \cup Q_j \\ |Q_j|^{-1} \int_{Q_j} f. & \text{if } x \in Q_j. \end{cases}$

$b(x) = \sum b_i(x)$. $b_i = (f - |Q_i|^{-1} \int_{Q_i} f) \chi_{Q_i}$

Rmk: It's like the technique of stopping time in probability.

g is mean part. b is error part.

Next, we will introduce a new decomposition method:

Prop. For $f \in L^1_{loc}(\mathbb{R}^n)$. Mf is l.s.c. so measurable.

Pf: i.e. prove: $\{ Mf > \lambda \}$ is open.

Note for $Mf(x) > \lambda$. $\exists\, B(x,r)$. st. $\frac{1}{|B(x,r)|} \int_B |f| > \lambda$

$\exists\, s > r$. $\frac{1}{|B(x,s)|} \int_{B(x,r)} |f| > \lambda$.

Then $\forall z : |z - x| < s - r$. $\Rightarrow B(x,r) \subset B(z,s)$

$\Rightarrow Mf(z) \ge |B(z,s)|^{-1} \int_{B(x,r)} |f| > \lambda$.

Prop. $f \in L^1_{loc}(\mathbb{R}^n)$. conti at x. $\Rightarrow Mf$ also conti at x.

Rmk: These also holds for M'.

It's easier to check $M_d f$ is l.s.c.

Lemma. $\Omega \underset{\text{open}}{\subseteq} \mathbb{R}^d$. There exists (Q_j) collection of disjoint dyadic cubes. s.t. $\Omega = \bigcup Q_j$. $\operatorname{diam}(Q_j) \le d(Q_j, \Omega^c) \le 4 \operatorname{diam}(Q_j)$. $\forall j$.

Pf. 1°) $\forall \bar{x} \in \Omega$. $\exists Q_{\bar{x}} \ni \bar{x}$. dyadic cube. holds

Set $\delta = d(\bar{x}, \Omega^c) > 0$.

Note the diameter of dyadic cube containing $\bar{x}$ varies over $(\sqrt{d}\, 2^{-k})_{k \in \mathbb{Z}}$.

$\Rightarrow \exists Q_{\bar{x}} \ni \bar{x}$. s.t. $\frac{\delta}{4} \le \operatorname{diam}(Q_{\bar{x}}) \le \frac{\delta}{2}$. ✓

2°) $\widetilde{Q} = \bigcup_{\bar{x} \in \Omega} Q_{\bar{x}}$ cover Ω. For disjoint:

We select the cubes in $\widetilde{Q}$. which are maximal. i.e. those cubes won't be contained in a larger one of $\widetilde{Q}$.

Rmk: i) Set $\Omega = \{ Mf(x) > \lambda \}$ (or $\{M_d f > \lambda\}$.)

We have a new decomposition of them.

$|\Omega| = |\bigcup Q_i| \le \frac{c}{\lambda} \|f\|_1$. by M is weak-(1,1).

"iii)" has a slight tweak: $\lambda \le |Q_j|^{-1} \int_{Q_j} |f| \le c_d \lambda$

by choosing $B \supseteq Q_j$. with $r = 5 \operatorname{diam} Q_j \Rightarrow B \cap \Omega^c \ne \emptyset$

ii) Actually, the 1st decomposition method can't indicates some property in Lemma:

Note for $\widetilde{Q}_j \supset Q_j$. with its twice side long. $\widetilde{Q}_j$ won't in $(Q_k)_{k \ne j}$. But $\widetilde{Q}_j$ may be cover by $\bigcup Q_j$. i.e. $\widetilde{Q}_j \subset \Omega = \bigcup Q_j$.

But interestingly. $\forall Q \notin (Q_j)$. $|Q|^{-1} \int_Q f \le \lambda$.

(3) Inequalities:

Lemma $f \geq 0$. m is Lebesgue measure.

Then: $m\{M'f > 4^n\lambda\} \leq 2^n m\{M_d f > \lambda\}$.

Pf: (Replace f by $|f|$ for general f)

Note that $\{M_d f > \lambda\} = \bigcup Q_j$. C-Z. decompose

Next. prove: $\{M'f > 4^n\lambda\} \subset \bigcup 2Q_j$

Fix $x \notin \bigcup_i 2Q_j$ and $Q^x \ni x$. $2^{k-1} \leq \ell(Q^x) < 2^k$.

Q^x intersects m cubes in Q_k, denote $(R_i)_1^m$.

$R_i \not\subseteq Q_j$. $\forall Q_j \in (Q_j)$. Otherwise $x \in Q^x \subset 2R_i \subset \bigcup 2Q_j$

$\Rightarrow |Q|^{-1}\int_Q f = |Q|^{-1}\sum_1^m \int_{Q\cap R_i} f$

$\leq \sum_1^m \frac{2^{kn}}{|Q|}\frac{1}{|R_i|}\int_{R_i} f \leq 2^n m\lambda \leq 4^n\lambda$.

Rmk: Replace m by $w\,dx$. $w \in A_1$, it still holds.

prop. $f \in L^1$. $f \not\equiv 0$. $\Rightarrow Mf \notin L^1$. ($M$ isn't strong-(1,1).)

Pf: $\exists R > 0$. st. $\int_{B_R} |f| \geq \varepsilon > 0$.

For $|x| > R$. $\Rightarrow B_R \subset B(x, 2|x|)$. Then:

$Mf(x) \geq \frac{1}{|2x|^n}\int_{B_R} |f| \geq \frac{\varepsilon}{2^n|x|^n} \notin L^1$.

Thm. $B \underset{\text{bdd}}{\subseteq} \mathbb{R}^n$. Then: $\int_B Mf \leq 2|B| + C\int |f|\log^+|f|$

Pf: WLOG. $f \geq 0$. $LHS = 2\int_0^\infty |\{x \in B \mid Mf > 2\lambda\}| d\lambda$.

$\leq 2|B| + 2\int_1^\infty \square \, d\lambda$

Set $f = f_1 + f_2$. $f_1 = f I\{f > \lambda\}$.

$\Rightarrow \{Mf > 2\lambda\} \cap B \subset \{Mf_1 > \lambda\} \cap B$.

So: $\int_1^\infty \square \le \int_1^\infty \frac{c}{\lambda} \int_{\{f > \lambda\}} f \, dx \, d\lambda$.

$$\overset{\text{Fubini}}{\le} c \int |f| \int_1^{\max\{f,1\}} \frac{d\lambda}{\lambda} \, dx$$

Thm. (Weighted Norm inequi.)

$w \ge 0$. measurable. For $1 \le p < \infty$. $\exists C_p$. s.t.

$\|Mf\|_{L^p(w)} \le C_p \|f\|_{L^p(Mw)}$. $\int_{\{Mf > \lambda\}} w(x) \, dx \le \frac{C_1}{\lambda} \|f\|_{L^1(Mw)}$.

Pf: By interpolation. prove: $\|Mf\|_{L^\infty(w)} \le \|f\|_{L^\infty(Mw)}$. and latter.

WLOG. suppose $Mw(x) > 0$. $\forall x$. otherwise $w \equiv 0$.

1°) $\forall a > \|f\|_{L^\infty(Mw)} \Rightarrow \int_{\{|f| > a\}} Mw(x) \, dx = 0$

$\Rightarrow |\{|f| > a\}| = 0$. $\Rightarrow Mf \le a$. m-a.e.

2°) WLOG. Suppose $f \ge 0$. $f \in L^1(\mathbb{R}^n)$.

since for $f \in L^1(Mw)$. $f \chi_{B(0,j)} \in L^1(\mathbb{R}^n) \uparrow f$.

By Lemma above: $\{M'f > 4^n \lambda\} \subset \bigcup 2Q_j$.

$$\int_{\{M'f > 4^n\lambda\}} w \, dx \le \sum_j \int_{2Q_j} w(x) \, dx$$

$$\le \frac{2^n}{\lambda} \sum_j \int_{Q_j} f(y) \left(\frac{1}{2|Q_j|} \int_{2Q_j} w \, dx \right) dy$$

$$\le \frac{c}{\lambda} \int f \, M''w \, dy.$$

combined with: $M''w \le C_n Mw$.

$\{Mf > \lambda\} \subseteq \{M'f > 4^n \lambda\}$.

Rmk: If $w \in A_1$. Then in both sides of inequality.

it holds with same weight w.