Fourier Series and Integral

(1) <u>Series:</u>

Consider $f \in L^1(\mathbb{T})$. $\mathbb{T} = \mathbb{R}^1/\mathbb{Z}$ one-dim Torous.

<u>Def:</u> i) $\hat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x} dx$. $(\hat{f}(k))$ is Fourier coefficients. $\sum_{\mathbb{Z}} e^{2\pi i k x} \hat{f}(k)$ is Fourier series of $f(x)$.

ii) N-th partial sum of Fourier series is:

$$S_N(f) = \sum_{-N}^{N} \hat{f}(k) e^{2\pi i k x}$$

iii) $(K_n(x))_n$ is good kernel if it satisfies:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} K_n(x) dx = 1. \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} |K_n(x)| \le M < \infty.$$

$$\int_{|x|>\delta} |K_n(x)| dx \xrightarrow{n\to\infty} 0. \quad \forall \delta > 0.$$

<u>Rmk:</u> i) $S_N(f)(x) = \int_0^1 f(t) D_N(x-t) dt$

$$D_N(t) = \sum_{-N}^{N} e^{2\pi i k t} = \sin(\pi(2N+1)t) / \sin \pi t$$

is called Dirichlet kernel satisfy:

$$\begin{cases} \int_{-\frac{1}{2}}^{\frac{1}{2}} D_N(t) dt = 1 \\ |D_N(t)| \le 1/\sin \pi\delta. \quad \delta \le |t| \le 1/2. \end{cases}$$

ii) Dirichlet kernel isn't good kernel.

<u>Lemma.</u> $L_N = \int_{-\frac{1}{2}}^{\frac{1}{2}} |D_N(t)| dt = \frac{4}{\pi^2} \log N + O(1)$.

<u>Pf:</u> $L_N = 2\int_0^{\frac{1}{2}} \left| \frac{\sin(\pi(2N+1)t)}{\pi t} \right| dt + O(1)$

$$= 2\sum_0^{N-1} \int_k^{k+1} \left| \frac{\sin \pi t}{\pi t} \right| dt + O(1)$$

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$$= \frac{2}{\pi} \int_0^1 \sin|\pi t| \sum_0^{N-1} \frac{1}{t+k} \, dt + O(1)$$

$$= \frac{4}{\pi^2} \log N + O(1).$$

① Criteria of Pointwise Converge:

Lemma (Riemann - Lebesgue)

$$f \in L^1(\mathbb{T}) \Rightarrow \lim_{|k|\to\infty} \hat{f}(k) = 0.$$

Pf: 1°) $f \in C(\mathbb{T})$. Note: $e^{2\pi i x}$ has period 1

$$\Rightarrow \hat{f}(k) = - \int_0^1 f(x - 1/2k) e^{-2\pi i k x} dx.$$

$$= \frac{1}{2} \int_0^1 [f(x) - f(x - \tfrac{1}{2k})] e^{2\pi i k x} \to 0.$$

2°) $\exists g_n \in C_c^\infty \xrightarrow{L^1} f$. $|(f - g_n)^\wedge(k)| \le \|f - g_n\|_1 \to 0$

Thm (Dini's Criterion)

If for some $x \in \mathbb{R}^1$. $\exists \delta > 0$. s.t. $\int_{|t|<\delta} \left| \frac{f(x-t) - f(x)}{t} \right| dt < \infty$.

Then: $S_N f(x) \xrightarrow{N\to\infty} f(x)$

Pf: $S_N f(x) - f(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} (f(x-t) - f(x)) \frac{\sin(\pi(2N+1)t)}{\sin \pi t} dt$

$$= \int_{|t|<\delta} + \int_{\delta<|t|<\frac{1}{2}} \to 0.$$

by $t \sim \sin t$ $(t\to 0)$. apply R-L Lemma.

Thm (Jordan's Criterion)

If f is BV in a nbd of x. Then:

$$S_N f(x) \xrightarrow{N\to\infty} \frac{1}{2}(f(x+) + f(x-)).$$

Pf: $S_N f(x) = \int_0^{\frac{1}{2}} (f(x-t)+f(x+t)) D_N(t)\,dt$

set $g(t) = (f(x-t)+f(x+t))$. BV in U_x. WLOG. $g \uparrow$

Prove: $\int_0^{\frac{1}{2}} g(t) D_N(t)\,dt \xrightarrow{N\to\infty} g(0+)/2$.

WLOG. $g(0+)=0$. $\int_0^{\frac{1}{2}} \square = \int_0^{\delta} + \int_{\delta}^{\frac{1}{2}}$. $g<\varepsilon$ if $t<\delta$.

1') $\int_{\delta}^{\frac{1}{2}} \square \to 0$. by R-L Lemma. directly.

2') $\int_0^{\delta} \square = g(\delta-) \int_v^{\delta} D_N(t)\,dt$. $\exists\, v \in (0,\delta)$.

$\left|\int_v^{\delta} D_N(t)\,dt\right| \sim \int_0^{\delta} \left|\frac{1}{\sin \pi t} - \frac{1}{\pi t}\right| + \sup_M \left|\int_0^M \frac{\sin \pi t}{t}\,dt\right|$

$\sim O(1)$. by $\sin t \sim t$ $(t\to 0)$.

Thm. (Riemann Localization Principle)

$f=0$ in U_x nbd of x. $\Rightarrow S_N f(x) \xrightarrow{N\to\infty} 0$.

Rmk: $f=g$ on U_x $\Rightarrow$ Their Fourier Series behave same.

Pf: suppose $f=0$ on $(x-\delta, x+\delta)$

$S_N f(x) = \int_{\delta\le|t|\le\frac{1}{2}} f(x-t) D_N(t)\,dt = (g e^{\pi i \cdot})^{\wedge}(N) + (g e^{-\pi i \cdot})^{\wedge}(-N)$

$g(t) = \frac{f(x-t)}{2i\sin \pi t} I_{\{\delta\le|t|\le\frac{1}{2}\}} \in L^1$. apply R-L Lemma.

Cor. $f \in L^1[0,2\pi] \Rightarrow \int_0^{2\pi} f \sin nx\,dx$. $\int_0^{2\pi} f \cos nx \xrightarrow{n\to\infty} 0$.

Rmk: Convergence of Fourier Series is a local property.

Note: if modify $f(x)$ slightly, $\hat{f}(k)$ changes a lot. outside a nbd of x. But behave at x doesn't change.

Thm. $\exists\, f \in C(\mathbb{T})$. its Fourier Series diverges at one point.

Rmk: Such f can't satisfy α-Hölder condition. Otherwise it satisfy Dini. Cri.

Pf: $(C(\mathbb{T}), \|\cdot\|_\infty) \xrightarrow{T_N} \mathbb{C}$. defined by:

$$T_N f = S_N f(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) D_N(t) dt.$$

1°) $\|T_N\| = L_N$.

$sgn(D_N(t))$ has finite jump. can be approxi. by $C(\mathbb{T}) \Rightarrow \|T_N\| \geq L_N$.

Conversely. $|T_N f| \leq \|T_N\| \|f\|_\infty$.

2°) $\|T_N\| = L_N \nearrow \infty$. Apply UBP.

$\exists f \in C(\mathbb{T})$. st. $\overline{\lim\limits_N} |S_N f(0)| = \infty$.

④ Order of Coefficients:

prop. i) $f \in C^k(\mathbb{T}) \Rightarrow \hat{f}(n) = o(n^{-k})$ $(n \to \infty)$

ii) f is BV $\Rightarrow \hat{f}(n) = O(\frac{1}{n})$ $(n \to \infty)$

iii) $|f(x+h) - f(x)| \leq C|h|^\alpha$. $0 < \alpha \leq 1$. Then:

$\hat{f}(n) = O(1/|n|^\alpha)$. $(n \to \infty)$

Pf: i) Integrate by part $\Rightarrow$ R-L. Lemma.

ii) WLOG. f is monotone. on $[-\frac{1}{2}, \frac{1}{2})$

Approxi. by Simple Func. $\sum \alpha_k \chi_{[a_k, a_{k+1}]}$

$\sim O(\frac{1}{n})$. $|\alpha_k| \leq M$. $\forall k$.

iii) By ii: $\hat{f}(0) = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} [f(x) - f(x - \frac{1}{2k})] e^{\square}$

Cor. If f satisfies iii) with $\frac{1}{2} < \alpha \le 1$.

Then $\sum |\hat{f}(n)| < \infty$.

Pf: 1°) By Parseval: $\|g_h\|_2^2 = 4\sum |\sin nh|^2 |\hat{f}(n)|^2$

$g_h(x) = f(x+h) - f(x-h)$.

$\Rightarrow \sum |\sin nh|^2 |\hat{f}(n)|^2 \lesssim h^{2\alpha}$.

2°) Set $h = \pi/2^{p+1}$. Then:

$\frac{1}{2}\sum_{2^{p-1} \le |n| \le 2^p} |\hat{f}(n)|^2 \le \sum_{[2^{p-1}, 2^p]} |\sin nh|^2 |\hat{f}(n)|^2$

$\lesssim 1/2^{2\alpha p}$.

With $\left(\sum_{2^{p-1} \le |n| \le 2^p} |\hat{f}(n)|\right)^2 \le 2^{p-1} \sum_{\square} |\hat{f}(n)|^2$

$\lesssim 1/2^{2\alpha p - p}$

$\Rightarrow \sum_{\mathbb{Z}} |\hat{f}(n)| = \sum_p \sum_{2^{p-1} \le |n| \le 2^p} |\hat{f}(n)| < \infty$.

③ Converge in Norm:

i) Does $\|S_N f - f\|_p \xrightarrow{N \to \infty} 0$. $\forall f \in L^p(\mathbb{T})$. hold ?

ii) Does $S_N f \to f$ a.e. $\forall f \in L^p(\mathbb{T})$. hold ?

Lemma: i) holds for $1 \le p < \infty \Leftrightarrow \exists C_p > 0$. st. $\|S_N f\|_p \le C_p \|f\|_p$ $\forall N$

Pf: $(\Rightarrow)$ It's direct.

$(\Leftarrow)$ $\exists g_n$ trigonometric poly $\xrightarrow{L^p} f$. Since density

Note $S_N g_n = g_n$ if $N \ge \deg g_n$.

Fix n. st. $\|f - g_n\|_p \le \varepsilon$. By Triangle inequi.

Rmk: i) $1 < p < \infty$. The inequility holds. But $p = 1$. not.

ii) $p = 2$ direct from $L^2(\mu) \underset{iso}{\cong} \ell^2$. if μ separable.

ii) : It holds when $1<p<\infty$. by Carleson.
but not when $p=1$. by Kolmogorov.

④ Summability Methods:

Lemma (Minkowski Inequi.)

$(X.\mathcal{M}.\mu)$. $(Y.\mathcal{N}.\nu)$ σ-finite measure spaces.
$f \in \mathcal{M}\otimes\mathcal{N}$. for $1\le p\le\infty$. If $f(\cdot,y)\in L^p(\mu)$
for a.e. y. $y\mapsto \|f(\cdot,y)\|_p \in L^1(\nu)$. Then:
$f(x,\cdot)\in L^1(\nu)$. a.e. x. $x\mapsto \int f(x,y)\,d\nu(y)\in L^p(\mu)$.
$\|\int f(\cdot,y)\,d\nu(y)\|_p \le \int \|f(\cdot,y)\|_p\,d\nu(y)$.

Pf: $p=1$ is trivial. For $1<p<\infty$. WLOG. $f\ge 0$.

$$\text{Note: } L^p(\mu) \underset{\text{iso}}{\overset{\Lambda}{\xrightarrow{\sim}}} L^q(\mu)^* \qquad \Lambda(f)(g)=\int fg\,d\mu.$$
$$f \longmapsto \Lambda(f)$$

$$|\Lambda(\int f(\cdot,y)\,d\nu(y))(g)| \le \int(\int|f|\,d\nu)|g|\,d\mu$$
$$\overset{\text{Fubini}}{=} \iint |fg|\,d\mu\,d\nu$$
$$\le \int \|f(\cdot,y)\|_p\|g\|_q\,d\nu$$
$$\Rightarrow \|\int f(\cdot,y)\,dy\|_p = \|\Lambda(\int f(\cdot,y)\,d\nu)\| \le \int \|f\|_p\,d\nu.$$

i) Cesàro Sum:

Def: $\sigma_N f(x) = \frac{1}{N+1}\sum_0^N S_k f(x) \overset{\Delta}{=} \int_0^1 f(t)F_N(x-t)\,dt$

$$F_N = \frac{1}{N+1}\left(\frac{\sin(\pi(N+1)t)}{\sin \pi t}\right)^2. \text{ Fejér kernel.}$$

Rmk: $F_N(t) \geq 0$. $\|F_N\|_1 = 1$. $\int_{\delta<|t|<\frac{1}{2}} F_N\, dt \xrightarrow{N\to\infty} 0$. $\forall \delta > 0$.

$\Rightarrow$ Fejér kernel is good kernel.

Thm. If $f \in L^p$. $1 \leq p < \infty$. or $f \in C(\mathbb{T})$. and $p = \infty$.

Then: $\|\sigma_N f - f\|_p \to 0 \quad (N \to \infty)$.

Pf: $LHS \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \|f(\cdot - t) - f(\cdot)\|_p F_N$

$\leq \int_{|t|<\delta} \|f(\cdot - t) - f(\cdot)\|_p F_N + 2\|f\|_p \int_{\delta<|t|\leq\frac{1}{2}} F_N \to 0$

Cor. i) Trigonometric Polynomials are dense in L^p. $1 \leq p < \infty$.

ii) $f \in L^1(\mathbb{T})$. $\hat{f}(k) = 0$. $\forall k \Rightarrow f \equiv 0$. a.e.

Pf: ii) $\hat{f}(k) = 0 \Rightarrow \sigma_N f \equiv 0$. $\forall N$.

ii) Poisson Sum:

Consider $u(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k + \sum_{-\infty}^{-1} \hat{f}(k) \bar{z}^{|k|}$. $z = re^{2\pi i\theta}$

Note $(\hat{f}(k))$ bdd $\Rightarrow |z| < 1$. $u(z)$ is well-def.

Def: $u(re^{2\pi i\theta}) = \sum_{\mathbb{Z}} \hat{f}(k) r^{|k|} e^{2\pi i k\theta} \underset{(r<1)}{\overset{\Delta}{=}} \int_0^1 f(t) P_r(\theta - t)\, dt$

$P_r(t) = \sum r^{|k|} e^{2\pi i k t} = \dfrac{1 - r^2}{1 - 2r\cos(2\pi t) + r^2}$ Poisson kernel.

Rmk: $P_r \geq 0$. $\|P_r\|_1 = 1$. $\int_{\delta<|t|<\frac{1}{2}} P_r\, dt \xrightarrow{r\to 1^-} 0$. $\forall \delta > 0$.

$\Rightarrow (P_r(t))$ is good kernel as $r \to 1$.

Thm. If $f \in L^p(\mathbb{T})$. $1 \leq p < \infty$. or $f \in C(\mathbb{T})$. $p = \infty$

Then: $\|P_r f - f\|_p \to 0$ as $r \to 1^-$.

iii) Relations:

Def: (c_k) is Abel summable if $\forall\, 0 \le r < 1$. $A(r) = \sum c_k r^k < \infty$ and $\lim\limits_{r \to 1^-} A(r) = s$. $\exists\, s \in \mathbb{R}^1$.

(c_k) is Cesàro summable if $\sum_1^N s_k / N$ converges as $N \to \infty$. $s_k = \sum_0^{k-1} c_i$

Prop. Summable $\Rightarrow$ Cesàro summable $\Rightarrow$ Abel summable.

(2) Fourier Transform:

Def: i) $\mathcal{M}(\mathbb{R}^n) = \{ |f(x)| \le A / |1+|x||^{n+\delta}.\ \exists\, \delta > 0 \}$. Moderate decrease functions family.

ii) $\mathcal{S}(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) \mid \sup\limits_x |x^\alpha D^\beta f(x)| = P_{\alpha,\beta}(f) < \infty$ $\forall\, \alpha, \beta \in \mathbb{N}^n \}$. Schwartz Class.

Rmk: Define a topo on $\mathcal{S}$:

Note $(P_{\alpha,\beta})$ is family of seminorms.

Set $(\phi_k) \subset \mathcal{S} \xrightarrow{k \to \infty} 0$ if $\lim\limits_{k \to \infty} P_{\alpha,\beta}(\phi_k) = 0$. $\forall\, \alpha, \beta \in \mathbb{N}^n$. $\Rightarrow \mathcal{S}$ is Fréchet space.

iii) $\mathcal{S}^*$ set of BLF on $\mathcal{S}$. called space of tempered distributions.

iv) $T \in \mathcal{S}^*$. $\hat{T}(f) =: T(\hat{f})$. for $f \in \mathcal{S}$.

Fourier Transform of tempered dist.

Thm. $\wedge : S^* \to S^*$. Fourier Transform between S^*.

is a bijective BLO. So as $\wedge^{-1} = \vee$.

<u>Pf</u>: Note Fourier Transform has period 4:

i.e. $(f^\wedge)^\wedge = \tilde{f}$. $\tilde{f}(x) = f(-x)$. $\Rightarrow \wedge^3 = \vee$.

As for $\wedge$ is BLO: $T_n \to T \Rightarrow \hat{T_n}(f) = T_n(\hat{f}) \to T(\hat{f})$

Note if $f \in L^p$, $1 \le p \le \infty$. then def: $T_f(\phi) = \int f\phi$. $\phi \in S$.

$\Rightarrow T_f \in S^*$. For $1 \le p \le 2$. We can define $\hat{f}$. directly:

Thm. $\wedge : L^2 \xrightarrow[iso]{\sim} L^2$. Besides, $\hat{f}(\xi) = \lim_{R \to \infty} \int_{|x|<R} f(x) e^{-2\pi i x^T \cdot \xi} dx$

$f(x) = \lim_{R \to \infty} \int_{|\xi|<R} \hat{f}(\xi) e^{2\pi i x^T \cdot \xi} d\xi$. (limit exists. $\in L^2$)

<u>Pf</u>: Extend from $S \underset{dense}{\subseteq} L^2$. The latter is by conti of $\wedge$.

① <u>Interpolation (Riesz)</u>:

Thm. $1 \le p_0, p_1, q_0, q_1 \le \infty$. for $0 < \theta < 1$. $\begin{cases} 1/p = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \\ 1/q = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \end{cases}$

If T is LO: $L^{p_0} + L^{p_1} \to L^{q_0} + L^{q_1}$. st.

$$\begin{cases} \|Tf\|_{q_0} \le M_0 \|f\|_{p_0}. & \forall f \in L^{p_0} \\ \|Tf\|_{q_1} \le M_1 \|f\|_{p_1}. & \forall f \in L^{p_1}. \end{cases}$$

Then: $\|Tf\|_q \le M_0^{1-\theta} M_1^{\theta} \|f\|_p$. $\forall f \in L^p$.

<u>Lemma.</u> $1 \le p, q \le \infty$. $1/p + 1/q = 1$.

i) If $g \in L^q$. Then $\|g\|_{L^q} = \sup_{\|f\|_p = 1} |\int g f \, dm|$

ii) If $g \in L^1(E)$. $\forall m(E) < \infty$. and $\sup\{|\int f g \, dm|$:

$\|f\|_p \le 1$. f is simple Func.$\} = M < \infty$.

Then $g \in L^q$. $\|g\|_q = M$.

Pf: i) by Riesz. Repre. ii) by duality.

Lemma (Three-Lines)

$\phi(z) \in \mathcal{H}(S)$. $S = \{z \in \mathbb{C} \mid 0 < Re(z) < 1\}$. and

ϕ is bdd. conti on $\bar{S}$. If $M_0 = \sup_{\eta \in \mathbb{R}} |\phi(i\eta)|$. $M_1 =$

$\sup_{\eta \in \mathbb{R}} |\phi(1+i\eta)|$ Then: $\sup_{\eta \in \mathbb{R}} |\phi(t+i\eta)| \le M_0^{1-t} M_1^{t}$.

for $\forall\ t \in (0,1)$.

Pf. By Lindelöf method:

1') Assume $M_0 = M_1 = 1$. $\sup_{0 \le x \le 1} |\phi(x+i\eta)| \xrightarrow{|\eta| \to \infty} 0$.

Set $M = \sup_{\bar{S}} |\phi(z)|$. $\exists (z_n) \subset S$ s.t.

$|\phi(z_n)| \xrightarrow{n \to \infty} M$. (z_n) lies on bdd domain

Apply MMP. $M = \sup_{\partial S/(\infty)} |\phi(z)| \le 1$.

2') Remove " $\sup_{x \in [0,1]} |\phi(x+i\eta)| \xrightarrow{|\eta| \to \infty} 0$ " in 1').

Def: $\phi_\varepsilon(z) = \phi(z) e^{\varepsilon(z^2-1)}$. $|z| \in [0,1]$.

S_1: $|\phi_\varepsilon| \le 1$. on $Re z = 0.1$. satisfies 1')

$\Rightarrow |\phi_\varepsilon| \le 1$ in $\bar{S}$. Set $\varepsilon \to 0$.

3') Remove " $M_0 = M_1 = 1$ " in 2'):

M_0 or $M_1 = 0$. trivial. by Uniqueness Thm. $\phi \equiv 0$

Otherwise. Set $\tilde{\phi}(z) = M_0^{z-1} M_1^{-z} \phi(z)$.

Rmk: $\phi(z) \in \mathcal{H}(S) \cap C(\bar{S})$. $|\phi| \le 1$. on $\{Re = 0.1\}$ but

ϕ is unbdd on $\bar{S}$ will happen : $\phi(z) = e^{-ie^{i\pi z}}$.

Return to the pf :

i) Assume $p < \infty$, $q > 1$:

1°) Assume f is simple. $\|f\|_p = 1$.

By Lemma, show : $|\int (Tf) g \, d\nu| \le M$. $\forall g$ simple $\|g\|_{q'} = 1$

Pf $\begin{cases} f_z = |f|^{\gamma(z)} f/|f|. & \gamma(z) = p(\frac{1-z}{p_0} + \frac{z}{p_1}) \\ g_z = |g|^{\delta(z)} g/|g|. & \delta(z) = q'(\frac{1-z}{q_0'} + \frac{z}{q_1'}) \end{cases}$

$\Rightarrow \begin{cases} \|f_z\|_{p_0} = 1. \text{ on } \{Re(z) = 0\}. \\ \|f_z\|_{p_1} = 1 \text{ on } \{Re(z) = 1\}. \end{cases}$ $f_t = f.\ \forall t \in \mathbb{R}$.

Similar for g_z. Consider $\phi(z) = \int (Tf_z) g_z \, d\nu$.

Write $f = \sum_1^n a_k \chi_{E_k}$. $g = \sum_1^m b_j \chi_{F_j}$. Express $\phi(z)$.

$\Rightarrow$ Check $\phi(z)$ satisfies three line Lemma.

2°) For general $f \in L^p$. $1 \le p < \infty$.

$\exists (f_n)$ simple $\xrightarrow{L^p} f \rightarrow (Tf_n)$ Cauchy in L^q

Next, show : $\lim_n Tf_n = Tf$. a.e. (So : $Tf_n \xrightarrow{L^q} Tf$)

$f = f^u + f^l$. $f^u = f I_{\{|f| \ge 1\}} \in L^t$. $1 \le t \le q$.

Similarly set : $f_n = f_n^u + f_n^l$. Suppose $p_0 \le p \le p_1$

$\|f_n^u - f^u\|_{p_0} \le \|f_n^u - f^u\|_p \le \|f_n - f\|_p \rightarrow 0$

$\Rightarrow f_n^u \xrightarrow{L^{p_0}} f^u$. Similar : $f_n^l \xrightarrow{L^{p_1}} f^l$. $\Rightarrow$ Select subseq.

So : We can use conti of T on L^{p_1}, L^{p_0}

ii) $p = \infty$, $q = 1$:

Then $p_0 = p_1 = \infty$. It's trivial to check.

iii) $p < \infty$. $q = 1$:

Then $q_0 = q_1 = 1$. set $q_z = q$. $\forall z$. Argue as $q > 1$.

Rmk: The essence of pf is simple func. We can extend T on $\forall L^p$. $1 \le p < \infty$.

Cor (Hausdorff - Young Inequal.)

$\forall f \in L^p$. $1 \le p \le 2 \Rightarrow \hat{f} \in L^{p'}$. $\|\hat{f}\|_{p'} \le \|f\|_p$.

Pf: $\|\hat{f}\|_2 = \|f\|_2$. $\|\hat{f}\|_\infty \le \|f\|_1$.

② Converge and Summation:

i) General:

Def: $(S_R f)^\wedge = \chi_{B_R} \hat{f}$. $B_R = \{Rx \mid x \in B\}$.

Determine: $S_R f \to f$ $(R \to \infty)$?

For converge in norm:

$n = 1$ ✓. $n > 1$. $p \ne 2$ holds.

Besides: $n = 1$. $S_R f = D_R * f(x)$. $D_R = \int_{-R}^{R} e^{2\pi i x y} dy$

$\Rightarrow D_R(x) = \sin(2\pi R x)/\pi x \in L^q$. $\forall q > 1$.

$\therefore S_R f$ is well-def if $f \in L^p$. $1 < p < \infty$.

For a.e. converge:

Depend on: $\|\sup_R |S_R f|\|_p \le C_p \|f\|_p$. $\forall 1 < p < \infty$.

Carleson Hunt Thm

ii) Cesàro Sum:

Def: For $n=1$. $\sigma_R f(x) = \frac{1}{R}\int_0^R S_t f(x)\,dt = F_R * f(x)$

$F_R = \frac{1}{R}\int_0^R D_t(x)\,dt = \sin^2(\pi R x)/R(\pi x)^2$. Fejér kernel.

Rmk: $F_R \in L^1$. $\sigma_R f(x) \xrightarrow{L^p} f(x)$. $\forall\, 1 \le p < \infty$.

iii) Abel-Poisson Sum:

Def: $u(x,t) = \int_{\mathbb{R}^n} e^{-2\pi t|\xi|}\hat{f}(\xi)\, e^{2\pi i x^T \cdot \xi}\, d\xi = P_t * f(x)$

$\hat{P}_t(\xi) = e^{-2\pi t|\xi|}$. Poisson kernel.

Rmk: $u(x,t)$ is harmonic in $\mathbb{R}^n \times (0,\infty)$

Thm. (Poisson Summation Formula)

$f, \hat{f} \in \mathcal{M}(\mathbb{R}^n)$. Then: $\sum_{v\in\mathbb{Z}^n} f(x+v) = \sum_v \hat{f}(v)\, e^{2\pi i x^T \cdot v}$.

Pf: Check Fourier coefficients of two sides are identical.

Cor. Set $x=0 \Rightarrow \sum_{\mathbb{Z}^n} f(v) = \sum_{\mathbb{Z}^n} \hat{f}(v)$.